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Factorization and extension of positive homogeneous polynomials

by

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To the memory of Aleksander Pełczyński

Abstract. We study the following problem: Given a homogeneous polynomial from a sublattice of a Banach lattice to a Banach lattice, under which additional hypotheses does this polynomial factorize through L_p-spaces involving multiplication operators? We prove that under some lattice convexity and concavity hypotheses, for polynomials certain vector-valued norm inequalities and weighted norm inequalities are equivalent. We combine these results and prove a factorization theorem for positive homogeneous polynomials which is a variant of a celebrated factorization theorem for linear operators due to Maurey and Rosenthal. Our main application is a Hahn–Banach extension theorem for positive homogeneous polynomials between Banach lattices.

1. Introduction. The study of polynomials and multilinear mappings between Banach spaces has recently gained importance within Banach space theory as well as other branches of analysis. The contemporary literature shows that multilinear functional analysis contributes to the solution of various problems in modern mathematics (see for example the recent article [18]).

This article has a twofold aim. We prove factorization theorems for homogeneous polynomials between Banach spaces, and we use these results in order to study when a given positive homogeneous polynomial defined on a subspace X_0 of a Banach function lattice X extends to the whole space X. Our main result is as follows: Let $1 \leq r, s, t < \infty$ and $m \in \mathbb{N}$ be such that $1 \leq t \leq r/m \leq r \leq s$. Then for every positive m-homogeneous polynomial $P\colon X_0 \to Y$ from a sublattice X_0 of an s-convex Banach function lattice X into a t-concave Banach function lattice Y there are multiplication operators $M_f\colon X \to L_r$ and $M_g\colon L_{r/m} \to Y$ as well as an m-homogeneous

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polynomial $Q\colon L_r \to L_{r/m}$ such that the composition $M_g Q M_f$ extends P from X_0 to X.

The problem of extending homogeneous polynomials from subspaces of Banach spaces to the whole space has gained the interest of many mathematicians—motivated by the seminal paper [1] by Aron and Berner where homogeneous polynomials on X are extended to the bidual. Since then, several works have appeared concerning the extension of polynomials (see, e.g., [2, 5, 6, 7, 19, 27] and the surveys given in [14, 30]).

Here we follow an alternative approach which was successfully applied within linear theory. In order to explain this idea recall that, given $1 \leq p < \infty$, a (linear) operator $T\colon E \to F$ between Banach spaces is said to be *p-summing* if there is a constant $C > 0$ such that for any finite collection of elements $x_1, \dots, x_n$ in X, we have

$$\Big(\sum_{k=1}^{n} \|Tx_k\|_F^p\Big)^{1/p} \leq C \sup_{x^* \in B_{E^*}} \Big(\sum_{k=1}^{n} |x^*(x_k)|^p\Big)^{1/p}, \tag{1.1}$$

where, as usual, we denote by B_{E^*} the unit ball of the Banach dual space E^* of E.

A. Pietsch discovered that the theory of p-summing operators (which is at the very heart of modern Banach space theory) is ruled by the following theorem (nowadays known as Pietsch's domination theorem, see [28, Theorem 2]): An operator $T\colon E \to F$ is p-summing if and only if there is a (regular) Borel probability measure μ on $(B_{E^*}, \sigma(E^*, E))$ and a constant $C > 0$ such that

$$\|Tx\|_F \leq C\Big(\int_{B_{E^*}} |x^*(x)|^p \, d\mu(x^*)\Big)^{1/p}, \quad x \in E. \tag{1.2}$$

In other terms, the *vector-valued norm inequality* (1.1) (through a nowadays standard separation argument) turns out to be equivalent to the *weighted norm inequality* (1.2). Since the work of Kwapień [20], Maurey [25], and Rosenthal [29] and many others there is a huge amount of similar results with a tremendous impact on operator theory in Banach spaces.

The general idea is as follows: If an operator T satisfies a vector-valued norm inequality (which in many interesting cases turns out to be for free!), then it even satisfies a weighted norm inequality, and this allows one to decode a lot of a priori hidden information on T.

To see an example, recall the fundamental Maurey–Rosenthal theorem which states (see again [25, 29] and also [8]): If ν is a σ-finite measure space on some measurable space (Ω, Σ), then an operator T from a quasi-Banach space E into $L_p(\nu)$ for $0 < p < r < \infty$ satisfies the vector-valued norm inequality

$$(1.3)\qquad \Big\| \Big(\sum_{k=1}^{n} |Tx_k|^r \Big)^{1/r} \Big\|_{L_p(\nu)} \le C \Big(\sum_{k=1}^{n} \|x_k\|_E^r \Big)^{1/r}, \quad x_1, \ldots, x_n \in E,$$

if and only if there exists a density function $w \in L_0(\nu)$ such that the following weighted norm inequality holds:

$$(1.4)\qquad \Big(\int_{\Omega} \frac{|Tx|^r}{w}\, d\nu \Big)^{1/r} \le C\|x\|_E, \quad x \in E$$

(with the implicit understanding that $Tx = 0$ a.e. on the set $\{\omega \in \Omega;\ w(\omega) = 0\}$).

Moreover, it can be proved that for $0 < p < r \le 2$ every operator $T\colon L_q(\mu) \to L_p(\nu)$ satisfies (1.3). This fact in combination with the equivalence of (1.3) and (1.4) has numerous applications in various different topics of analysis (see, e.g., [13] or [16]).

In the present article we wish to clarify to which extent this circle of ideas generalizes to m-homogeneous polynomials between Banach lattices—exploiting mainly ideas of [8] and [10] we will see that at least for positive homogeneous polynomials the outcome seems satisfying. Let us point out that in the recent article [23] it is proved that certain vector-valued norm inequalities for multilinear operators are equivalent to domination theorems, which then under some mild additional assumptions can be expressed in terms of factorization through Orlicz spaces.

To give a first flavor of the factorization results we aim at, we recall a well-known substitute of Pietsch's domination theorem for homogeneous polynomials: Given $1 \le r < \infty$, an m-homogeneous polynomial $P\colon E \to F$ between Banach spaces is said to be r-*dominated* if there is a constant $C > 0$ such that for each choice of vectors $x_1, \ldots, x_n \in E$,

$$(1.5)\qquad \Big(\sum_{k=1}^{n} \|Px_k\|_F^{r/m} \Big)^{m/r} \le C \sup_{\|x^*\| \le 1} \Big(\sum_{k=1}^{n} |x^*(x_k)|^r \Big)^{m/r}.$$

Geiss [17] (see also [3, 4, 24, 26]) proved that an m-homogeneous polynomial $P\colon E \to F$ is r-dominated if and only if there is a Borel probability measure μ on $(B_{E^*}, \sigma(E^*, E))$ and a constant $C > 0$ such that for every $x \in E$,

$$(1.6)\qquad \|Px\|_F \le C \Big(\int_{B_{E^*}} |x^*(x)|^r \, d\mu(x^*) \Big)^{m/r}.$$

Clearly, an operator is r-dominated if and only if it is r-summing, and hence this equivalence at least formally includes the equivalence of (1.1) and (1.2) as a special case.

2. Preliminaries. We shall use standard notation and notions from Banach space theory, as presented, e.g., in [9, 13, 21]. Recall that for an integer

m a mapping $P\colon E \to F$ between Banach spaces is said to be a (bounded) *m-homogeneous polynomial* whenever there is a (bounded) m-linear mapping $\varphi\colon E \times \cdots \times E \to F$ such that $P(x) = \varphi(x, \ldots, x)$ for all $x \in E$. For the theory of homogeneous polynomials we refer to [15]. An m-homogeneous polynomial $P\colon X \to Y$ between Banach lattices is said to be *positive* if its unique associated symmetric m-linear mapping $L\colon X \times \cdots \times X \to Y$ is positive in the sense that $L(x_1, \ldots, x_m) \geq 0$ for all choices of positive elements $x_1, \ldots, x_m \in X$. Note that an m-homogeneous polynomial $P\colon \mathbb{R}^m \to \mathbb{R}$ is positive if and only if all coefficients in its expansion are positive, and also that there are m-homogeneous polynomials which are positive on the positive cone of X but are not positive in the sense of the definition given here. For a careful study of positive polynomials on Riesz spaces see [22].

Let (Ω, Σ, μ) be a complete measure space and let $L_0(\mu)$ denote the space (of all equivalence classes) of real-valued measurable functions on Ω. A *quasi-Banach* (*function*) *lattice* X on (Ω, Σ, μ) (on (Ω, μ) for short) is a subspace of $L_0(\mu)$, which is complete with respect to a quasi-norm $\|\cdot\|$ and which has the property: $x \in L_0(\mu)$, $y \in X$, $|x| \leq |y|$ μ-a.e. implies $x \in X$ and $\|x\|_X \leq \|y\|_X$; moreover we will assume that there exists $u \in X$ with $u > 0$ μ-a.e.

A quasi-Banach function space X on (Ω, μ) is said to be *σ-order continuous* if $\|x_n\|_X \to 0$ for every sequence (x_n) in X such that $0 \leq x_n \downarrow 0$ μ-a.e. We use without reference the well-known fact that if X is a σ-order continuous Banach lattice on a σ-finite measure space (Ω, μ), then the Banach dual space X^* can be identified with the Köthe function lattice X' of all $x \in L_0(\mu)$ such that

$$\|x\|_{X'} = \sup_{\|y\|_X \leq 1} \int_{\Omega} |xy| \, d\mu < \infty.$$

A quasi-Banach lattice $X = (X, \|\cdot\|)$ is said to be *p-convex* $(0 < p < \infty)$, respectively *q-concave*, $0 < q < \infty$, if there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{k=1}^{n} |x_k|^p\Big)^{1/p}\Big\| \leq C\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{1/p},$$

respectively,

$$\Big(\sum_{k=1}^{n} \|x_k\|^q\Big)^{1/q} \leq C\Big\|\Big(\sum_{k=1}^{n} |x_k|^q\Big)^{1/q}\Big\|$$

for every choice of elements $x_1, \ldots, x_n \in X$. The optimal constant C in this inequality is called the *p-convexity* (respectively, *q-concavity*) *constant* of X, and is denoted by $M^{(p)}(X)$ (respectively, by $M_{(q)}(X)$). A quasi-Banach lattice is said to have *nontrivial convexity* whenever it is p-convex for some $0 < p < \infty$.

Suppose we are given a quasi-Banach lattice X on (Ω, μ). For any $0 < r < \infty$ we define X^r to be the quasi-Banach lattice of all $x \in L_0(\mu)$ such that $|x|^{1/r} \in X$ equipped with the quasi-norm $\|x\|_{X^r} = \big\||x|^{1/r}\big\|_X^r$. It is well known and easy to verify that if X is r-convex, then there exists a Banach function lattice Y on (Ω, μ) such that $x \in X$ if and only if $|x|^r \in Y$ and

$$(M^{(r)}(X))^{-1}\|x\|_X \le \|x\|_{Y^r} \le \|x\|_X.$$

Thus when we consider a quasi-Banach lattice X with nontrivial convexity, we can assume without loss of generality that there exists $0 < t < \infty$ such that $M^{(t)}(X) = 1$, which is equivalent to

$$\big\|(|x|^t + |y|^t)^{1/t}\big\|_X \le (\|x\|_X^t + \|x\|_X^t)^{1/t}, \quad x, y \in X. \tag{2.1}$$

Throughout the paper we always assume that the quasi-Banach lattices we consider have nontrivial convexity, satisfy (2.1), and in addition are maximal. We recall that a quasi-Banach lattice X is said to be *maximal* (or X has the *Fatou property*) whenever $0 \le x_n \uparrow x$ a.e., $x_n \in X$, and $\sup_{n\ge1} \|x_n\|_X < \infty$ implies that $x \in X$ and $\|x_n\|_X \to \|x\|_X$.

Examples of maximal Banach function lattices are L_p-spaces, mixed L_p-spaces, Lorentz and Marcinkiewicz spaces as well as Orlicz spaces.

3. Weighted norm inequalities for polynomials. The following notion is taken from [8]. We call a nonempty set U together with a map $\circ\colon \mathbb{R}_+ \times U \to U$, $\circ(\lambda, x) := \lambda x$, a *homogeneous set* (with respect to $\circ$). If then X is a quasi-Banach space and $\phi\colon U \to X$ a homogeneous mapping in the sense that $\phi(\lambda x) = \lambda\phi(x)$ for all $\lambda \ge 0$ and $x \in U$, then U is said to be *homogeneously represented* by ϕ in X.

Here we will be mainly interested in the following three special cases.

Example 3.1.

(1) $U = X$ a quasi-Banach lattice and $\phi\colon U \to X$ the identity map.
(2) $U = E$ a quasi-Banach space, $X = L_r(\mu)$ with $0 < r \le \infty$ and μ a Dirac measure, and $\phi\colon U \to L_r(\mu)$, $\phi x := \|x\|_E\, 1$.
(3) $U = E$ a quasi-Banach space, $X = \ell_\infty(B_{E^*})$, and $\phi\colon U \to \ell_\infty(B_{E^*})$, $(\phi x)(x^*) := x^*(x)$.

We call a mapping $P\colon U \to V$ between two homogeneous sets U and V *m-homogeneous* if $P(\lambda x) = \lambda^m P(x)$ for every $\lambda \ge 0$ and $x \in U$.

For $m = 1$ the following theorem is the main result [8, Theorem 2]—its proof reduces the case of arbitrary m to $m = 1$.

Theorem 3.2. *Let $0 < r < \infty$ and $m \in \mathbb{N}$. Let U be a set homogeneously represented by ϕ in the r-convex quasi-Banach lattice X on (Ω_1, μ), and V a set homogeneously represented by ψ in the r/m-concave quasi-Banach* *lattice Y on (Ω_2,ν). Suppose that $P\colon U\to V$ is an m-homogeneous mapping such that for every finite sequence $(x_k)_{k=1}^n$ in U we have*

$$\Big\|\Big(\sum_{k=1}^n|\psi(Px_k)|^{r/m}\Big)^{m/r}\Big\|_Y\le\Big\|\Big(\sum_{k=1}^n|\phi(x_k)|^r\Big)^{1/r}\Big\|_X^m.$$

Then there exist a positive functional $\varphi\colon X^r\to\mathbb{R}$ and a positive function $w_2\in L_0(\nu)$ such that

$$\sup_{\|x\|_X\le1}\varphi(|x|^r)^{1/r}\le M^{(r)}(X),\qquad\sup_{\|y\|_{L_{r/m}(\nu)}\le1}\|w_2^{m/r}y\|_Y\le M_{(r/m)}(Y),$$

and

$$\Big(\int_{\Omega_2}\frac{|\psi(Px)|^{r/m}}{w_2}\,d\nu\Big)^{m/r}\le\varphi(|\phi(x)|^r)^{m/r},\qquad x\in U.$$

If in addition X is σ-order continuous on a σ-finite measure space, then there exists a positive function $w_1\in L_0(\mu)$ with $\sup_{\|x\|_X\le1}\|w_1^{1/r}x\|_{L_r(\mu)}\le M^{(r)}(X)$ such that

$$\Big(\int_{\Omega_2}\frac{|\psi(Px)|^{r/m}}{w_2}\,d\nu\Big)^{m/r}\le\Big(\int_{\Omega_1}|\phi(x)|^rw_1\,d\mu\Big)^{m/r},\quad x\in U.$$

Proof. Define a new scalar multiplication on U by

$$\lambda\circ u:=\lambda^{1/m}u,\quad(\lambda,u)\in\mathbb{R}_+\times U.$$

Then the mapping

$$\phi^m=\prod_{k=1}^m\phi\colon(U,\circ)\to X^m$$

defines a homogeneous representation of $(U,\circ)$ in the r/m-convex Banach function space $X^m(\mu)$, and (see, e.g., [8, Lemma 2])

$$M^{(r/m)}(X^m)=M^{(r)}(X)^m<\infty.$$

Moreover, the mapping $P\colon(U,\circ)\to V$ is homogeneous, and

$$\Big\|\Big(\sum_{k=1}^n|\psi(Px_k)|^{r/m}\Big)^{m/r}\Big\|_Y\le\Big\|\Big(\sum_{k=1}^n|\phi(x_k)|^r\Big)^{1/r}\Big\|_X^m=\Big\|\Big(\sum_{k=1}^n|\phi^m(x_k)|^{r/m}\Big)^{m/r}\Big\|_{X^m}.$$

By [8, Theorem 2] there exist a positive functional $\varphi\colon(X^m)^{r/m}\to\mathbb{R}$ with

$$\sup_{\|x\|_{X^m}\le1}\varphi(|x|^{r/m})^{m/r}\le M^{(r)}(X)^m$$

and a positive function $w_2 \in L_0(\nu)$ with

$$\sup_{\|y\|_{L_{r/m}(\nu)} \le 1} \|w_2^{m/r} y\|_Y \le M_{(r/m)}(Y),$$

which in addition satisfy

$$\Big(\int_{\Omega_2} \frac{|\psi(Px)|^{r/m}}{w_2}\, d\nu\Big)^{m/r} \le \varphi(|\phi^m(x)|^{r/m})^{m/r} = \varphi(|\phi(x)|^r)^{m/r}, \quad x \in U.$$

In particular, $\varphi\colon X^r \to \mathbb{R}$ and

$$\sup_{\|x\|_X \le 1} \varphi(|x|^r)^{1/r} \le M^{(r)}(X).$$

To complete the proof it is enough to note the obvious fact that if X is σ-order continuous, then so is X^r. ■

According to our three basic examples from 3.1 (of homogeneous representations of homogeneous sets in quasi-Banach function spaces) we present three corollaries of the preceding theorem representing the apparently most important special cases. The proof of the implication (1)⇒(2) in all three corollaries is an immediate consequence of Theorem 3.2 and the corresponding example in 3.1, whereas the proof of the converse (2)⇒(1) in each case only needs a straightforward calculation.

Corollary 3.3. *Let $0 < r < \infty$ and $m \in \mathbb{N}$. Suppose that X is an r-convex quasi-Banach lattice on (Ω_1, μ), Y is an r/m-concave quasi-Banach lattice on (Ω_2, ν), and X_0 a sublattice of X. Then for every m-homogeneous polynomial $P\colon X_0 \to Y$ the following two statements are equivalent:*

(1) *For every finite sequence $(x_k)_{k=1}^n$ in X_0,*

$$\Big\|\Big(\sum_{k=1}^n |Px_k|^{r/m}\Big)^{m/r}\Big\|_Y \le \Big\|\Big(\sum_{k=1}^n |x_k|^r\Big)^{1/r}\Big\|_X^m.$$

(2) *There exist a positive functional $\varphi\colon X^r \to \mathbb{R}$ and a positive function $w_2 \in L_0(\nu)$ such that*

$$\sup_{\|x\|_X \le 1} \varphi(|x|^r)^{1/r} \le M^{(r)}(X), \qquad \sup_{\|y\|_{L_{r/m}(\nu)} \le 1} \|w_2^{m/r} y\|_Y \le M_{(r/m)}(Y),$$

and

$$\Big(\int_{\Omega_2} \frac{|Px|^{r/m}}{w_2}\, d\nu\Big)^{m/r} \le \varphi(|x|^r)^{m/r}, \qquad x \in U.$$

If in addition X is σ-order continuous on a σ-finite measure space, then there exists a positive function $w_1 \in L_0(\mu)$ such that

$$\Big(\int_{\Omega_2} \frac{|Px|^{r/m}}{w_2}\,d\nu\Big)^{m/r} \le \Big(\int_{\Omega_1} |x|^r w_1\,d\mu\Big)^{m/r}, \quad x\in U.$$

The second corollary is the polynomial analog of the Maurey–Rosenthal theorem—for the linear case see again the equivalence of (1.3) and (1.4) in the introduction.

Corollary 3.4. *Let $0 < r < \infty$ and $m \in \mathbb{N}$. Let E be a Banach space and Y an r/m-concave quasi-Banach lattice on (Ω,ν). Then for every m-homogeneous polynomial $P\colon E \to Y$ the following two statements are equivalent:*

(1) *For every finite sequence $(x_k)_{k=1}^n$ in X_0,*

$$\Big\|\Big(\sum_{k=1}^n |Px_k|^{r/m}\Big)^{m/r}\Big\|_Y \le \Big(\sum_{k=1}^n \|x_k\|_E^r\Big)^{m/r}.$$

(2) *There is a positive function $w_2 \in L_0(\nu)$ such that*

$$\Big(\int_{\Omega} \frac{|Px|^{r/m}}{w_2}\,d\nu\Big)^{m/r} \le \|x\|_E^m, \qquad x\in E.$$

Finally, we present an extension of the domination theorem for r-dominated polynomials $P\colon E\to F$ given in the introduction (equivalence of (1.5) and (1.6)); this result follows immediately from the following corollary if we represent E in the canonical way homogeneously in $X = X_0 = \ell_\infty(B_{E^*})$.

Corollary 3.5. *Let $0 < r < \infty$ and $m \in \mathbb{N}$. Let X_0 be a sublattice of an r-convex quasi-Banach lattice X on (Ω,μ), and F a Banach space. Then for every m-homogeneous polynomial $P\colon X_0 \to F$ the following statements are equivalent:*

(1) *For every finite sequence $(x_k)_{k=1}^n$ in X_0,*

$$\Big(\sum_{k=1}^n \|Px_k\|_F^{r/m}\Big)^{m/r} \le \Big\|\Big(\sum_{k=1}^n |x_k|^r\Big)^{m/r}\Big\|_X.$$

(2) *There exists a positive functional $\varphi\colon X^r \to \mathbb{R}$ such that*

$$\sup_{\|x\|_X\le 1} \varphi(|x|^r)^{1/r} \le M^{(r)}(X)$$

and

$$\|P(x)\|_F \le \varphi(|x|^r)^{m/r}, \quad x\in X_0.$$

If in addition X is σ-order continuous on a σ-finite measure space, then there is a positive function $w \in L_0(\mu)$ such that

$$\|P(x)\|_F \le \Big(\int_{\Omega} |x|^r w\,d\mu\Big)^{m/r}, \quad x\in X_0.$$

4. Norm estimates for positive polynomials. Recall the following well-known theorem of Krivine (see, e.g., [21, II, 1.d.9]): For every positive operator $T\colon X \to Y$ between Banach lattices, every $1 \le r < \infty$, and every choice of finitely many $x_1, \dots, x_n \in X$ we have

$$\Big\| \Big(\sum_{k=1}^{n} |Tx_k|^r \Big)^{1/r} \Big\|_Y \le \|T\| \Big\| \Big(\sum_{k=1}^{n} |x_k|^r \Big)^{1/r} \Big\|_X .$$

Provided $r \ge m$ there is a polynomial counterpart of this vector-valued norm inequality.

PROPOSITION 4.1. *Let X and Y be Banach lattices, $1 \le r < \infty$, and $m \in \mathbb{N}$ with $r \ge m$. Then for every positive m-homogeneous polynomial $P\colon X \to Y$ and for every finite sequence $(x_k)_{k=1}^n$ in X we have*

$$\Big\| \Big(\sum_{k=1}^{n} |Px_k|^{r/m} \Big)^{m/r} \Big\|_Y \le \|P\| \frac{m^m}{m!} \Big\| \Big(\sum_{k=1}^{n} |x_k|^r \Big)^{1/r} \Big\|_X^m .$$

Proof. The required inequality follows from its m-linear analog from [10, Theorem 6.2] which states: If $\phi\colon X_1 \times \dots \times X_m \to Y$ is a positive m-linear operator between Banach lattices, and $1 \le s \le \infty$ such that $1/s = 1/r_1 + \dots + 1/r_m$ with $1 \le r_j \le \infty$, $1 \le j \le m$, then for every choice of finitely many elements $x_1^{(j)}, \dots, x_n^{(j)}$ in X_j, $1 \le j \le m$, we have

$$\Big\| \Big(\sum_{i=1}^{n} |\phi(x_i^{(1)}, \dots, x_i^{(m)})|^s \Big)^{1/s} \Big\|_Y \le \|\phi\| \prod_{j=1}^{m} \Big\| \Big(\sum_{i=1}^{n} |x_i^{(j)}|^{r_j} \Big)^{1/r_j} \Big\|_{X_j} . \tag{4.1}$$

Given a positive m-homogeneous polynomial $P\colon X \to Y$ between Banach lattices, its associated symmetric m-linear mapping $L\colon X \times \dots \times X \to Y$ is positive by our hypothesis, i.e., $L(x_1, \dots, x_m) \ge 0$ for all choices of positive elements $x_1, \dots, x_m \in X$. Applying (4.1) to L and $s = r/m$, $r_1 \dots = r_m = r$ we obtain, for every choice of finitely many $x_1, \dots, x_n \in X$,

$$\Big\| \Big(\sum_{k=1}^{n} |L(x_k, \dots, x_k)|^{r/m} \Big)^{m/r} \Big\|_Y \le \|L\| \Big\| \Big(\sum_{k=1}^{n} |x_k|^r \Big)^{1/r} \Big\|_X^m .$$

Since $L(x_k, \dots, x_k) = Px_k$ for each $1 \le k \le m$, and by polarization $\|L\| \le \frac{m^m}{m!}\|P\|$ (see, e.g., [15, 2.1]), we obtain the desired estimate. ∎

This result in particular implies that for $r \ge m$ the second statement of Corollary 3.3 always holds, a fact which we are going to exploit in the next section.

5. Factorization and extension of positive polynomials. The following extension as well as factorization theorem for positive m-homogeneous polynomials between Banach function spaces is our main result.

THEOREM 5.1. *Let $1 \le r, s, t < \infty$ and $m \in \mathbb{N}$ be such that $1 \le t \le r/m \le r \le s$. Let X be a σ-order continuous and s-convex Banach lattice on (Ω_1, μ), X_0 a sublattice of X, and Y a t-concave Banach lattice on (Ω_2, ν). Then for every m-homogeneous positive polynomial $P\colon X_0 \to Y$ there are multiplication operators $M_f\colon X \to L_r(\mu)$ and $M_g\colon L_{r/m}(\nu) \to Y$ as well as an m-homogeneous polynomial $Q\colon L_r(\mu) \to L_{r/m}(\nu)$ such that $M_g Q M_f$ extends P from X_0 to X:*

$$\begin{array}{ccc} X_0 \hookrightarrow X & \xrightarrow{\quad M_g Q M_f \quad} & Y \\ \Big\downarrow{\scriptstyle M_f} & & \Big\uparrow{\scriptstyle M_g} \\ L_r(\mu) & \xrightarrow{\quad Q \quad} & L_{r/m}(\nu) \end{array}$$

Proof. Assume without loss of generality that $\|P\| = 1$. Moreover, by the fact that u-convexity is a property decreasing in u, and u-concavity a property increasing in u, we know that X is r-convex and Y is r/m-concave. Then by Corollary 3.3 and Proposition 4.1 there exist positive functions $w_1 \in L_0(\mu)$ and $w_2 \in L_0(\nu)$ such that

$$\sup_{\|x\|_X \le 1} \|w_1^{1/r} x\|_{L_r(\mu)} \le M^{(r)}(X), \qquad \sup_{\|y\|_{L_{r/m}(\nu)} \le 1} \|w_2^{m/r} y\|_Y \le M_{(r/m)}(Y)$$

and

$$\Big(\int_{\Omega_2} \frac{|Px|^{r/m}}{w_2}\, d\nu \Big)^{m/r} \le \Big(\int_{\Omega_1} |x|^r w_1 \, d\mu \Big)^{m/r}, \quad x \in X_0.$$

We may assume without loss of generality that both w_1 and w_2 are strictly positive. Let $L\colon X_0 \times \cdots \times X_0 \to Y$ be the unique symmetric, bounded m-linear mapping such that for all $x \in X_0$ we have

$$P(x) = L(x, \ldots, x).$$

Define, for $x_1, \ldots, x_m \in X_0$,

$$Q_0(x_1 w_1^{1/r}, \ldots, x_m w_1^{1/r}) := L(x_1, \ldots, x_m) \frac{1}{w_2^{m/r}},$$

and let $M_{w_1^{1/r}}\colon X_0 \to L_r(\mu)$ be multiplication by $w_1^{1/r}$. We are going to prove that

$$Q_0\colon M_{w_1^{1/r}} X_0 \times \cdots \times M_{w_1^{1/r}} X_0 \to Y$$

is a well-defined, bounded and m-linear mapping, where the range $M_{w_1^{1/r}} X_0$

is considered as a subspace of $L_r(\mu)$. Indeed, by the polarization formula we have, for all choices of $x_1, \ldots, x_m \in X_0$,

$$L(x_1, \ldots, x_m) = \frac{1}{m!} \int_0^1 r_1(t) \ldots r_m(t) P\Big(\sum_{i=1}^m r_i(t) x_i\Big)\, dt,$$

where r_k denotes the kth Rademacher function (see, e.g., [15, 1.4 and 1.12]). Hence by (1) and the continuous Minkowski inequality (by assumption $1 \le r/m$)

$$\begin{aligned}
&\Big(\int_{\Omega_2} |Q_0(x_1 w_1^{1/r}, \ldots, x_m w_1^{1/r})|^{r/m} d\nu\Big)^{m/r} \\
&\qquad = \Big(\int_{\Omega_2} \Big|\frac{1}{m!}\int_0^1 r_1(t) \ldots r_m(t) \frac{P(\sum_{i=1}^m r_i(t) x_i)}{w_2^{m/r}}\, dt\Big|^{r/m} d\nu\Big)^{m/r} \\
&\qquad \le \frac{1}{m!}\int_0^1 \Big(\int_{\Omega_2} \Big|\frac{P(\sum_{i=1}^m r_i(t) x_i}{w_2^{m/r}}\Big|^{r/m} d\nu\Big)^{m/r} dt \\
&\qquad \le \frac{1}{m!}\int_0^1 \Big(\int_{\Omega_1} \Big|\sum_{i=1}^m r_i(t) x_i\Big|^r w_1 d\mu\Big)^{m/r} dt \\
&\qquad \le \frac{1}{m!}\int_0^1 \Big(\sum_{i=1}^m \|x_i w_1^{1/r}\|_r\Big)^m dt = \frac{1}{m!}\Big(\sum_{i=1}^m \|x_i w_1^{1/r}\|_r\Big)^m.
\end{aligned}$$

But then Q_0 is well-defined, and being obviously m-linear and continuous at zero, it is bounded. Since it can be easily seen that $M_{w_1^{1/r}} X_0$ is a dense subspace of $L_r(\mu)$ (see also [11, Lemma 3.3]), the mapping Q_0 extends to a bounded m-linear mapping $L_r(\mu) \times \cdots \times L_r(\mu) \to L_{r/m}(\nu)$. Define the bounded m-homogeneous polynomial $Q\colon L_r(\mu) \to L_{r/m}(\nu)$ to be the restriction of Q_0 to the diagonal. By construction we see that as desired $M_{w_2^{m/r}} Q M_{w_1^{1/r}}$ extends P. Putting $f = w_1^{1/r}$ and $g = w_2^{m/r}$ completes the proof. ■

In the scale of L_p-spaces we obtain the following corollary which is well-known in the linear case $m = 1$.

Corollary 5.2. *Let $1 \le r, s, t < \infty$ and $m \in \mathbb{N}$ be such that $1 \le t \le r/m \le r \le s$.*

(1) *For every positive m-homogeneous polynomial $P\colon L_s(\mu) \to L_t(\nu)$ there is a factorization*

$$\begin{array}{ccc} L_s(\mu) & \xrightarrow{\quad P \quad} & L_t(\nu) \\ \Big\downarrow{\scriptstyle M_f} & & \Big\uparrow{\scriptstyle M_g} \\ L_r(\mu) & \xrightarrow{\quad Q \quad} & L_{r/m}(\nu) \end{array}$$

where Q is an m-homogeneous polynomial and M_f as well as M_g are multiplication operators.

(2) *Let X_0 be a sublattice of $L_s(\mu)$. Then every positive m-homogeneous polynomial $P\colon X_0 \to L_t(\nu)$ extends to an m-homogeneous polynomial $Q\colon L_s(\mu) \to L_t(\nu)$.*

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