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Approximation of a symmetric α-stable Lévy process by a Lévy process with finite moments of all orders

by

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Abstract. In this paper we consider a symmetric α-stable Lévy process Z. We use a series representation of Z to condition it on the largest jump. Under this condition, Z can be presented as a sum of two independent processes. One of them is a Lévy process Y_x parametrized by $x > 0$ which has finite moments of all orders. We show that Y_x converges to Z uniformly on compact sets with probability one as $x \downarrow 0$. The first term in the cumulant expansion of Y_x corresponds to a Brownian motion which implies that Y_x can be approximated by Brownian motion when x is large. We also study integrals of a nonrandom function with respect to Y_x and derive the covariance function of those integrals. A symmetric α-stable random vector is approximated with probability one by a random vector with components having finite second moments.

1. Introduction. Stable Lévy processes play an important role among stable processes, similar to that of Brownian motion among Gaussian processes. Thus we start with the definition of a symmetric α-stable Lévy process (for a comprehensive treatment of stable variables and processes see e.g. Janicki and Weron [2] or Samorodnitsky and Taqqu [8]). We will consider stochastic processes on the time interval $[0, 1]$.

DEFINITION 1.1. A stochastic process $\{Z(t), 0 \le t \le 1\}$ is called a *symmetric α-stable Lévy process* $(0 < \alpha \le 2)$ if

1. $Z(0) = 0$ a.s.
2. Z has independent increments.
3. $Z(t)-Z(s)$ has distribution $S_\alpha(\sigma(t-s)^{1/\alpha}, 0, 0)$ for any $0 \le s < t \le 1$, that is, α-stable distribution with scale parameter $\sigma(t-s)^{1/\alpha}$, and skewness and shift parameters equal to zero.

For $\alpha = 2$ we get Brownian motion. If the first two assumptions are satisfied and the process has stationary increments we call it a *Lévy process*. In this paper we will consider $0 < \alpha < 2$.

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The *Lévy–Itô integral representation* for a symmetric Lévy process Y is the following:

$$Y(t) = \int_{\mathbb{R}\setminus 0} y\, N_t(dy) \tag{1}$$

where N is the point process of jumps of Y: $N = \sum_{s\,:\,\Delta Y(s)\neq 0} \delta_{(s,\Delta Y(s))}$ (see e.g. Kallenberg [3]). N is a Poisson point process with the mean measure $ds \times \nu(dy)$ where $\nu(dy)$ is a Lévy measure on $\mathbb{R}\setminus 0$. Series representations for Lévy processes can be derived from the Lévy–Itô integral representation (see Rosiński [7]). The series representation for α-stable Lévy processes is called LePage's representation (see LePage [4] or Samorodnitsky and Taqqu [8]).

2. The series representation of a symmetric α-stable Lévy process. A symmetric α-stable Lévy process can be represented as the following series:

$$Z(t) = \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} \tag{2}$$

where $0 \le t \le 1$ and $\{\Gamma_k\}_{k=1}^{\infty}$ is a sequence of arrival epochs in a Poisson process with unit arrival rate, $\{\gamma_k\}_{k=1}^{\infty}$ is a sequence of iid random variables satisfying

$$\mathbb{P}(\gamma_k = 1) = \mathbb{P}(\gamma_k = -1) = 1/2,$$

and $\{U_k\}_{k=1}^{\infty}$ is a sequence of iid random variables uniformly distributed on $[0,1]$. These sequences are independent and

$$C_\alpha = \Big(\int_0^\infty s^{-\alpha} \sin s\, ds\Big)^{-1} = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ 2/\pi & \text{if } \alpha = 1. \end{cases} \tag{3}$$

A symmetric α-stable Lévy process can be regarded as a pure jump process. The epochs of its jumps are described by $\{U_k\}_{k=1}^{\infty}$, the direction of a jump is governed by $\{\gamma_k\}_{k=1}^{\infty}$ and the heights of the jumps are presented in decreasing order by $\{\sigma C_\alpha^{1/\alpha} \Gamma_k^{-1/\alpha}\}_{k=1}^{\infty}$.

Let us introduce the Poisson random measure and the Lévy measure generated by the process Z. Define

$$N_t(A) = N([0,t], A) = \sum_{0\le s\le t} \boldsymbol{I}_A(\Delta Z_s) \tag{4}$$

for $0 \le t \le 1$ and any Borel subset A of $\mathbb{R}$ such that $0 \notin \overline{A}$. The process $N_t(A)$, $0 \le t \le 1$, is a Poisson process with arrival rate

$$\nu(A) = \mathbb{E}N_1(A).$$

The measure ν is called the *Lévy measure* of the Lévy process Z. In fact the measure $N([0,t], A)$ can be extended to a Poisson measure on $[0,1] \times \mathbb{R} \setminus 0$

with the mean measure $ds \times \nu(dy)$. Thus using the series representation we obtain

$$N_t(A) = \sum_{k=1}^{\infty} \boldsymbol{I}_A\{\sigma C_\alpha^{1/\alpha} \Gamma_k^{-1/\alpha} \gamma_k\} \boldsymbol{I}\{U_k \le t\} \tag{5}$$

and

$$\nu(A) = \sum_{k=1}^{\infty} \mathbb{E} \boldsymbol{I}_A\{\sigma C_\alpha^{1/\alpha} \Gamma_k^{-1/\alpha} \gamma_k\}. \tag{6}$$

This Lévy measure has the following form:

$$\nu(dy) = \frac{P}{y^{1+\alpha}} \boldsymbol{I}_{(0,\infty)}(y)dy + \frac{P}{|y|^{1+\alpha}} \boldsymbol{I}_{(-\infty,0)}(y)dy \tag{7}$$

where

$$P = \frac{1}{2} \alpha \sigma^\alpha C_\alpha.$$

3. Conditioning on the largest jump. Consider a symmetric α-stable Lévy process under the condition $\Gamma_1 = x$ where $x > 0$ (Γ_1 has exponential distribution with parameter 1). Then we get

$$\begin{aligned} Z_x(t) &\overset{d}{=} \sigma C_\alpha^{1/\alpha} x^{-1/\alpha} \gamma_1 \boldsymbol{I}\{U_1 \le t\} \\ &\quad + \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_{k+1} \boldsymbol{I}\{U_{k+1} \le t\}. \end{aligned} \tag{8}$$

Therefore let us investigate the following process:

$$\begin{aligned} Y_x(t) &= \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_{k+1} \boldsymbol{I}\{U_{k+1} \le t\} \\ &\overset{d}{=} \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} \end{aligned} \tag{9}$$

and note that

$$Z_x(t) \overset{d}{=} A_x(t) + Y_x(t) \tag{10}$$

where $A_x(t) = \sigma C_\alpha^{1/\alpha} x^{-1/\alpha} \gamma_1 \boldsymbol{I}\{U_1 \le t\}$ and the processes A_x and Y_x are independent.

PROPOSITION 3.1. *The process Y_x is well-defined, that is, the sum in* (9) *converges almost surely.*

Proof. This is a special case of Proposition 3.3 below. ∎

Let us define the following auxiliary process:

$$Y_{x,z}(t) = \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_k \boldsymbol{I}\{U_k \le t\} \boldsymbol{I}\{\Gamma_k < z\}. \tag{11}$$

Notice that the series in (11) consists of finite random number of nonzero terms a.s. (by the strong law of large numbers for the sums Γ_k) and $Y_{x,z}(t) \to Y_x(t)$ a.s. as $z \to \infty$ for all $0 \le t \le 1$. The process (11) can be written using the Poisson measure N defined in (4) in the following way:

$$Y_{x,z}(t) = \int_{A_z} f_x(y)\, N_t(dy) \tag{12}$$

where $A_z = (-\infty, -z^{-1/\alpha}) \cup (z^{-1/\alpha}, \infty)$ and (for simplicity we assume here that $\sigma C_\alpha^{1/\alpha} = 1$)

$$f_x(y) = \frac{1}{(|y|^{-\alpha} + x)^{1/\alpha}} \operatorname{sgn}(y). \tag{13}$$

It follows that $Y_{x,z}$ is a Lévy process (see e.g. Protter [6, Th. 36]).

THEOREM 3.1. *The process* Y_x *defined in* (9) *is a Lévy process with finite moments of all orders and*

$$Y_x(t) = \int_{\mathbb{R}\setminus 0} f_x(y)\, N_t(dy) \tag{14}$$

in the sense of the limit of $Y_{x,z}(t)$ *given in* (12) *as* $z \to \infty$ *a.s.*

Proof. The stationarity of the increments follows from the fact that

$$\boldsymbol{I}\{U_k \le t\} - \boldsymbol{I}\{U_k \le s\} = \boldsymbol{I}\{s < U_k \le t\} \stackrel{d}{=} \boldsymbol{I}\{U_k \le t - s\}$$

for all $0 \le s < t \le 1$. From the convergence $Y_{x,z}(t) \to Y_x(t)$ a.s. as $z \to \infty$ for all $0 \le t \le 1$ we obtain the independence of the increments of Y_x. Since Y_x has bounded jumps, it has finite moments of all orders (see Protter [6, Th. 34]). ∎

We derive the exact form of the Lévy measure of the process Y_x and the characteristic function of the random variable $Y_x(t)$.

THEOREM 3.2. *The Lévy measure of the process* Y_x *has the form*

$$\nu_x(dy) = \frac{P}{y^{1+\alpha}} \boldsymbol{I}_{(0\,,\,\sigma C_\alpha^{1/\alpha} x^{-1/\alpha})}(y)dy + \frac{P}{|y|^{1+\alpha}} \boldsymbol{I}_{(-\sigma C_\alpha^{1/\alpha} x^{-1/\alpha}\,,\,0)}(y)dy. \tag{15}$$

The characteristic function of the random variable $Y_x(t)$ *is*

$$\mathbb{E}\exp(iuY_x(t)) = \exp\Bigg(-t|u|^\alpha 2P \int_0^{|u|\sigma C_\alpha^{1/\alpha} x^{-1/\alpha}} \frac{1-\cos s}{s^{1+\alpha}}\, ds\Bigg) \tag{16}$$

$$= \exp\Bigg(-tx\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-\alpha)(2k)!} (u\sigma C_\alpha^{1/\alpha} x^{-1/\alpha})^{2k}\Bigg). \tag{17}$$

Proof. Let ν_x be the Lévy measure of the process Y_x. Then

$$\nu_x(A) = \sum_{k=1}^{\infty} \mathbb{E}\boldsymbol{I}_A(\sigma C_\alpha^{1/\alpha}(\Gamma_k + x)^{-1/\alpha}\gamma_k) \tag{18}$$

for any $A \subset \mathbb{R}$ with $0 \notin \overline{A}$. For simplicity we assume that $\sigma C_\alpha^{1/\alpha} = 1$. Take $A = (y, \infty)$ where $y > 0$ and $y < x^{-1/\alpha}$ because $(\Gamma_k + x)^{-1/\alpha} < x^{-1/\alpha}$ a.s. Since $(\Gamma_k + x)^{-1/\alpha} > y$ is equivalent to $\Gamma_k^{-1/\alpha} > (y^{-\alpha} - x)^{-1/\alpha}$, using (6), (7) and (18) we get

$$\nu_x(A) = \int_{(y^{-\alpha}-x)^{-1/\alpha}}^{\infty} \frac{P}{z^{1+\alpha}}\, dz = \int_{y}^{x^{-1/\alpha}} \frac{P}{w^{1+\alpha}}\, dw.$$

For $y < 0$ we proceed similarly. Thus we obtain

$$\nu_x(dy) = \frac{P}{y^{1+\alpha}}\boldsymbol{I}_{(0,x^{-1/\alpha})}(y)dy + \frac{P}{|y|^{1+\alpha}}\boldsymbol{I}_{(-x^{-1/\alpha},0)}(y)dy.$$

From the Lévy–Khinchin formula we can write

$$\mathbb{E}\exp(iuY_x(t)) = \exp(-t\psi(u))$$

where for $u \geq 0$,

$$\begin{aligned}\psi(u) &= \int_{\mathbb{R}\setminus 0} (1 - e^{iuy})\, \nu_x(dy) = \int_{\mathbb{R}\setminus 0} (1 - \cos(uy))\, \nu_x(dy) \\ &= \int_0^{x^{-1/\alpha}} (1 - \cos(uy))\, \frac{2P}{y^{1+\alpha}}\, dy = 2Pu^\alpha \int_0^{ux^{-1/\alpha}} \frac{1 - \cos(s)}{s^{1+\alpha}}\, ds\end{aligned}$$

where in the last equality we have substituted $s = uy$. Arguing similarly for $u < 0$, we arrive at (16). To obtain the second form of the characteristic function we expand $\cos s$ in Taylor series and integrate each summand. ∎

In the next proposition we derive the covariance structure of the process Y_x.

Proposition 3.2. *Let Y_x be defined in* (9). *Then*

$$\mathbb{E}Y_x(t) = 0\,, \tag{19}$$

$$\mathbb{E}Y_x^2(t) = \mathbb{E}\Big(\int_{\mathbb{R}\setminus 0} f_x(y)\, N_t(dy)\Big)^2 = t\int_{\mathbb{R}\setminus 0} f_x^2(y)\, \nu(dy) = t\sigma^2(x) \tag{20}$$

and

$$\mathbb{E}Y_x(s)Y_x(t) = \sigma^2(x)\min\{s, t\} \tag{21}$$

where

$$\sigma^2(x) = \frac{\alpha\sigma^2 C_\alpha^{2/\alpha}}{(2-\alpha)x^{(2-\alpha)/\alpha}}.$$

Proof. The expectation is zero because the process Y_x is symmetric. The form of the variance follows from the well known fact that for Lévy processes without Gaussian component, $\mathbf{Var}\, Y_x(t) = t \int y^2\, \nu_x(dy)$. (Twice differentiating the characteristic function in the form (17) also gives the second moment of the random variable $Y_x(t)$ in the form (20).) To get the desired form of covariance function we use the equality $Y(s)Y(t) = Y(s)(Y(t)-Y(s)) + Y^2(s)$ for $s < t$. ∎

REMARK 3.1. Taking only the first term in the series (17) we obtain the characteristic function of Brownian motion B_x with variance $\sigma^2(x) = \frac{\alpha\sigma^2 C_\alpha^{2/\alpha}}{(2-\alpha)x^{(2-\alpha)/\alpha}}$ the same as that of Y_x, that is,

$$\mathbb{E}\exp(iuB_x(t)) = \exp\biggl(-\frac{tu^2}{2}\,\frac{\alpha\sigma^2 C_\alpha^{2/\alpha}}{(2-\alpha)x^{(2-\alpha)/\alpha}}\biggr).$$

Now we investigate the integral of a non-random measurable function g with respect to the process Y_x, that is,

$$\int_0^1 g(s)\, dY_x(s). \tag{22}$$

We impose assumptions on g to ensure that the second moment of the integral is finite. The integral of a non-random function g with respect to a symmetric α-stable Lévy process and the process Y_x is well defined if $g \in L^\alpha([0,1], ds)$ (this follows from Urbanik and Woyczyński [9]).

THEOREM 3.3. *If the functions g^2 and h^2 are integrable on $[0,1]$ then*

$$\mathbb{E}\Bigl(\int_0^1 g(s)\, dY_x(s)\Bigr) = 0,$$

$$\mathbb{E}\Bigl(\int_0^1 g(s)\, dY_x(s)\Bigr)^2 = \sigma^2(x)\int_0^1 g^2(s)\, ds,$$

and

$$\mathbb{E}\Bigl(\int_0^1 g(s)\, dY_x(s)\Bigr)\Bigl(\int_0^1 h(s)\, dY_x(s)\Bigr) = \sigma^2(x)\int_0^1 g(s)h(s)\, ds. \tag{23}$$

Proof. Since Y_x is a square integrable martingale, by the Doob inequality we are able to define the above integrals in L^2 sense and get the desired equalities (see e.g. Kallenberg [3]). ∎

In fact the integral $\int_0^1 g(s)\, dY_x(s)$ can be defined for $g \in L^\alpha([0,1], ds)$ in the following way:

$$\int_0^1 g(s)\, dY_x(s) = \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} (\Gamma_k + x)^{-1/\alpha} \gamma_k g(U_k)\,. \tag{24}$$

PROPOSITION 3.3. *If $g \in L^\alpha([0,1], ds)$ then the series in* (24) *is convergent a.s.*

Proof. This is an easy application of [7, Th. 4.1]. ∎

REMARK 3.2. If $g \in L^2([0,1], ds)$ then the integral $\int_0^1 g(s)\, dY_x(s)$ defined in (24) coincides (in the finite-dimensional distribution sense) with the L^2 integral appearing in Theorem 3.3.

4. Approximation of a symmetric α-stable Lévy process and Brownian motion. The process Y_x approximates the symmetric α-stable Lévy process Z.

THEOREM 4.1. *The process Y_x converges uniformly on compact sets to the process Z with probability one as $x \downarrow 0$; more precisely,*

$$\sup_{0\le t\le 1} |Y_x(t) - Z(t)| \le x\alpha^{-1}\sigma C_\alpha^{1/\alpha} Z_{\alpha/(\alpha+1)} \tag{25}$$

a.s. for $x \ge 0$, where

$$Z_{\alpha/(\alpha+1)} = \sum_{k=1}^{\infty} \frac{1}{\Gamma_k^{(\alpha+1)/\alpha}} \tag{26}$$

is an $\alpha/(\alpha+1)$-stable random variable with skewness parameter one and shift parameter zero.

Proof. Consider the function $g(x) = a^{-1/\alpha} - (a+x)^{-1/\alpha}$ for fixed $a > 0$ and $x \ge 0$. Since

$$\frac{d}{dx}[a^{-1/\alpha} - (a+x)^{-1/\alpha}] = \frac{1}{\alpha(a+x)^{1+1/\alpha}} \le \frac{1}{\alpha a^{1+1/\alpha}}$$

it is easy to notice that

$$|a^{-1/\alpha} - (a+x)^{-1/\alpha}| \le \frac{1}{\alpha a^{1+1/\alpha}}\, x. \tag{27}$$

Thus we obtain

$$\begin{aligned}
&\sup_{0\le t\le 1} \sigma C_\alpha^{1/\alpha} \Big| \sum_{k=1}^{\infty} [\Gamma_k^{-1/\alpha} - (\Gamma_k + x)^{-1/\alpha}] \gamma_k \boldsymbol{I}\{U_k \le t\} \Big| \\
&\qquad \le \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} |\Gamma_k^{-1/\alpha} - (\Gamma_k + x)^{-1/\alpha}| \le \frac{x}{\alpha}\, \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \frac{1}{\Gamma_k^{1+1/\alpha}} \\
&\qquad = \frac{x}{\alpha}\, \sigma C_\alpha^{1/\alpha} \sum_{k=1}^{\infty} \frac{1}{\Gamma_k^{(\alpha+1)/\alpha}} = \frac{x}{\alpha}\, \sigma C_\alpha^{1/\alpha} Z_{\alpha/(\alpha+1)}
\end{aligned}$$

where $Z_{\alpha/(\alpha+1)}$ is an $\alpha/(\alpha+1)$-stable random variable with skewness parameter one and shift parameter zero (the series is a.s. convergent, see Samorodnitsky and Taqqu [8]). ∎

Similarly we can prove the following proposition.

PROPOSITION 4.1. *Let*

$$X(t)=\int_0^1 g_t(s)\,dY_x(s)=\sigma C_\alpha^{1/\alpha}\sum_{k=1}^{\infty}(\Gamma_k+x)^{-1/\alpha}\gamma_k g_t(U_k)$$

where $g_t(s)$ *is uniformly bounded. Then*

$$\int_0^1 g_t(s)\,dY_x(s)\to\int_0^1 g_t(s)\,dZ(s)$$

uniformly on compact sets a.s. as $x\downarrow 0$.

Let us consider a symmetric α-stable random vector and recall the following representation theorem (see Samorodnitsky and Taqqu [8]).

THEOREM 4.2. *If* X *is a symmetric* α*-stable random vector then there are bounded measurable functions* $g_1,\ldots,g_d$ *such that*

$$X\overset{d}{=}\Big(\int_0^1 g_1(s)\,dZ(s),\ldots,\int_0^1 g_d(s)\,dZ(s)\Big). \tag{28}$$

Thus we are able to approximate symmetric α-stable random vectors by vectors with finite second moments.

THEOREM 4.3. *Let* $g_1,\ldots,g_d$ *be bounded measurable functions. Then*

$$\begin{aligned}X_x&=\Big(\int_0^1 g_1(s)\,dY_x(s),\ldots,\int_0^1 g_d(s)\,dY_x(s)\Big)\\&\to\Big(\int_0^1 g_1(s)\,dZ(s),\ldots,\int_0^1 g_d(s)\,dZ(s)\Big)=X\end{aligned} \tag{29}$$

a.s. as $x\downarrow 0$ *and*

$$\|X_x-X\|_d\le\frac{x}{\alpha}M\sigma C_\alpha^{1/\alpha}Z_{\alpha/(\alpha+1)} \tag{30}$$

for $x\ge 0$, *where* $\|\cdot\|_d$ *is Euclidean norm,* $Z_{\alpha/(\alpha+1)}$ *is the random variable defined in* (26) *and* $M=\sqrt{\sum_{k=1}^{d}\sup_{0\le s\le 1}|g_k(s)|^2}$.

Proof. Since the functions g_l are bounded they belong to $L^\alpha([0,1],ds)$ and the integral can be defined by (24). Similarly to the proof of Theorem 4.1 we can write

$$\begin{aligned}&\Big|\sum_{k=1}^{\infty}[\Gamma_k^{-1/\alpha}-(\Gamma_k+x)^{-1/\alpha}]\gamma_k g_l(U_k)\Big|\\&\le M_l\sum_{k=1}^{\infty}|\Gamma_k^{-1/\alpha}-(\Gamma_k+x)^{-1/\alpha}|\le M_l\frac{x}{\alpha}\sum_{k=1}^{\infty}\frac{1}{\Gamma_k^{(\alpha+1)/\alpha}}=M_l\frac{x}{\alpha}Z_{\alpha/(\alpha+1)}\end{aligned} \tag{31}$$

for $x\ge 0$, where $\sup_{0\le s\le 1}|g_l(s)|=M_l$ and $Z_{\alpha/(\alpha+1)}$ is defined in (26). ■

REMARK 4.1. Since $g_1, \ldots, g_d$ are bounded, they belong to L^2 and by Theorem 3.3 the components of X_x have finite second moments.

The next question is what happens with the processes Y_x when $x \to \infty$. The answer is the following.

THEOREM 4.4.

$$\sqrt{\frac{2-\alpha}{\alpha\sigma^2 C_\alpha^{2/\alpha}}}\, x^{1/\alpha-1/2}\, Y_x \Rightarrow B \tag{32}$$

as $x \to \infty$ weakly in the Skorokhod space D equipped with the uniform metric where B is a standard Brownian motion.

Proof. Since Y_x is a Lévy process, it is enough to show that $Y_x(1) \Rightarrow B(1)$ in distribution as $x \to \infty$ (this follows easily from [5, Th. V.19]). Taking the representation (17) of the characteristic function of $Y_x(t)$ we find that for

$$u = u' \sqrt{\frac{2-\alpha}{\alpha\sigma^2 C_\alpha^{2/\alpha}}}\, x^{1/\alpha-1/2}$$

the first term in the series converges to $tu'^2/2$ and the other terms tend to zero, which yields

$$\mathbb{E}\exp\left(iu\sqrt{\frac{2-\alpha}{\alpha\sigma^2 C_\alpha^{2/\alpha}}}\, x^{1/\alpha-1/2}\, Y_x(t)\right) \to \exp\left(-\frac{tu^2}{2}\right)$$

as $x \to \infty$. ∎

REMARK 4.2. The assertion of the above theorem can be obtained from Theorem 2.1 of [1] by substituting $\varepsilon = \sigma C_\alpha^{1/\alpha} x^{-1/\alpha}$.

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