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An upper bound for the Lipschitz retraction constant in l_1

by

Marco Annoni and Emanuele Casini (Como)

Abstract. We construct a new lipschitzian retraction from the closed unit ball of the Banach space l_1 onto its boundary, with Lipschitz constant 8.

1. Introduction and notation. Given a Banach space $(X, \|\cdot\|)$, let $B(X) = \{x \in X : \|x\| \leq 1\}$ denote the closed unit ball, and $S(X) = \{x \in X : \|x\| = 1\}$ the unit sphere.

We know, by Benyamini–Sternfeld's theorem (see [2]), that X is infinite-dimensional if and only if there exists a lipschitzian retraction from $B(X)$ onto $S(X)$, i.e., a lipschitzian function which fixes all points in $S(X)$. Let $\mathcal{R}_X(k)$ (resp. $\mathcal{L}_X(k)$) denote the set of k-lipschitzian retractions (resp. maps) from $B(X)$ onto $S(X)$ (resp. into $B(X)$). If X is infinite-dimensional, we define the *retraction constant* of X to be

$$k_0(X) = \inf\{k \in \mathbb{R}^+ : \mathcal{R}_X(k) \neq \emptyset\}.$$

The problem to exactly determine $k_0(X)$ for at least one Banach space X is still open. Some bounds are however known:

- $k_0(X) \geq 3$ for every Banach space (see [5]);
- $k_0(H) \geq 4.55$ for every Hilbert space H (see [5]);
- $k_0(l_1) \geq 4$ (see [3]);
- $k_0(X) \leq 30.84$ for every Banach space X in which $\psi_X(k) = 1 - 1/k$ (see [1]), where

$$\psi_X(k) = \sup_{T \in \mathcal{L}_X(k)} \inf_{x \in B(X)} \|x - Tx\|,$$

  in particular, $k_0(c_0) \leq 30.84$;
- $k_0(H) \leq 28.99$ (see [1]);
- $k_0(\mathcal{C}([0,1])) \leq 4(1+\sqrt{2})^2 \simeq 23.31$ (see [4]);
- $k_0(l_1) \leq 22.45$ (see [1]);

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- $k_0(\mathcal{C}_0([0,1])) \leq 17.38$ (see [5]), where $\mathcal{C}_0([0,1]) := \{f \in \mathcal{C}([0,1]) : f(0) = 0\}$ is endowed with its natural norm;
- $k_0(L_1([0,1])) \leq 9.43$ ([5]).

The last upper bound has been the best known so far for any Banach space. Here we improve this result by exhibiting an 8-lipschitzian retraction from $B(l_1)$ onto $S(l_1)$.

For further results about the retraction constant and the function ψ_X we refer the reader to [5].

2. The retraction in l_1. Let

$$B_1 = \{x \in l_1 : \|x\| \leq 1/2\}, \qquad B_2 = \{x \in l_1 : 1/2 \leq \|x\| < 1\},$$

and define $i : B_2 \to \mathbb{N}$ by setting, for $x = (x_j)_{j=0}^{\infty} \in l_1$,

$$i(x) = \min\Big\{j \in \mathbb{N} : \sum_{k=j+1}^{\infty} |x_k| < 1 - \|x\|\Big\}.$$

Moreover, let $\mu : B_2 \to (0,1]$ be such that, for all $x \in B_2$,

$$\mu(x)|x_{i(x)}| + \sum_{k=i(x)+1}^{\infty} |x_k| = 1 - \|x\|,$$

and define $Q : \overline{B}_2 \to B_1$ by

$$Q(x) = \begin{cases} \mu(x)x_{i(x)}e_{i(x)} + \displaystyle\sum_{k=i(x)+1}^{\infty} x_k e_k & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| = 1, \end{cases}$$

where $\{e_j\}_{j=0}^{\infty}$ is the standard basis of l_1 and $\overline{B}_2$ is the closure of B_2. Observe that, for any $x \in B_2$, we have

$$\|Q(x)\| = 1 - \|x\|$$

and so

$$\|(I - Q)(x)\| = \|x\| - \|Q(x)\| = 2\|x\| - 1,$$

where I is the identity map.

In the following proposition we give the main properties of the map Q.

Proposition 1. *Q is 3-lipschitzian and $I - Q$ is 2-lipschitzian.*

Proof. If $\|x\| = 1$ and $y \in \overline{B}_2$ then

$$\begin{aligned} \|(I-Q)(x) - (I-Q)(y)\| &\leq \|x - y\| + \|Q(x) - Q(y)\| \\ &= \|x - y\| + \|0 - Q(y)\| = \|x - y\| + (1 - \|y\|) \\ &= \|x - y\| + \|x\| - \|y\| = 2\|x - y\|. \end{aligned}$$

Now suppose that $x, y \in B_2$. We will discuss only the case $i(x) \neq i(y)$ (the case $i(x) = i(y)$ is analogous and left to the reader). We can suppose that $i(x) < i(y)$. Assuming that $\sum_{k=n}^{m} a_k = 0$ if $m < n$, we have

$$\begin{aligned}
&\|(I-Q)(x) - (I-Q)(y)\| \\
&= \sum_{k=0}^{i(x)-1} |x_k - y_k| + |(1-\mu(x))x_{i(x)} - y_{i(x)}| + \sum_{k=i(x)+1}^{i(y)-1} |y_k| + (1-\mu(y))|y_{i(y)}| \\
&\leq \sum_{k=0}^{i(x)} |x_k - y_k| + \mu(x)|x_{i(x)}| + \sum_{k=i(x)+1}^{i(y)} |y_k| - \mu(y)|y_{i(y)}| \\
&= \sum_{k=0}^{i(x)} |x_k - y_k| + \Big(1 - \sum_{k=0}^{\infty} |x_k| - \sum_{k=i(x)+1}^{\infty} |x_k|\Big) + \sum_{k=i(x)+1}^{i(y)} |y_k| \\
&\quad - \Big(1 - \sum_{k=0}^{\infty} |y_k| - \sum_{k=i(y)+1}^{\infty} |y_k|\Big) \\
&= \sum_{k=0}^{i(x)} |x_k - y_k| + \sum_{k=0}^{\infty} (|y_k| - |x_k|) + \sum_{k=i(x)+1}^{\infty} (|y_k| - |x_k|) \leq 2\|x - y\|.
\end{aligned}$$

So $I - Q$ is 2-lipschitzian. It follows that Q is 3-lipschitzian (just write $Q = I - (I - Q)$). ∎

Finally, if σ denotes the "right-shift" map, we define $R : B(l_1) \to S(l_1)$ by

$$R(x) = \begin{cases} (1 - 2\|x\|)e_0 + 2\sigma(x) & \text{if } x \in B_1, \\ (I - Q)(x) + 2\sigma(Q(x)) & \text{if } x \in \overline{B}_2, \end{cases}$$

and let $R_1 :\equiv R_{|B_1}$ and $R_2 :\equiv R_{|\overline{B}_2}$. It can be easily verified that:

- R is well defined;
- R_1 is 4-lipschitzian and R_2 is 8-lipschitzian;
- R is lipschitzian with Lipschitz constant $\mathrm{lip}(R) = \max\{\mathrm{lip}(R_1), \mathrm{lip}(R_2)\} \leq 8$;
- $R(x) \in S(l_1)$ for all $x \in B(l_1)$;
- $R_2(x) = x$ if $\|x\| = 1$, so that R is a retraction onto $S(l_1)$.

This obviously shows that $k_0(l_1) \leq 8$.

References

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Marco Annoni
Via Salita Cappuccini 16
22100 Como, Italy
E-mail: marcoannoni.81@gmail.com

Emanuele Casini
Dipartimento di Fisica e Matematica
Università degli Studi dell'Insubria
Via Valleggio 11
22100 Como, Italy
E-mail: emanuele.casini@uninsubria.it

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