An upper bound for the Lipschitz retraction constant in $l_1$

by

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Abstract. We construct a new lipschitzian retraction from the closed unit ball of the Banach space $l_1$ onto its boundary, with Lipschitz constant 8.

1. Introduction and notation. Given a Banach space $(X, \| \cdot \|)$, let $B(X) = \{ x \in X : \| x \| \leq 1 \}$ denote the closed unit ball, and $S(X) = \{ x \in X : \| x \| = 1 \}$ the unit sphere.

We know, by Benyamini–Sternfeld’s theorem (see [2]), that $X$ is infinite-dimensional if and only if there exists a lipschitzian retraction from $B(X)$ onto $S(X)$, i.e., a lipschitzian function which fixes all points in $S(X)$. Let $\mathcal{R}_X(k)$ (resp. $\mathcal{L}_X(k)$) denote the set of $k$-lipschitzian retractions (resp. maps) from $B(X)$ onto $S(X)$ (resp. into $B(X)$). If $X$ is infinite-dimensional, we define the retraction constant of $X$ to be

$$k_0(X) = \inf\{ k \in \mathbb{R}^+ : \mathcal{R}_X(k) \neq \emptyset \}.$$

The problem to exactly determine $k_0(X)$ for at least one Banach space $X$ is still open. Some bounds are however known:

- $k_0(X) \geq 3$ for every Banach space (see [5]);
- $k_0(H) \geq 4.55$ for every Hilbert space $H$ (see [5]);
- $k_0(l_1) \geq 4$ (see [3]);
- $k_0(X) \leq 30.84$ for every Banach space $X$ in which $\psi_X(k) = 1 - 1/k$ (see [1]), where

$$\psi_X(k) = \sup_{T \in \mathcal{L}_X(k)} \inf_{x \in B(X)} \| x - Tx \|,$$

in particular, $k_0(c_0) \leq 30.84$;
- $k_0(H) \leq 28.99$ (see [1]);
- $k_0(C([0,1])) \leq 4(1 + \sqrt{2})^2 \simeq 23.31$ (see [4]);
- $k_0(l_1) \leq 22.45$ (see [1]);

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• \( k_0(C_0([0,1])) \leq 17.38 \) (see [5]), where \( C_0([0,1]) := \{ f \in C([0,1]) : f(0) = 0 \} \) is endowed with its natural norm;  
• \( k_0(L_1([0,1])) \leq 9.43 \) ([5]).

The last upper bound has been the best known so far for any Banach space. Here we improve this result by exhibiting an 8-lipschitzian retraction from \( B(l_1) \) onto \( S(l_1) \).

For further results about the retraction constant and the function \( \psi_X \) we refer the reader to [5].

2. The retraction in \( l_1 \). Let  
\[ B_1 = \{ x \in l_1 : \| x \| \leq 1/2 \}, \quad B_2 = \{ x \in l_1 : 1/2 \leq \| x \| < 1 \}, \]

and define \( i : B_2 \to \mathbb{N} \) by setting, for \( x = (x_j)_{j=0}^{\infty} \in l_1 \),  
\[ i(x) = \min \left\{ j \in \mathbb{N} : \sum_{k=j+1}^{\infty} |x_k| < 1 - \| x \| \right\}. \]

Moreover, let \( \mu : B_2 \to (0,1] \) be such that, for all \( x \in B_2 \),  
\[ \mu(x)|x_{i(x)}| + \sum_{k=i(x)+1}^{\infty} |x_k| = 1 - \| x \|, \]

and define \( Q : \overline{B_2} \to B_1 \) by  
\[ Q(x) = \begin{cases} 
\mu(x)x_{i(x)}e_{i(x)} + \sum_{k=i(x)+1}^{\infty} x_ke_k & \text{if } \| x \| < 1, \\
0 & \text{if } \| x \| = 1,
\end{cases} \]

where \( \{e_j\}_{j=0}^{\infty} \) is the standard basis of \( l_1 \) and \( \overline{B_2} \) is the closure of \( B_2 \). Observe that, for any \( x \in B_2 \), we have  
\[ \| Q(x) \| = 1 - \| x \| \]

and so  
\[ \| (I - Q)(x) \| = \| x \| - \| Q(x) \| = 2\| x \| - 1, \]

where \( I \) is the identity map.

In the following proposition we give the main properties of the map \( Q \).

**Proposition 1.** \( Q \) is 3-lipschitzian and \( I - Q \) is 2-lipschitzian.

**Proof.** If \( \| x \| = 1 \) and \( y \in \overline{B_2} \) then  
\[ \| (I - Q)(x) - (I - Q)(y) \| \leq \| x - y \| + \| Q(x) - Q(y) \| \]
\[ = \| x - y \| + \| 0 - Q(y) \| = \| x - y \| + (1 - \| y \|) \]
\[ = \| x - y \| + \| x \| - \| y \| = 2\| x - y \|. \]
Now suppose that $x, y \in B_2$. We will discuss only the case $i(x) \neq i(y)$ (the case $i(x) = i(y)$ is analogous and left to the reader). We can suppose that $i(x) < i(y)$. Assuming that $\sum_{k=n}^{m} a_k = 0$ if $m < n$, we have

$$
\|(I - Q)(x) - (I - Q)(y)\|
\leq \sum_{k=0}^{i(x)-1} |x_k - y_k| + \sum_{k=i(x)+1}^{i(y)-1} |y_k| + (1 - \mu(y))|y_{i(y)}|
$$

Finally, if $\sigma$ denotes the “right-shift” map, we define $R : B(l_1) \to S(l_1)$ by

$$
R(x) = \begin{cases}
(1 - 2\|x\|)e_0 + 2\sigma(x) & \text{if } x \in B_1, \\
(I - Q)(x) + 2\sigma(Q(x)) & \text{if } x \in \overline{B}_2,
\end{cases}
$$

and let $R_1 := R|_{B_1}$ and $R_2 := R|_{\overline{B}_2}$. It can be easily verified that:

- $R$ is well defined;
- $R_1$ is 4-lipschitzian and $R_2$ is 8-lipschitzian;
- $R$ is lipschitzian with Lipschitz constant $\lip(R) = \max\{\lip(R_1), \lip(R_2)\} \leq 8$;
- $R(x) \in S(l_1)$ for all $x \in B(l_1)$;
- $R_2(x) = x$ if $\|x\| = 1$, so that $R$ is a retraction onto $S(l_1)$.

This obviously shows that $k_0(l_1) \leq 8$.

References


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