

Carleson measure and monogenic functions

by

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Abstract. We present necessary and sufficient conditions for a measure to be a p -Carleson measure, based on the Poisson and Poisson–Szegő kernels of the n -dimensional unit ball.

1. Introduction. The study of H^p -theory started in 1915 with the work of G. Hardy on the mean value of an analytic function on the unit disk. It soon proved to have a deep connection with L_p -boundary results and it required techniques of both real and complex analysis. Attempts to extend this theory to higher dimensions began in the seventies and coincided with the discovery of the identification of the dual of H^1 with BMO, the space of functions of bounded mean oscillation. The important result that $\varphi \in \text{BMO}$ iff its harmonic extension u to the unit ball has the property that $|\nabla u|^2(1 - |z|)dxdy$ is a Carleson measure (C. Fefferman) shows a close relation between this space and Carleson measures.

For a higher dimensional counterpart there are several problems arising. In the first place, the group of automorphisms of the unit ball (a Lie group) no longer preserves the Laplace operator, but instead, it preserves the Laplace–Beltrami operator. Thus, the conformally invariant kernel is no longer the Poisson kernel, but the Poisson–Szegő kernel. Second, there exist two possible generalizations to higher dimensions of the classical complex function theory, both dating from the 30’s: 1) the several complex variables’ approach which has the strong drawback that the Cauchy kernel strongly depends on the shape of the domain under consideration, 2) the Clifford analysis’ approach which started with the work of Fueter (1934) on quaternions and later on was developed by the group led by Delanghe to deal with the universal real

2000 *Mathematics Subject Classification*: Primary 30G35; Secondary 31B10, 30C40.

Key words and phrases: Carleson measure, Poisson kernel, Poisson–Szegő kernel.

Research of P. Cerejeiras partially supported by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro, through *Programa Operacional “Ciência, Tecnologia, Inovação”* (POCTI) of the *Fundação para a Ciência e a Tecnologia (FCT)*, cofinanced by the European Community fund FEDER.

Clifford algebras $\mathcal{C}_{p,q}$. It presents the advantage of providing a Cauchy kernel independent of the domain and a series expansion, the so-called Cauchy–Kovalevskaya extension, which is in general linked to the Euclidean metric. Thus, in order to consider Carleson measures in a higher dimensional setting we are led to an approach via Clifford analysis with an extra consideration of the tensor metric effect on the arising reproducing kernels.

The layout of the paper is as follows: in Section 2 we present some basic definitions and results concerning Clifford analysis. Section 3 is dedicated to a short review of the automorphic group of the n -dimensional unit ball, followed by a description of the main properties of the Poisson, Poisson–Szegő and Szegő kernels in Section 4. Finally, in Section 5 we shall consider p -Carleson measures, together with necessary and sufficient conditions for a measure to be a p -Carleson measure.

2. Clifford analysis

2.1. Clifford algebras. Let e_1, \dots, e_n be an orthonormal basis of the real vector space \mathbb{R}^n equipped with the negative definite quadratic form,

$$Q(x) = -|x|^2,$$

where $|\cdot|$ denotes the standard Euclidean norm in \mathbb{R}^n . We define the universal real-valued Clifford algebra $\mathcal{C}_{0,n}$ as the 2^n -dimensional associative algebra generated by \mathbb{R}^n modulo the quadratic form $Q(x)$. That is, in the algebra $\mathcal{C}_{0,n}$ we have the anticommutation relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

and a basis for $\mathcal{C}_{0,n}$ consists of the identity $e_0 = 1$, together with the elements of the form $e_A = e_{i_1} \cdots e_{i_k}$, where $A = \{(i_1, \dots, i_k) : 1 \leq i_1 < \dots < i_k \leq n\}$. Hence, each element $a \in \mathcal{C}_{0,n}$ can be written as a linear combination of the basis elements $a = \sum_A a_A e_A$, $a_A \in \mathbb{R}$. Henceforward, we will identify \mathbb{R}^n with the linear subspace $\mathbb{R}^{0,n} \subset \mathcal{C}_{0,n}$ spanned by e_1, \dots, e_n .

We define an anti-automorphism called *conjugation* (or *main anti-involution*) in $\mathcal{C}_{0,n}$ by its action on the basis elements:

$$\bar{e}_0 = e_0, \quad \bar{e}_j = -e_j, \quad j = 1, \dots, n, \quad \overline{e_{i_1} e_{i_2} \cdots e_{i_l}} = \bar{e}_{i_l} \cdots \bar{e}_{i_1}.$$

In particular, for a vector $x \in \mathbb{R}^n$ we have $\bar{x} = x_1 \bar{e}_1 + \dots + x_n \bar{e}_n = -x$ and $x\bar{x} = -x^2 = |x|^2$, the square of the Euclidean norm of x .

2.2. Clifford analysis. Clifford analysis offers a function theory which is a higher dimensional analogue to the theory of holomorphic functions of one complex variable (see e.g. [6]).

Let Ω denote an open region of \mathbb{R}^n . Properties such as continuity, differentiability, integrability, and so on, are ascribed to a Clifford-valued function

$f = \sum_A e_A f_A : \Omega \rightarrow \mathcal{C}_{0,n}$ by requiring that all its real-valued components f_A should possess them.

A Clifford-valued function $f : \Omega \rightarrow \mathcal{C}_{0,n}$ is called *left monogenic* (*right monogenic*, or *two-sided monogenic*) in Ω if $Df = 0$ ($fD = 0$, or $Df = fD = 0$, respectively). Here, D denotes the *Dirac operator*

$$Df = \sum_{j=1}^n e_j \frac{\partial f}{\partial x_j}.$$

An important function which is both left and right monogenic is the generalized Cauchy kernel

$$e(x) = \frac{1}{\omega_n} \frac{\bar{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit sphere in \mathbb{R}^n .

The higher dimensional Cauchy's theorem and Cauchy's integral formula are as follows:

THEOREM 2.1 (Cauchy's Theorem). *Let \mathcal{B}^n denote the open unit ball in \mathbb{R}^n . If f and g are right and left monogenic in a neighborhood of $\overline{\mathcal{B}^n}$, then*

$$(1) \quad \int_{\mathcal{S}^{n-1}} f(y)n(y)g(y) d\mathcal{S}_y^{n-1} = \int_{\mathcal{B}^n} [(fD_y)(y)g(y) + f(y)(D_yg)(y)] dy = 0$$

where $d\mathcal{S}_y^{n-1}$ is the surface element of the boundary \mathcal{S}^{n-1} of \mathcal{B}^n , $n(y)$ is the exterior unit normal to \mathcal{S}^{n-1} , defined for almost all $y \in \mathcal{S}^{n-1}$, and dy is the volume element of the unit sphere.

COROLLARY 2.2 (Cauchy's Integral Formula). *Under the previous conditions,*

$$C_r f(x) = \int_{\mathcal{S}^{n-1}} f(y)n(y)e(x-y) d\mathcal{S}_y^{n-1} = \begin{cases} f(x), & x \in \mathcal{B}^n, \\ 0, & x \notin \overline{\mathcal{B}^n}, \end{cases}$$

$$C_l g(x) = \int_{\mathcal{S}^{n-1}} e(x-y)n(y)g(y) d\mathcal{S}_y^{n-1} = \begin{cases} g(x), & x \in \mathcal{B}^n, \\ 0, & x \notin \overline{\mathcal{B}^n}. \end{cases}$$

These results can be generalized to any strongly Lipschitz and bounded open subset Ω of \mathbb{R}^n with boundary $\partial\Omega$ and exterior unit normal $n(y)$ defined for almost all $y \in \partial\Omega$. For details see [9], [8] or [7].

As an immediate consequence we find that both left and right monogenics can be represented in terms of Cauchy integrals.

3. Conformal mappings of the unit ball. It is well known that a complex-valued conformal mapping preserving the unit disk \mathcal{B}^2 can be writ-

ten as the composition of a Möbius transformation of type

$$(2) \quad \varphi_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1,$$

with a rotation $w = e^{i\phi}z$, with $0 \leq \phi < 2\pi$. The Möbius transformation φ_a can be characterized as the unique conformal transformation which preserves the unit disk, maps a to the origin and its derivative satisfies $\varphi'_a(a) > 0$. It can be proved that $\varphi_a \circ \varphi_{-a}$ is the identity so that $\mathcal{G}(\mathbb{C}) = \{\varphi_a : |a| < 1\}$ (endowed with the composition) is identified with a subgroup of $\mathrm{SL}(2, \mathbb{C})$, the special linear group. This particular subgroup has the property of leaving the 2-dimensional Laplacian invariant.

Moreover, the derivative of (2) satisfies

$$\varphi'_a(z) = \frac{1 - a\bar{a}}{(1 - \bar{a}z)^2} = \frac{1 - |a|^2}{(1 - \bar{a}z)^2},$$

hence proving the invariance of the Poincaré metric in the complex disk

$$\frac{1}{2}|ds| = \frac{|dz|}{1 - |z|^2} = \frac{|d\varphi_a(z)|}{1 - |\varphi_a(z)|^2}$$

under the action of this subgroup.

The higher dimensional counterpart of (2) is the transform

$$(3) \quad \varphi_a(x) = (x - a)(1 - \bar{a}x)^{-1} = \frac{(x - a)(1 - \bar{a}a)}{|1 - \bar{a}x|^2},$$

where x, a are vectors of \mathcal{B}^n , the n -dimensional unit ball (see e.g. [1], [3]). In what follows we shall use the Möbius transformations defined via the right inverse of $1 - \bar{a}x$. By duality, analogous results hold for Möbius transformations defined via the left inverse. As in the complex case we have (see [3], [4])

COROLLARY 3.1. *Every conformal transformation χ leaving the unit ball invariant can be written as the composition of a Möbius transformation of type φ_a with an orthogonal transformation. Moreover, if χ is direct then the orthogonal transformation is a rotation.*

Several properties are directly extendable to the n -dimensional case. In fact, simple calculations show that

$$(4) \quad 1 - |\varphi_a(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{|1 - \bar{a}x|^2},$$

proving that the group $\mathcal{G}(\mathbb{R}^n)$ of transformations of type (3) preserves the unit ball. This, combined with the relation between the differentials

$$(5) \quad |d\varphi_a(x)| = \frac{1 - |a|^2}{|1 - \bar{a}x|^2}|dx|,$$

where $|dx| = \sqrt{dx_1^2 + \cdots + dx_n^2}$, proves the invariance of the n -dimensional Poincaré metric

$$|ds| = \frac{2|dx|}{1 - |x|^2} = \frac{2|d\varphi_a(x)|}{1 - |\varphi_a(x)|^2}$$

under the action of $\mathcal{G}(\mathbb{R}^n)$. Therefore, the change of variable $y = \varphi_a(x)$, $|a| < 1$, yields for the integral over the unit ball the formula

$$\int_{\mathcal{B}^n} f(y) dy = \int_{\mathcal{B}^n} f(\varphi_a(x)) \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^n dx.$$

Note that both (3) and the differential form (5) have a singularity at $z_0 = a/|a|^2$. However, there is an important difference between the complex and the n -dimensional versions of the group preserving the unit ball. While in \mathbb{C} this group preserves the Laplacian, in the sense that

$$\Delta(f \circ \varphi_a) = (\Delta f) \circ \varphi_a, \quad \forall \varphi_a \in \mathcal{G}(\mathbb{C}),$$

this is no longer true in higher dimensions. On the other hand, the group $\mathcal{G}(\mathbb{R}^n)$ preserves the Laplace–Beltrami operator

$$\Delta_{\text{LB}} f = \operatorname{div} \operatorname{grad} f = \frac{1}{\sqrt{|g|}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|} \sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x_j} \right),$$

related to the metric tensor $ds^2 = \sum_{i,j} g_{ij} dx_i dx_j$, where (g^{ij}) stands for the inverse matrix of (g_{ij}) and $|g| = \det(g_{ij})$. In the present case, we have for the Laplace–Beltrami operator the expression

$$(6) \quad \Delta_{\text{LB}} f = (1 - r^2)[(1 - r^2)\Delta + 2(n - 2)r\partial_r]f,$$

and it satisfies the invariance condition

$$\Delta_{\text{LB}}(f \circ \varphi_a) = (\Delta_{\text{LB}} f) \circ \varphi_a$$

for all φ_a in $\mathcal{G}(\mathbb{R}^n)$. The solutions of the Laplace–Beltrami operator are called *hyperbolically harmonic functions* or *\mathcal{M} -harmonic functions*.

4. Poisson and Poisson–Szegő kernels for \mathcal{B}^n . In the two-dimensional case, it is well known that the Poisson and Poisson–Szegő kernels coincide, both integrals being conformally invariant. As a consequence, harmonic complex-valued functions are uniquely reconstructed from their values on the unit sphere.

In higher dimensions, however, that is no longer true, since the Laplacian is no longer a conformally invariant operator; therefore, the Poisson kernel, which is defined by means of the outward derivative of the Green function for the Laplacian in the unit ball \mathcal{B}^n , does not coincide with the Poisson–Szegő kernel, a positive conformally invariant kernel arising out of the Szegő kernel

for the unit ball. Moreover, both the Poisson–Szegő and the Szegő kernels are connected to the Laplace–Beltrami operator (6).

In fact, let $G(x, y)$ be the Green’s function for the Laplacian with respect to the unit ball \mathcal{B}^n . This function is uniquely determined by the following properties:

- it is smooth for all $x \in \mathcal{B}^n, y \in \overline{\mathcal{B}^n}$ such that $x \neq y$;
- $\Delta_y G(x, y) = \delta(x - y)$;
- for a fixed $x \in \mathcal{B}^n$, $G(x, y) - \frac{1}{\omega_n |x - y|^{n-2}}$ is harmonic in the y variable;
- $G(x, y) = 0$ for all $x \in \mathcal{B}^n$ and $y \in \mathcal{S}^{n-1}$.

Denoting by $n_y = y$ the outward unit normal to \mathcal{S}^{n-1} at y we have the *Poisson kernel* in the unit ball

$$P(x, y) = -\frac{\partial G(x, y)}{\partial n_y} = \frac{1}{\omega_n} \frac{1 - |x|^2}{|1 - \bar{x}y|^n},$$

defined in $\mathcal{B}^n \times \mathcal{S}^{n-1}$. Note that P has a singularity of the same order as the one of the fundamental solution of the Laplace operator.

Now, let us define the *Szegő kernel* in \mathcal{B}^n . From the differential relation (5) we derive the change of coordinates for the surface element $d\mathcal{S}^{n-1}$ on \mathcal{S}^{n-1} (unique rotationally invariant measure on the unit sphere) as

$$\int_{\mathcal{S}^{n-1}} d\mathcal{S}_{\varphi_a(x)}^{n-1} = \int_{\mathcal{S}^{n-1}} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{n-1} d\mathcal{S}_x^{n-1},$$

leading to the following expression for the (positive) Poisson–Szegő kernel in \mathcal{B}^n (which is related to the Szegő kernel $S(x, \xi)$):

$$(7) \quad PS(x, \xi) = \left(\frac{1 - |x|^2}{|1 - \bar{x}\xi|^2} \right)^{n-1} = \frac{|S(x, \xi)|^2}{S(x, x)},$$

hence, for this last kernel we have

$$(8) \quad S(x, \xi) = \left(\frac{1 - \bar{\xi}x}{|1 - \bar{x}\xi|^2} \right)^{n-1}.$$

The behaviour of the Szegő kernel under the action of $\mathcal{G}(\mathbb{R}^n)$ is given by

$$S(\varphi_a(x), \varphi_a(\xi)) = \left(\frac{1}{1 - |a|^2} \frac{(1 - \bar{a}\xi)(1 - \bar{\xi}x)(1 - \bar{x}a)}{|1 - \bar{x}\xi|^2} \right)^{n-1}.$$

Moreover,

$$|S(\varphi_a(x), \varphi_a(\xi))| = \frac{|S(a, a)|}{|S(a, \xi)| |S(x, a)|} |S(x, \xi)|,$$

and in the particular case of $\xi = x$ we have $(1 - \bar{a}x)(1 - \bar{x}x)(1 - \bar{x}a) \in \mathbb{R}$. Therefore,

$$S(\varphi_a(x), \varphi_a(x)) = \frac{S(a, a)S(x, x)}{|S(a, x)|^2}.$$

These results lead to the transformation rule for the Poisson–Szegő kernel

$$\begin{aligned} PS(\varphi_a(x), \varphi_a(\xi)) &= \frac{|S(\varphi_a(x), \varphi_a(\xi))|^2}{S(\varphi_a(x), \varphi_a(x))} = \frac{S(a, a)}{|S(a, \xi)|^2} \frac{|S(x, \xi)|^2}{S(x, x)} \\ &= \frac{PS(x, \xi)}{PS(a, \xi)}. \end{aligned}$$

Now, we obtain the following important result:

THEOREM 4.1. *For each fixed $x \in \mathcal{B}^n$ the measure $d\mu_x(\xi)$ on the boundary \mathcal{S}^{n-1} given by*

$$d\mu_x(\xi) = PS(x, \xi) d\mathcal{S}^{n-1}(\xi)$$

is invariant under the transformations of type (3), that is,

$$d\mu_{\varphi_a(x)}(\varphi_a(\xi)) = d\mu_x(\xi).$$

Proof. Using relation (5) we have

$$d\mathcal{S}^{n-1}(\varphi_a(\xi)) = \left(\frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \right)^{n-1} d\mathcal{S}^{n-1}(\xi).$$

Now it is easily seen that

$$\begin{aligned} PS(\varphi_a(x), \varphi_a(\xi)) d\mathcal{S}^{n-1}(\varphi_a(\xi)) &= \frac{PS(x, \xi)}{PS(a, \xi)} \left(\frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \right)^{n-1} d\mathcal{S}^{n-1}(\xi) \\ &= PS(x, \xi) d\mathcal{S}^{n-1}(\xi). \blacksquare \end{aligned}$$

A simple calculation shows that $-DD = \Delta$, where Δ denotes the Laplacian in \mathbb{R}^n . Therefore, one can regard monogenic functions as an analogue to holomorphic functions. The linkage of monogenic functions with harmonic functions tells us that every monogenic function is harmonic. It is well-known that harmonic functions can be represented by Poisson integrals. Unfortunately, in higher dimensions the Poisson integral is no longer invariant under conformal transformations. This leads to the consideration of hyperbolically harmonic functions which are not only conformally invariant but can be represented by Poisson–Szegő integrals.

5. Carleson measures. In this section we shall consider the space of all real-valued $f \in L_p(\mathcal{S}^{n-1})$, $p \geq 1$, such that the Poisson–Szegő transform

$$\mathcal{PS}[f](x) = \int_{\mathcal{S}^{n-1}} f(\xi) PS(x, \xi) d\mathcal{S}^{n-1}(\xi), \quad |x| < 1,$$

is a hyperbolically harmonic function. By abuse of language we shall denote henceforth these spaces as h^p , that is,

$$\begin{aligned} h^p &= \{f \in L_p(\mathcal{S}^{n-1}) : (\exists g : \mathcal{B}^n \rightarrow \mathcal{C}_{0,n}) \Delta_{\text{LB}} g = 0 \\ &\quad \& \sup_{0 \leq r < 1} \|g(r \cdot)\|_{L_p(\mathcal{S}^{n-1})} < \infty\}. \end{aligned}$$

For the consistence of such a construction, we refer to [10, Theorem 3.3.4(b) and Theorem 4.4.3]. Finally, we shall prove that for every $f \in h^p$, its L_p -norm in the unit ball (with respect to a given measure ν) can be estimated in terms of its L_p -norm on the unit sphere if ν is a Carleson measure.

DEFINITION 5.1. A positive measure ν defined on the unit ball \mathcal{B}^n in \mathbb{R}^n is a p -Carleson measure if there exists a constant $A > 0$ such that

$$\nu(Q_h(\xi)) \leq Ah^{(n-1)p}$$

for any subset

$$Q_h(\xi) := \{x \in \mathcal{B}^n : |x - \xi| < h\},$$

where $h > 0$ and $\xi \in \mathcal{S}^{n-1}$.

In the case of complex functions defined on the unit circle we make use of a Carleson box

$$\{z = re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\}$$

instead of $Q_h(\xi)$.

The importance of Carleson measures is highlighted in the theorems to be proved later in this section. We will also need a covering lemma which can be found in [11].

LEMMA 5.2. Let $F = \{B(x_\sigma, r_\sigma) : \sigma \in \Lambda\}$ be a family of balls with bounded radii. Then for each constant $c > 0$ there exists a countable subfamily $\{B(x_i, r_i) : i \in \mathbb{N}\}$ consisting of pairwise disjoint balls such that each ball in F is contained in one of the balls $B(x_i, cr_i)$.

We define the *surface ball* $E(x, r)$, $x \in \mathcal{S}^{n-1}$ and $r > 0$, as the intersection of the ball $B(x, r)$ with the unit sphere \mathcal{S}^{n-1} . For each $a \in \mathcal{B}^n$ we now consider the particular family of surface balls

$$\begin{aligned} E^m(a) &:= E(a/|a|, 2^m(1 - |a|)) \\ &= B(a/|a|, 2^m(1 - |a|)) \cap \mathcal{S}^{n-1}, \quad m = 0, 1, 2, \dots, \end{aligned}$$

with $E^m(a) = \mathcal{S}^{n-1}$ for all $m \geq M$, where M is the smallest natural number satisfying $2^M(1 - |a|) \geq 2$ (see Figure 1). Moreover, it can be proved that this family has the following additional properties (see [2]):

$$(9) \quad \frac{C(n)2^{n-2}(1 - |a|)^{n-1}}{(n-1)\pi^{n-2}} \leq \mu(E^0(a)) \leq \frac{C(n)2^{n-1}(1 - |a|)^{n-1}}{n-1}$$

and

$$\mu(E(x_0, tr)) \leq ct^{n-1}\mu(E(x_0, r)),$$

from which it follows that $\mu(E^m(a)) \leq c2^{m(n-1)}\mu(E^0(a))$.

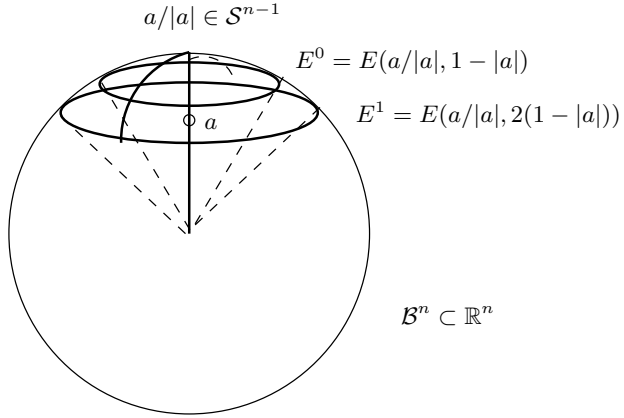


Fig. 1. Surface balls

The following theorem provides a necessary and sufficient condition for a positive measure to be a Carleson measure, given in terms of the Poisson–Szegő kernel.

THEOREM 5.3. *A positive measure ν defined on the unit ball \mathcal{B}^n is a p -Carleson measure if and only if*

$$\sup_{a \in \mathcal{B}^n} \int_{\mathcal{B}^n} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{(n-1)p} d\nu < \infty.$$

Proof. In a first step we assume

$$\sup_{a \in \mathcal{B}^n} \int_{\mathcal{B}^n} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{(n-1)p} d\nu < C_p < \infty.$$

For all $a \in \mathcal{B}^n$ we write $a = |a| \frac{a}{|a|} = (1 - \delta)\xi$, where $\xi \in \mathcal{S}^{n-1}$ and $0 < \delta < 1$. Then for all $x \in Q_\delta(\xi)$ we have

$$(10) \quad |1 - \bar{a}x| = |1 - \bar{\xi}x + \delta \bar{\xi}x| \leq |1 - \bar{\xi}x| + \delta|x| = |\xi - x| + \delta|x| < \delta + \delta|x| \leq 2\delta$$

and, on the other hand,

$$1 - |a|^2 = 1 - (1 - \delta)^2 = 2\delta - \delta^2 = \delta(2 - \delta) \geq \delta.$$

Therefore, we obtain the estimate

$$\nu(Q_\delta(\xi)) \leq \left(\frac{(2\delta)^2}{\delta} \right)^{(n-1)p} \int_{\mathcal{B}^n} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{(n-1)p} d\nu \leq 4^{(n-1)p} C_p \delta^{(n-1)p}.$$

This ends the first part of the proof.

Conversely, we now assume ν to be a Carleson measure, i.e.

$$(11) \quad \sup_{\delta, \xi} \frac{\nu(Q_\delta(\xi))}{\delta^{(n-1)p}} = K(n, p) < \infty,$$

and therefore ν is a finite measure.

We first assume $|a| \leq 3/4$, so that

$$(12) \quad |1 - \bar{a}x| \geq 1 - \frac{3}{4}|x| \geq \frac{1}{4}.$$

Note that $\mathcal{B}^n \subseteq Q_\delta(\xi)$ for all $\delta \geq 2$. Now

$$(13) \quad \int_{\mathcal{B}^n} d\nu \leq \nu(Q_2(\xi)) \leq 2^{(n-1)p} K(n, p),$$

and we obtain

$$\begin{aligned} \sup_{|a| \leq 3/4} \int_{\mathcal{B}^n} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{(n-1)p} d\nu &\leq 4^{2(n-1)p} \sup_{|a| \leq 3/4} \int_{\mathcal{B}^n} d\nu \\ &\leq 2^{4(n-1)p} \nu(Q_2(\xi)) \\ &\leq 2^{5(n-1)p} K(n, p) < \infty. \end{aligned}$$

It remains to consider the case of $|a| > 3/4$. We define for $\xi = a/|a|$ the family of sets

$$(14) \quad F_0 = Q_{2(1-|a|)}(\xi) \quad \text{and} \quad F_k = Q_{2^{k+1}(1-|a|)}(\xi) \setminus Q_{2^k(1-|a|)}(\xi).$$

Hence, we get

$$\nu(F_k) \leq \nu(Q_{2^{k+1}(1-|a|)}(\xi)) \leq K(n, p)(2^{k+1}(1-|a|))^{(n-1)p}.$$

Furthermore,

$$(15) \quad \begin{aligned} |1 - \bar{a}x| &= \left| \frac{a}{|a|} - |a|x \right| = |\xi - x + x - |a|x| \\ &\geq |\xi - x| - (1 - |a|)|x| \geq 2^k(1 - |a|) - (1 - |a|)|x| \\ &\geq 2^{k-1}(1 - |a|), \quad \forall x \in F_k, \quad k \geq 1, \end{aligned}$$

while for $x \in F_0$ we have $1/2 < |x| \leq 1$ so that

$$|1 - \bar{a}x| \geq 1 - |a||x| = (1 - |x|) + |x|(1 - |a|) \geq \frac{1}{2}(1 - |a|),$$

thus proving the validity of estimate (15) for all $k \geq 0$. We then obtain

$$\begin{aligned} \int_{F_k} \left(\frac{1 - |a|^2}{|1 - \bar{a}x|^2} \right)^{(n-1)p} d\nu &\leq \frac{(1 + |a|)^{(n-1)p}(1 - |a|)^{(n-1)p}}{(2^{k-1}(1 - |a|))^{2(n-1)p}} \int_{F_k} d\nu \\ &\leq \frac{2^{p(n-1)(3-2k)}}{(1 - |a|)^{(n-1)p}} K(n, p) 2^{p(n-1)} (1 - |a|)^{(n-1)p} \\ &= K(n, p) 2^{2p(n-1)(2-k)} \end{aligned}$$

and we conclude the proof with

$$\begin{aligned} \sup_{|a|>3/4} \int_{\mathcal{B}^n} \left(\frac{1-|a|^2}{|1-\bar{a}x|^2} \right)^{(n-1)p} d\nu &= \sup_{|a|>3/4} \sum_{k=0}^{\infty} \int_{F_k} \left(\frac{1-|a|^2}{|1-\bar{a}x|^2} \right)^{(n-1)p} d\nu \\ &\leq K(n,p) 2^{4p(n-1)} \sum_{k=0}^{\infty} (2^{-2p(n-1)})^k < \infty. \quad \blacksquare \end{aligned}$$

Finally, we present a necessary criterion for a positive measure to be a Carleson measure, based on estimates of the L_p -norm of the boundary values of $f \in h^p$. For that purpose, we first prove the following result regarding an estimation of the values of a hyperbolically harmonic function by means of its boundary values.

THEOREM 5.4. *For every $f \in h^1$,*

$$(16) \quad |f(a)| \lesssim \tilde{f}(a) + \|f\|_1, \quad \forall |a| < 1,$$

where

$$(17) \quad \tilde{f}(a) := \sup_{m=0,1,\dots,M} \frac{1}{\mu(E^m(a))} \int_{E^m(a)} |f(y)| d\mathcal{S}_y^{n-1}.$$

Proof. Under the theorem's conditions we have (see e.g. [5, p. 34] for a proof in the harmonic complex case)

$$|f(a)| \leq \int_{\mathcal{S}^{n-1}} PS(a,y) |f(y)| d\mathcal{S}_y^{n-1}.$$

For simplicity, write $\theta = \angle(x, a)$. For each $x \in \mathcal{S}^{n-1}$ and $a \in \mathcal{B}^n$ we have the initial estimate

$$\frac{1-|a|^2}{1+|a|^2-2|a|\cos\theta} \leq \frac{1+|a|}{1-|a|} \leq \frac{2}{1-|a|},$$

which can be improved, for $|a| \geq 1/2$ and $x \in E^{m+1}(a) \setminus E^m(a)$, to

$$\frac{1-|a|^2}{1+|a|^2-2|a|\cos\theta} \leq \frac{2^{2-2m}}{1-|a|}.$$

Applying the Poisson–Szegő formula and inequality (9) we obtain, for $|a| \geq 1/2$,

$$\begin{aligned} |f(a)| &\leq \frac{1}{\omega_n} \int_{\mathcal{S}^{n-1}} |f| \left(\frac{1-|a|^2}{1+|a|^2-2|a|\cos\theta} \right)^{n-1} d\mathcal{S}^{n-1} \\ &\leq \frac{1}{\omega_n} \left[\int_{E^0(a)} \frac{2^{n-1}|f|}{(1-|a|)^{n-1}} d\mathcal{S}^{n-1} + \sum_{m=0}^{M-1} \int_{E^{m+1}(a) \setminus E^m(a)} \frac{2^{2(n-1)(1-m)}|f|}{(1-|a|)^{n-1}} d\mathcal{S}^{n-1} \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\omega_n} \left[\frac{C(n)2^{2(n-1)}}{\mu(E^0(a))(n-1)} \int_{E^0(a)} |f| d\mathcal{S}^{n-1} \right. \\
&\quad \left. + \sum_{m=0}^{M-1} \frac{c2^{(m+1)(n-1)}C(n)2^{n-1}}{\mu(E^{m+1}(a))(n-1)} \int_{E^{m+1}(a) \setminus E^m(a)} 2^{2(n-1)(1-m)} |f| d\mathcal{S}^{n-1} \right] \\
&= \frac{C(n)2^{2(n-1)}}{(n-1)\omega_n} \left[\frac{1}{\mu(E^0(a))} \int_{E^0(a)} |f| d\mathcal{S}^{n-1} \right. \\
&\quad \left. + \sum_{m=0}^{M-1} c2^{2-m/2} \frac{1}{\mu(E^{m+1}(a))} \int_{E^{m+1}(a)} |f| d\mathcal{S}^{n-1} \right] \\
&\leq \frac{C(n)2^{2(n-1)}}{(n-1)\omega_n} \left(1 + 4c \sum_{m=0}^{M-1} 2^{-m/2} \right) \tilde{f}(a).
\end{aligned}$$

In the case of $|a| < 1/2$, we have the following estimate for the Poisson–Szegő kernel:

$$PS(a, y) \leq \left[\frac{1 - |a|^2}{(1 - |a|)^2} \right]^{n-1} \leq 2^{n-1}, \quad y \in \mathcal{S}^{n-1},$$

and, therefore,

$$|f(a)| \leq \int_{\mathcal{S}^{n-1}} PS(a, y) |f(y)| d\mathcal{S}^{n-1} \leq \frac{2^n}{\omega_n} \int_{\mathcal{S}^{n-1}} |f(y)| d\mathcal{S}^{n-1} \leq K \|f\|_1.$$

Combining the above estimates we get the desired result

$$|f(a)| \lesssim \tilde{f}(a) + \|f\|_1 \quad \text{for all } |a| < 1. \quad \blacksquare$$

THEOREM 5.5. *If the positive measure ν defined on the unit ball \mathcal{B}^n is a Carleson measure then for all $1 < p < \infty$ there exists a constant $C > 0$ such that*

$$\left(\int_{\mathcal{B}^n} |f(x)|^p d\nu(x) \right)^{1/p} \leq C \|f\|_p \quad \text{for all } f \in h^p.$$

Proof. In a first step, we shall prove that the mapping $f \mapsto \tilde{f}$ is of weak type $(1, 1)$, i.e. the Carleson measure of the set $M_s := \{x \in \mathcal{B}^n : \tilde{f}(x) > s\}$ satisfies

$$(18) \quad \nu(M_s) \lesssim s^{-1} \|f\|_1.$$

Let K_s be a compact subset of M_s . Consider the covering of K_s by the family of balls $B(x/|x|, r)$, where $x \in K_s$ and $r = 2(1 - |x|)$. Note that

- $B(x/|x|, r) \cap \mathcal{S}^{n-1} = E(x/|x|, r)$;
- $\left| \frac{x}{|x|} - x \right| = \left| \frac{x}{|x|} (1 - |x|) \right| = (1 - |x|)$, hence $x \in B(x/|x|, r)$.

Due to Lemma 5.2 (we will assume the constant $c = 4$) and because K_s is compact we can extract a finite subfamily $\{B(x_i/|x_i|, r_i) : i = 1, \dots, l\}$ satisfying

$$\bigcup_{i=1}^l B(x_i/|x_i|, r_i) \cap \mathcal{B}^n \subseteq K_s \subseteq \bigcup_{i=1}^l B(x_i/|x_i|, 4r_i) \cap \mathcal{B}^n.$$

In accordance with Definition 5.1 we write $Q_{r_i}(x_i/|x_i|) = B(x_i/|x_i|, r_i) \cap \mathcal{B}^n$. Since ν is a Carleson measure we have

$$\begin{aligned} \nu(K_s) &\leq \nu\left(\bigcup_{i=1}^l Q_{4r_i}(x_i/|x_i|)\right) \leq \sum_{i=1}^l \nu(Q_{4r_i}(x_i/|x_i|)) \\ &\leq A \sum_{i=1}^l (4r_i)^{n-1} = A8^{n-1} \sum_{i=1}^l (1 - |x_i|)^{n-1}. \end{aligned}$$

Due to estimate (9), we get

$$(19) \quad A8^{n-1} \sum_{i=1}^l (1 - |x_i|)^{n-1} \leq \frac{A(n-1)\pi^{n-2}}{2^{1-2n}C(n)} \sum_{i=1}^l \mu(E^0(x_i)).$$

For each $x_i \in K_s \subset M_s$ we have

$$\begin{aligned} s &< \sup_{m=0, \dots, M} \frac{1}{\mu(E^m(x_i))} \int_{E^m(x_i)} |f| d\mathcal{S}^{n-1} = \tilde{f}(x_i), \\ \mu(E^0(x_i)) &\leq \frac{1}{s} \sup_{m=0, \dots, M} \frac{\mu(E^0(x_i))}{\mu(E^m(x_i))} \int_{E^m(x_i)} |f| d\mathcal{S}^{n-1} \leq \frac{1}{s} \|f\|_1. \end{aligned}$$

Inserting now this estimate in (19), we obtain

$$\begin{aligned} \nu(K_s) &\leq \frac{A(n-1)\pi^{n-2}}{2^{1-2n}C(n)} \sum_{i=1}^l \mu(E(x_i/|x_i|, 1 - |x_i|)) \\ &< \frac{A(n-1)\pi^{n-2}}{2^{1-2n}C(n)} \frac{l}{s} \|f\|_1, \end{aligned}$$

and, therefore, $\nu(M_s) = \sup_{K_s \subset M_s} \nu(K_s) \lesssim s^{-1} \|f\|_1$, which concludes the first part of the proof.

In a second step, we shall prove that the mapping $f \mapsto \tilde{f}$ is of weak type (p, p) for $p > 1$, that is, $\nu(M_s) \lesssim \|f\|_p^p$. For that we introduce the auxiliary function

$$f_1(x) = \begin{cases} f(x), & |f(x)| > s/2, \\ 0, & \text{otherwise,} \end{cases}$$

and we obtain

$$\frac{s}{2} < \sup_{m=0,\dots,M} \frac{1}{\mu(E^m(x))} \int_{E^m(x)} |f_1(y)| d\mathcal{S}_y^{n-1} \leq \tilde{f}(x),$$

that is, $M_s \subset \widetilde{M}_{s/2} = \{x \in \mathcal{B}^n : \tilde{f}_1 > s/2\}$. Therefore, $\nu(M_s) \leq \nu(\widetilde{M}_{s/2})$ implies

$$\nu(M_s) \leq c \|f_1\|_1 \leq \frac{c}{s} \int_{\mathcal{S}^{n-1}} |f_1| d\mathcal{S}^{n-1} = \frac{c}{s} \int_{\{x \in \mathcal{S}^{n-1} : |f| > s/2\}} |f| d\mathcal{S}^{n-1}.$$

Using (18) we have

$$s^{p-1} \nu(M_s) \leq C s^{p-2} \|f\|_1,$$

which leads to

$$\begin{aligned} \int_0^{2|f|} s^{p-1} \nu(M_s) ds &\leq C \int_0^{2|f|} s^{p-2} \|f\|_1 ds = C \int_{\mathcal{S}^{n-1}} |f| \int_0^{2|f|} s^{p-2} ds d\mathcal{S}^{n-1} \\ &= C \frac{2^{p-1}}{p-1} \int_{\mathcal{S}^{n-1}} |f|^p d\mathcal{S}^{n-1} = C \frac{2^{p-1}}{p-1} \|f\|_p^p \end{aligned}$$

for all $p > 1$. Since \tilde{f} is ν -measurable in M_s we now apply the formula

$$\frac{s^p}{p} \nu(M_s) \leq \frac{1}{p} \int_{M_s} |\tilde{f}|^p d\nu = \int_0^{2|f|} s^{p-1} \nu(M_s) ds$$

to obtain the desired result.

By the Marcinkiewicz interpolation theorem ([10]), we conclude that the mapping $f \mapsto \tilde{f}$ is of strong type (p, p) for all $1 < p < \infty$, that is,

$$(20) \quad \|\tilde{f}\|_{\nu, p}^p = \int_{\mathcal{B}^n} |\tilde{f}|^p d\nu \lesssim \|f\|_p^p.$$

Finally, we conclude our proof with the help of Theorem 5.4 and of (20). In fact,

$$\begin{aligned} \left(\int_{\mathcal{B}^n} |f|^p d\nu \right)^{1/p} &= \|f\|_{\nu, p} \leq K \|\tilde{f}\| + \|f\|_1 \|f\|_{\nu, p} \\ &\leq K (\|\tilde{f}\|_{\nu, p} + \|f\|_1 \nu(\mathcal{B}^n)) \lesssim \|f\|_p, \end{aligned}$$

since $\|f\|_1 \leq \|f\|_p$ for $p > 1$. ■

6. Conclusion. We must remark that this paper leaves several open questions. First, it remains to establish a generalization of the classical harmonic Hardy spaces to $\mathcal{C}_{0,n}$ -valued functions. Second, one should investigate the possible connection between these spaces and the classical harmonic h^p , as well as its relation to already existing monogenic Hardy-type spaces.

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Received July 26, 2005
Revised version March 14, 2007

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