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ε-Kronecker and I_0 sets in abelian groups, III: interpolation by measures on small sets

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Abstract. Let U be an open subset of a locally compact abelian group G and let E be a subset of its dual group Γ. We say E is $I_0(U)$ if every bounded sequence indexed by E can be interpolated by the Fourier transform of a discrete measure supported on U. We show that if $E \cdot \Delta$ is I_0 for all finite subsets Δ of a torsion-free Γ, then for each open $U \subset G$ there exists a finite set $F \subset E$ such that $E \setminus F$ is $I_0(U)$. When G is connected, F can be taken to be empty. We obtain a much stronger form of that for Hadamard sets and ε-Kronecker sets, and a slightly weaker general form when Γ has torsion. This extends previously known results for Sidon, ε-Kronecker, and Hadamard sets.

1. Introduction. This paper is a continuation of [4]. We refer the reader to that paper for further background and motivation. We give here only the absolute essentials from [4].

G denotes a locally compact abelian group and Γ its dual group. Group operations will be written multiplicatively, except for explicit elements of $\mathbb{N}$ (non-negative integers), $\mathbb{R}$ (real line), $\mathbb{T}$ (circle group), and $\mathbb{Z}$ (integers). The duality will be denoted $\langle x, \gamma\rangle$, $x \in G$, $\gamma \in \Gamma$ or by e^{ixy} in the case of the three classical groups.

Recall that an increasing sequence of real numbers $1 \le n_1 < n_2 < \cdots$ is an *Hadamard set* if there is a real number $q > 1$ such that $q \le n_{j+1}/n_j$ for all $j \ge 1$. We will call any such number q a “ratio” of the set.

We now define ε-Kronecker sets, I_0 sets, and the other classes of sets needed here. These are elaborations of the definitions given in [4].

DEFINITION. (1) Let $U \subseteq G$ and $E \subseteq \Gamma$. We say E is $I_0(U)$ if there is a constant K such that for every bounded, continuous function ϕ on E there

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is a discrete measure μ supported on U such that ϕ is the restriction of the Fourier–Stieltjes transform of μ to E and $\|\mu\| \le K\|\phi\|_{\ell^\infty(E)}$. The infimum of all such K is called the *interpolation constant* and is denoted $K(E,U)$. The set is called I_0 if it is $I_0(G)$.

(2) If E is $I_0(U)$, we say E has *bounded* $I_0(U)$ *constants* if the infimum of the interpolation constants $K(E\setminus F, U)$ as F varies over the finite subsets of E is bounded over all open subsets U of G ([1]).

(3) Let $\varepsilon > 0$ be given. A set E is ε-*Kronecker*(U) if for every continuous $\phi : E \to \mathbb{T}$ there exists $x \in U$ such that

$$|\langle x, \gamma\rangle - \phi(\gamma)| < \varepsilon \quad \text{for all } \gamma \in E. \tag{1.0.1}$$

We say that a set E is ε-*Kronecker* if it is ε-Kronecker(G).

Summary of results. This paper addresses the question: if G is connected, is every I_0 set $I_0(U)$ for all open $U \subset G$? We show that some classes of I_0 sets are $I_0(U)$ for all open $U \subset G$. That is, these I_0 sets have the required interpolation using only discrete measures concentrated in arbitrarily small open sets U.

These classes include ε-Kronecker sets in the duals of compact connected groups (with bounded constants—Corollary 3.3), Hadamard sets (bounded constants—Theorem 3.10), and I_0 sets in the duals of compact connected groups having the property that $E\cdot\Delta$ is I_0 for all finite sets Δ (Theorem 4.4).

Our conclusion about the $I_0(U)$ property for general I_0 sets is rather unsatisfactory since Example 5.1 and Proposition 5.2 show that the assumption that $E \cdot \Delta$ is I_0 for all finite sets Δ is not necessary. At the present time we are unable to decide if $I_0 \Rightarrow I_0(U)$ for all open $U \subset G$, even in the case of $\Gamma = \mathbb{Z}$.

Background. The results here are motivated by work of Déchamps-Gondim [1] and Méla [15]. Déchamps-Gondim showed that every Sidon set E in a discrete torsion-free group is a *Sidon*(U) *set* ("U associated with E" in [1]): every bounded function on E is the restriction of the Fourier–Stieltjes transform of a (not necessarily discrete) measure supported on U. Obviously, $I_0(U)$ sets in discrete groups are Sidon(U). Much earlier, Weiss [20, Theorem 2] had proved that Hadamard sets are Sidon(U) (in a quantitative way) for all open intervals U.

Méla [15, Théorème 5] showed that Hadamard sets in $\mathbb{R}$ are Sidon(U) for all open U with bounded constants and that each I_0 set in $\mathbb{R}$ is a finite union of Sidon(U) sets each with bounded constants [15, Théorème 6] (Méla used the terminology "Sidon set of the first type").

([1]) In the definition of bounded constants, we delete finite sets F because otherwise we would never have "bounded constants": for fixed $0 \ne m \in \mathbb{Z}$, the interpolation constants from $M_d((-\delta,\delta))$ to $\ell^\infty(\{0,m\})$ increase as $\delta \to 0^+$ (see [15, p. 31]).

See [1, 4, 5, 12, 14, 15, 17] and their references for further background.

At several points we have the hypotheses that G is connected and locally connected. For a characterization of the connected, compact abelian groups that are locally connected, see [8, Theorem 8.36].

2. Notation and preliminary results

2.1. *Notation for this paper.* Let $U \subset G$, $E \subset \Gamma$, $N \geq 1$ and $\varepsilon > 0$. We write

$$D(N,U) = \Big\{ \sum_{j=1}^{N} a_j \delta_{x_j} : |a_j| \leq 1, x_j \in U,\, 1 \leq j \leq N \Big\} \tag{2.1.1}$$

and

$$\begin{aligned} &\mathrm{AP}(E,N,U,\varepsilon) \\ &\quad = \{\phi \in \mathbb{T}^E : \exists \mu \in D(N,U) \text{ with } \|\phi - \widehat{\mu}_{|E}\|_{\ell^\infty(E)} \leq \varepsilon\}. \end{aligned} \tag{2.1.2}$$

Then $\mathrm{AP}(E,N,U,\varepsilon) \subset \mathrm{AP}(E,N+1,U,\varepsilon)$ for all N.

Given $E \subset \Gamma$, we write $\ell^\infty(E)$ for the set of bounded functions on a set E and $B(\ell^\infty(E))$ for the unit ball of $\ell^\infty(E)$. If the mapping $\mu \mapsto \widehat{\mu}_{|E}$ sends $M_d(U)$ onto $\ell^\infty(E)$, then there is an *interpolation constant* C, the infimum of the numbers K such that for each $\phi \in B(\ell^\infty(E))$ there exists $\mu \in M_d(U)$ with $\|\mu\| \leq K$ and $\widehat{\mu}_{|E} = \phi$.

The *Bohr compactification* of Γ will be denoted by $\overline{\Gamma}$. The *Bohr topology* on $E \subset \Gamma$ is the restriction to E of the topology of $\overline{\Gamma}$. That is the topology of pointwise convergence on G.

If $f \in L^1(\Gamma)$, its Fourier transform is denoted by $\widehat{f}$ and $\|\widehat{f}\|_A = \|f\|_1$, as usual. Similarly, $\widehat{\mu}$ denotes the Fourier–Stieltjes transform of a measure.

2.2. *General preliminary results.* An easy observation is this:

LEMMA 2.1. *Let G be a locally compact abelian group, $U \subset G$ be open, and $E \subset \Gamma$. If E is $I_0(U)$, then*

(1) *$E \cdot \gamma$ is $I_0(U)$ for every $\gamma \in \Gamma$, and*
(2) *E is $I_0(U \cdot x)$ for every $x \in G$.*

If $E \subset \Gamma$ is ε-Kronecker(U) then

(3) *E is ε-Kronecker$(U \cdot x)$ for every $x \in G$.*

With the notation of the preceding subsection, we give the following useful result, adapted from [6, Prop. 2.1].

PROPOSITION 2.2. *Let G be a locally compact group, U a σ-compact subset of G and $E \subset \Gamma$. Consider the following properties:*

(1) *E is $I_0(U)$.*
(2) *For every $\phi \in B(\ell^\infty(E))$ there exists $\mu \in M_d(U)$ such that $\widehat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$.*
(3) *There exists some $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) and integer N such that for every $\phi \in B(\ell^\infty(E))$ there exists $\mu \in M_d(U)$ with $\|\mu\| \le N$ and $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*
(4) *There exists some $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) such that for every $\phi \in B(\ell^\infty(E))$ there exists $\mu \in M_d(U)$ with $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*
(5) *There exists some $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) and integer N such that for every $\phi \in B(\ell^\infty(E))$ there exists $\mu \in D(N, U^2)$ with $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.*

Then properties (1)–(3) are equivalent. Property (1) implies (4) implies (5) implies E is $I_0(U^2)$.

Finally,

(6) *Under the conditions of property (3) one can show $K(E, U) \le N/(1-\varepsilon)$.*

We will need the following results.

LEMMA 2.3 ([10]). *Let A, B be subsets of the connected, locally compact group G. Then the lower Haar measure m_* on G satisfies $m_*(A \cdot B) \ge m_*(A) + m_*(B)$, unless $m(G) \le m_*(A \cdot B)$ in which case $A \cdot B = G$.*

LEMMA 2.4. *Let G be a compact connected group and U a Borel subset of G of non-zero Haar measure. Then there exists $N > 0$ such that $G = U^N$.*

Proof. Because $m(U) > 0$, U^2 has non-empty interior V (see [7, 20.17]). By Lemma 2.3, there exists an integer M such that $m(V^M) > 1/2$. Then $m(V^{2M}) = 1$. That means V^{2M} is dense in G. Of course, V^{2M+1} contains the closure of V^{2M}, so $V^{2M+1} = G$. Hence, $N = 4M + 1$ will do. ∎

LEMMA 2.5. *Let G be a locally compact abelian group. Let $E \subset \Gamma$, $U \subset G$, $N \ge 1$ and $\varepsilon > 0$. Then $\mathrm{AP}(E, N, U, \varepsilon)$ is closed in $\mathbb{T}^E$ if U is compact and is an F_σ in $\mathbb{T}^E$ if U is σ-compact.*

Proof. The first assertion is [16, Appendix, Lemma 6]. The second follows from the first by writing

$$\mathrm{AP}(E, N, U, \varepsilon) = \bigcup_n \mathrm{AP}(E, N, U_n, \varepsilon),$$

where the U_n are compact subsets of U with $U_n \subset U_{n+1}$ for all n and $U = \bigcup U_n$. ∎

Proof of Proposition 2.2. The equivalence of (1)–(2) is the closed graph theorem.

Clearly (1) implies (3) and (4).

The implication (3)⇒(1) is an iteration argument as follows: let $\phi \in B(\ell^\infty(E))$. Choose $\mu_1 \in M_d(U)$ such that $\|\phi(\gamma) - \widehat{\mu}_1(\gamma)\|_\infty < \varepsilon$ for all $\gamma \in E$ and $\|\mu_1\| \le N$. Now choose $\mu_2 \in M_d(U)$ such that

$$|\phi(\gamma) - \widehat{\mu}_1(\gamma) - \widehat{\mu}_2(\gamma)| < \varepsilon^2 \quad \text{for all } \gamma \in E$$

and $\|\mu_2\| \le \varepsilon N$. Continuing in this way, we find $\mu = \sum \mu_j \in M_d(U)$ such that $\|\mu\| \le N/(1-\varepsilon)$ and $\phi(\gamma) = \widehat{\mu}(\gamma)$, $\gamma \in E$.

The claim (6) about the interpolation constant follows from the proof.

This iteration argument, applied finitely many times, also shows the equivalence of the statements "there exists some $0 < \varepsilon < 1$" and "for every $0 < \varepsilon < 1$", at the cost of a larger constant N.

In (5) the condition $\mu \in D(N, U^2)$ ensures $\|\mu\| \le N$; thus (5) implies that E is $I_0(U^2)$ (via (3)).

To prove (4) implies (5), we appeal to an argument similar to that given in [16, pp. 127 ff.]: Suppose that (4) holds for ε. Fix $\varepsilon_1 > 0$. We claim that we may assume that $\varepsilon < \varepsilon_1/4$. Choose any $\phi \in \mathbb{T}^E$. Then the hypothesis of (4) says that we can find $\mu_1 \in M_d(U)$ such that $\|\phi - \widehat{\mu}_{1|E}\|_\infty < \varepsilon$. Let $\phi_2 = \phi - \widehat{\mu}_{1|E}$. Then there exists $\mu_2 \in M_d(U)$ such that $|\phi_2(\gamma) - \widehat{\mu}_2(\gamma)| < \varepsilon^2$ for all $\gamma \in E$, so

$$|\phi(\gamma) - (\widehat{\mu}_1 + \widehat{\mu}_2)(\gamma)| < \varepsilon^2$$

for all $\gamma \in E$. Iterating in this way, we eventually find a $\nu \in M_d(U)$ such that

$$|\phi(\gamma) - \widehat{\nu}(\gamma)| < \varepsilon^k < \varepsilon_1/4 \tag{2.2.1}$$

for all $\gamma \in E$. This can be done for all $\phi \in \mathbb{T}^E$. Hence, we may assume that $\varepsilon < \varepsilon_1/4$.

By our hypothesis, $\bigcup_N \mathrm{AP}(E, N, U, \varepsilon) = \mathbb{T}^E$. By Lemma 2.5, the sets $\mathrm{AP}(E, N, U, \varepsilon)$ are Borel sets, so there exists an integer M such that $\mathrm{AP}(E, M, U, \varepsilon)$ has $\mathbb{T}^E$-Haar measure greater than 1/2. Therefore

$$\mathrm{AP}(E, M, U, \varepsilon) \cdot \mathrm{AP}(E, M, U, \varepsilon) = \mathbb{T}^E$$

by Lemma 2.3, that is, for each $\phi \in \mathbb{T}^E$ there exists $\phi_1, \phi_2 \in \mathrm{AP}(E, M, U, \varepsilon)$ such that $\phi = \phi_1\phi_2$. Let $\mu_j \in D(M, U)$ be such that

$$|\phi_j(\gamma) - \widehat{\mu}_j(\gamma)| \le \varepsilon \quad \text{for } j = 1, 2 \text{ and all } \gamma \in E.$$

Then

$$|\phi - (\widehat{\mu}_1\widehat{\mu}_2)| \le \varepsilon + \varepsilon(1+\varepsilon) < 3\varepsilon < \varepsilon_1 \quad \text{for all } \gamma \in E. \tag{2.2.2}$$

Hence, $\mathrm{AP}(E, M^2, U^2, 3\varepsilon) = \mathbb{T}^E$. Since every element of $B(\ell^\infty(E))$ is the average of two elements of $\mathbb{T}^E$, (5) holds with $N = 2M^2$, and ε_1. ■

REMARK 2.6. Item (3) implies that ε-Kronecker sets with $0 < \varepsilon < 1$ are I_0 (see [19]), but [4, Prop. 2·8] shows that the inequality can be weakened to $\varepsilon < \sqrt{2}$.

2.3. *Preliminaries on $I_0(U)$ sets.* The following was proved for Sidon sets in discrete groups Γ in [1, Lemme 6.5].

LEMMA 2.7. *Let G be a connected locally compact abelian group with dual group Γ. Let $U \subset G$ be open. If $E \subset \Gamma$ is $I_0(U)$, $\lambda \in \Gamma$ and $W \subset G$ any neighbourhood of the identity, then $E \cup \{\lambda\}$ is $I_0(U \cdot W)$.*

Proof. By Lemma 2.1(1) we may assume that λ is the identity $\mathbf{1}$ of Γ and U is a neighbourhood of $\mathbf{1}$.

Let V be a neighbourhood of the identity of G such that $V \cdot V^{-1} \subset W$.

Since E is an I_0 set, [5, Cor. 4·2] tells us that E does not cluster in the Bohr topology at any element of Γ and that $E \cup \{\mathbf{1}\}$ is also an I_0 set. Therefore, there exists a finitely supported measure $\mu \in M_d(G)$ such that $|\widehat{\mu}| < 1/100$ on E and $\widehat{\mu}(\mathbf{1}) = 1$. Suppose $\mu = \sum_{j=1}^{J} a_j \delta_{x_j}$. Then $\widehat{\mu}(\mathbf{1}) = \sum_j a_j = 1$.

Since G is connected and V is an open neighbourhood of the identity, $\bigcup_{N=1}^{\infty} V^N = G$. Therefore there exists an integer N such that $x_j \in V^N$ for all $1 \le j \le J$. That is, there exist $w_{j,k} \in V$ such that $x_j = \prod_{k=1}^{N} w_{j,k}$ for all j. Now, a continuity argument shows that for some $\varepsilon > 0$,

$$\sum_{j=1}^{J}\sum_{k=1}^{N} |\langle w_{j,k}, \gamma\rangle - \langle w_{j,k}, \mathbf{1}\rangle| \ \ge \varepsilon \quad \text{for } \gamma \in E. \tag{2.3.1}$$

Indeed, if (2.3.1) failed for all $\varepsilon > 0$, then μ could not be uniformly away from 1 on E. Thus, for $\gamma \in E$, (2.3.1) and the Cauchy–Schwarz inequality imply that

$$\Big[\sum_{j=1}^{J}\sum_{k=1}^{N} (\delta_{w_{j,k}} - \delta_{\mathbf{1}}) * (\delta_{w_{j,k}^{-1}} - \delta_{\mathbf{1}})\Big]^{\widehat{\ }}(\gamma) \ge \frac{\varepsilon^2}{JN}. \tag{2.3.2}$$

Let $\omega = \sum_{j=1}^{J}\sum_{k=1}^{N} (\delta_{w_{j,k}} - \delta_{\mathbf{1}}) * (\delta_{w_{j,k}^{-1}} - \delta_{\mathbf{1}})$. Then ω has Fourier–Stieltjes transform 0 at the identity of Γ, and transform at least ε^2/JN on E. Of course, ω is supported on $V \cdot V^{-1} \subset W$.

Since E is $I_0(U)$, there exists $\nu \in M_d(U)$ such that $\widehat{\nu} = 1/\widehat{\omega}$ on E. Therefore, $\tau = \omega * \nu \in M_d(U \cdot W)$, and

$$\widehat{\tau} = \begin{cases} 1 & \text{on } E, \\ 0 & \text{at } \mathbf{1}. \end{cases} \tag{2.3.3}$$

Hence, we can interpolate on $E \cup \{\mathbf{1}\}$ with discrete measures from $U \cdot W$. ∎

COROLLARY 2.8. *Let G be a connected, locally compact abelian group with dual group $\widehat{\Gamma}$. Let $U \subset G$. If $E \subset \Gamma$ is $I_0(U)$, F is a finite subset of Γ and $W \subset G$ any neighbourhood of the identity, then $E \cup F$ is $I_0(U \cdot W)$.*

Proof. This is a straightforward induction on the number of elements of F, using Lemma 2.7, where the V in the proof of Lemma 2.7 is replaced with a neighbourhood V_1 such that $(V_1 \cdot V_1^{-1})^{\#F} \subset W$. ∎

LEMMA 2.9. *Let G be a connected locally compact abelian group. Suppose that $E \subset \Gamma$ is such that for every open subset $U \subset G$, there exists a finite subset $F_U \subset E$ with $E \setminus F_U \in I_0(U)$. Then $E \in I_0(U)$ for all open $U \subset G$.*

Proof. Fix the set U. We may assume that U is a neighbourhood of the identity. Choose $U_1 \subset U$ and a neighbourhood W of the identity $\mathbf{1}_G$ such that $U_1 \cdot W \subset U$. The assumptions tell us that there is a finite set $F \subset E$ such that $E \setminus F \in I_0(U_1)$. By Corollary 2.8, $E = (E \setminus F) \cup F \in I_0(U_1 \cdot W)$. Hence, $E \in I_0(U)$. ∎

COROLLARY 2.10. *Let G be a connected locally compact abelian group. Let E be a finite subset of its dual Γ. Then E is $I_0(U)$ for all open sets $U \subset G$.*

In §3 we will need a strengthening of [4, proof of Prop. 2·8]. The proof of the version here follows that of [4], with some changes necessitated by the substitution of the set U for the group G.

PROPOSITION 2.11. *Let Γ be a locally compact abelian group. Let $0 < \varepsilon < \sqrt{2}$. Let $E \subset \Gamma$ be an ε-Kronecker(U) set. Then $E \cup E^{-1}$ is $I_0(U \cup U^{-1})$. The $I_0(U \cup U^{-1})$ constant $K(E, U \cup U^{-1})$ depends only on ε and not on U.*

Proof. Let $\phi_1 : E \to [-1, 1]$ be given. Let

$$\psi(\gamma) = \begin{cases} 1 & \text{if } \phi_1(\gamma) \geq 0, \\ -1 & \text{if } \phi_1(\gamma) < 0. \end{cases}$$

Let $x \in U$ be such that $|\widehat{\delta}_x(\gamma) - \psi(\gamma)| < \varepsilon$ on E. Let $\mu = \frac{1}{4}(\delta_x + \delta_{x^{-1}})$. Then $\widehat{\mu}(\gamma) \in \mathbb{R}$; in fact, there exists $0 < \delta < 1/2$ such that

$$\widehat{\mu}(\gamma) \in [\delta, 1/2] \quad \text{if } \psi(\gamma) = 1 \quad \text{and} \quad \widehat{\mu}(\gamma) \in [-1/2, -\delta] \quad \text{if } \psi(\gamma) = -1.$$

Hence,

$$|\widehat{\mu} - \phi_1| \leq \max(1 - \delta, 1/2) = \lambda < 1 \quad \text{on } E. \tag{2.3.4}$$

Iterating, we find a real $\nu_1 \in M_d(U)$ with $\widehat{\nu}_1 = \phi_1$ on E. We note that

$$\widehat{\nu}_1(\gamma) = \widehat{\nu}_1(\gamma^{-1}) \tag{2.3.5}$$

for all γ. Similarly, if $\phi_2 : E \to [-1, 1]$, by taking sums of measures of the form $(i/4)(\delta_y - \delta_{y^{-1}})$, we can find a purely imaginary $\nu_2 \in M_d(U)$ such that

$\widehat{\nu}_2(\gamma) = \phi_2$ on E and

$$\widehat{\nu_2(\gamma)} = -\widehat{\nu_2(\gamma^{-1})} \tag{2.3.6}$$

for all γ. Functions on $E \cup E^{-1}$ satisfying (2.3.5) are "Hermitian", and every function satisfying (2.3.6) is "anti-Hermitian". Of course, every $\phi : E \cup -E \to \{z \in \mathbb{C} : |z| \leq 1\}$ can be written $\phi = \phi_1 + \phi_2$, where ϕ_1 is Hermitian and ϕ_2 is anti-Hermitian. Thus, every bounded real-valued ϕ on E is the Fourier–Stieltjes transform of a discrete measure concentrated on $U \cup U^{-1}$. Repeating this for bounded imaginary-valued ϕ, we see that $E \cup -E$ is indeed $I_0(U \cup U^{-1})$.

The statement about the interpolation constant K follows immediately from the above: $K(E \cup E^{-1}, (U \cup U^{-1})) \leq 8/(1 - \lambda)$, where λ is given by (2.3.4). ∎

3. Sets that are $I_0(U)$ with bounded constants

3.1. *Bounded constants and unions of translates.* There is a connection between being $I_0(U)$ with bounded constants and having $E \cdot \Delta$ being I_0. Here is one direction. The other direction is in Theorem 4.4. Unfortunately, there is not an equivalence; see Example 5.1 and Proposition 5.2.

THEOREM 3.1. *Let G be a connected compact abelian group and $E \subset \Gamma$. Suppose that E is $I_0(U)$ for all open U with bounded constants. Then $E \cdot \Delta$ is I_0 for all finite sets Δ.*

Proof. Let K be a bound for the $I_0(U)$ constants for E. Fix $\gamma \neq \lambda \in \Gamma$. It will suffice to show that $E \cdot \gamma$ and $E \cdot \lambda$ have disjoint closures for all $\gamma \neq \lambda \in \Gamma$.

Let $x \in G$ be such that $z = \langle x, \lambda\gamma^{-1}\rangle \neq 1$. Let U be a neighbourhood of x such that

$$|\langle u, \lambda\gamma^{-1}\rangle - z| < \frac{|1 - z|}{10K}, \quad u \in U.$$

Let F be a finite subset of E such that for some $\nu \in M_d(U)$ we have $\|\nu\| \leq 2K$ and $\widehat{\nu} = 1$ on $(E \setminus F) \cdot \gamma$. Such a ν exists by our hypothesis and both parts of Lemma 2.1 (we have to translate both U and E). Then

$$|\widehat{\nu}(\omega\lambda) - z\widehat{\nu}(\omega\gamma)| < \|\nu\|\,|1 - z|/(10K) < \frac{|1 - z|}{5} \quad \text{for all } \omega \in E \setminus F.$$

Hence, $|\widehat{\nu} - z| < |1 - z|/5$ on $(E \setminus F) \cdot \lambda$, so $(E \setminus F) \cdot \lambda$ and $(E \setminus F) \cdot \gamma$ have disjoint closures in $\overline{\Gamma}$. Now apply [4, Lemma 2·1] and [5, Cor. 4·2]. ∎

3.2. *ε-Kronecker sets are ε-Kronecker(U)*

THEOREM 3.2. *Let $0 < \varepsilon' < \varepsilon < 2$ and let E be a discrete ε'-Kronecker subset of the locally compact abelian group Γ whose dual group is σ-compact.*

Then for each open $U \subset G$ there exists a finite set F such that $E \setminus F$ is ε-Kronecker(U).

We state and prove some corollaries before proving Theorem 3.2.

COROLLARY 3.3. *Let $0 < \varepsilon < \sqrt{2}$. Suppose that G is connected. Let E be an ε-Kronecker set. Then $E \cup E^{-1}$ is $I_0(U)$ for all open $U \subset G$ with bounded constants.*

Proof. This is immediate from Lemma 2.9, Proposition 2.11, and Theorem 3.2. ∎

The next corollary follows from Theorems 3.2 and 3.1.

COROLLARY 3.4. *Let $0 < \varepsilon < \sqrt{2}$. Suppose that G is connected. Let E be an ε-Kronecker set. Then $(E \cup E^{-1}) \cdot \Delta$ is I_0 for all finite sets $\Delta \subset \Gamma$.*

Proof of Theorem 3.2. Let $U \subset G$ be open. For each precompact $V \subset G$, let

$$\Phi(V) = \{\phi : E \to \mathbb{T} : \exists x \in V \text{ such that } |\phi(\gamma) - \langle \gamma, x\rangle| \le \varepsilon' \ \forall \gamma \in E\}.$$

We give $\mathbb{T}^E$ the product topology. Then the closure of $\Phi(V)$ in $\mathbb{T}^E$ is $\Phi(\overline{V})$. Indeed, suppose that $\phi_\alpha \in \Phi(V)$ and that $x_\alpha \in V$ satisfy $|\phi_\alpha(\gamma) - \langle \gamma, x_\alpha\rangle| \le \varepsilon'$ for all $\gamma \in E$. Suppose also that $\phi_\alpha \to \phi$ pointwise on E (this being the topology on $\mathbb{T}^E$). Let $x \in G$ be an accumulation point of the x_α. Then clearly ϕ is in $\Phi(\overline{V})$.

We now assume that V is compact, has non-empty interior, and $V \subset U$. Then the σ-compactness of G implies there exist a countable number of translates $V \cdot x_j$ such that $\bigcup_j V \cdot x_j = G$. As E is ε'-Kronecker, $\bigcup \Phi(V \cdot x_j) = \mathbb{T}^E$. Because $\mathbb{T}^E$ is a compact Hausdorff space, the Baire category theorem applies: there exists j such that $\Phi(V \cdot x_j)$ has interior in $\mathbb{T}^E$.

That means $\Phi(V \cdot x_j)$ contains a set of the form $(z_1, \ldots, z_n) \times \mathbb{T}^{E \setminus F}$ for some finite set F. In particular, for every $\phi : (E \setminus F) \to \mathbb{T}$ there exists $x \in V \cdot x_j \subset U \cdot x_j$ such that $|\phi(\gamma) - \langle \gamma, x\rangle| \le \varepsilon'$ for all $\gamma \in E \setminus F$. Because $\varepsilon' < \varepsilon$, $E \setminus F$ is ε-Kronecker$(V \cdot x_j)$. By Lemma 2.1(3), $E \setminus F$ is ε-Kronecker(V), and therefore ε-Kronecker(U). ∎

REMARK 3.5. A slightly sharper version of Corollary 3.4 appears in [4, Theorem 3·1(2)]: suppose that $0 < \varepsilon < \sqrt{2}$, that E is an ε-Kronecker subset of the discrete abelian group Γ, and $\gamma, \lambda \in \Gamma$, $\gamma \ne \lambda$. Then $\overline{E \cdot \gamma} \cap \overline{E \cdot \lambda} = E \cdot \gamma \cap E \cdot \lambda$, and this set is finite.

COROLLARY 3.6. *Let G be compact, locally connected, and connected. Then Γ contains an infinite set that is $I_0(U)$ for all open U with bounded constants.*

Proof. Every such group contains an infinite $\frac{1}{2}$-Kronecker set [2, Lemma 3.2]. ∎

We say that a subset X of a Banach space Y is *w-ε-dense* in Y if for every $y \in Y$ there exists $x \in X$ such that

$$\|x - y\| \leq \varepsilon. \tag{3.2.1}$$

It is obvious that if E is Hadamard and $m \in \mathbb{N}$, $m \neq 0$, then there exists a finite set Δ (of cardinality at most two) such that $(E \setminus \Delta) \cup \{m\}$ is Hadamard with the same ratio as E.

We have the following analogue for ε-Kronecker sets.

Corollary 3.7. *Let $\varepsilon, \delta > 0$. Let $E \subset \Gamma$ be an ε-Kronecker set, where G is σ-compact. Suppose that $\gamma \in \Gamma$ is such that $\langle G, \gamma\rangle$ is w-δ-dense in $\mathbb{T}$. Then there exists a finite set $\Delta \subset \Gamma$ such that $(E \setminus \Delta) \cup \{\gamma\}$ is ε'-Kronecker for all $\varepsilon' > \varepsilon + \delta$.*

Proof. Let $\tau = \frac{1}{4}(\varepsilon' - \varepsilon - \delta)$. Choose a finite subset $X \subset G$ such that $\langle X, \gamma\rangle$ is $\tau + \delta$-dense in T. For each $x \in X$, choose a neighbourhood U_x of x such that $|\langle xu^{-1}, \gamma\rangle - 1| < \tau$ for all $u \in U_x$. For each $x \in X$, there exists a finite subset Δ_x such that $E \setminus \Delta_x$ is ε'-Kronecker(U_x), by Theorem 3.2.

Let $\Delta = \bigcup_{x \in X} \Delta_x$. Then $(E \setminus \Delta) \cup \{\gamma\}$ has the required properties. ∎

Corollary 3.8. *Let $0 < \varepsilon < \varepsilon'$. Let $E \subset \Gamma$ be an ε-Kronecker set, G be σ-compact, and suppose that F is a finite ε-Kronecker set. Then there exists a finite set Δ such that $(E \setminus \Delta) \cup F$ is ε'-Kronecker.*

We remark that we may perturb an ε-Kronecker set and still have such a set (less an initial segment and with slightly larger ε), rather like topological Sidon sets [1].

Corollary 3.9. *Let G be a σ-compact abelian group. Suppose $0 \leq \varepsilon < \varepsilon_1 < 2$, that $E \subset \Gamma$ is a discrete ε-Kronecker set, $\Delta \subset \Gamma$ is finite, and $\phi : E \to \Delta$ is any function. Then there exists a finite set $F \subset E$ such that $E' = \{\gamma\phi(\gamma) : \gamma \in E \setminus F\}$ is ε_1-Kronecker.*

Proof. Let $U \subset G$ be an open neighbourhood of the identity such that $|\langle u, \delta\rangle - 1| < \varepsilon_1 - \varepsilon$ for all $u \in U$ and $\delta \in \Delta$. Choose a finite F so that $E \setminus F$ is ε-Kronecker(U). Then E' is ε_1-Kronecker. ∎

3.3. *Hadamard sets.* It has been known for a long time that if $E \subset \mathbb{R}$ is an Hadamard set, then $E \cup -E$ is I_0. That was proved originally by Strzelecki [18]; other proofs can be found in [9, 11] and [15]. We note that some authors (e.g., [9, 21]) use "Hadamard" for symmetric subsets E such that $E \cap [0, \infty)$ is Hadamard in our sense.

Theorem 3.10. *Let $E \subset \mathbb{R}$ be Hadamard with ratio q. Then $E \cup -E$ is $I_0(U)$ for every open $U \subset \mathbb{R}$ with bounded constants.*

COROLLARY 3.11. *Let $E \subset \mathbb{N}$ be an Hadamard set. Then $(E \cup -E) + \Delta$ is I_0 for all finite sets $\Delta \subset \mathbb{Z}$.*

In the proof of Theorem 3.10, we will use the sets $D(N, U)$ as defined in (2.1.1).

LEMMA 3.12. *Let G be a σ-compact locally compact abelian group. Let $E \subset \Gamma$ be I_0 and $0 < \varepsilon < 1$. Then there exists $N \geq 1$ and a compact set $U \subset G$ such that $D(N, U)\widehat{\ }$ is w-ε-dense in $B(\ell^\infty(E))$.*

For the term w-ε-dense see formula (3.2.1).

Proof. Let U_n be compact subsets of G such that $U_n \subset U_{n+1}$ for all $n \geq 1$ and $G = \bigcup U_n$. Because U_n is compact, $A_n = \mathrm{AP}(E, n, U_n, \varepsilon/3)$ is closed in $\mathbb{T}^E$. Because E is I_0, $\bigcup A_n = \mathbb{T}^E$. Since $\mathbb{T}^E$ is a compact abelian group and the A_n are increasing, some A_n must have Haar measure greater than 1/2. Then $A_n \cdot A_n = \mathbb{T}^E$ by Lemma 2.3. A computation shows that

$$A_n \cdot A_n \subset \mathrm{AP}(E, n^2, U_n \cdot U_n, 2\varepsilon/3 + \varepsilon^2/9) \subset \mathrm{AP}(E, n^2, U_n \cdot U_n, \varepsilon), \tag{3.3.1}$$

since $0 < \varepsilon < 1$. Since every element of $B(\ell^\infty(E))$ is the average of two elements of $\mathbb{T}^E$, $N = 2n^2$ and $U = U_n^2$ will do. ∎

LEMMA 3.13. *Let $q > 1$ and $\varepsilon < 1$. Then there exists an integer $N \geq 1$ and a compact set $U \subset \mathbb{R}$ such that for every Hadamard set E of ratio q, there exists a finite set $\Delta \subset E$ such that $D(N, U)\widehat{\ }$ is w-ε-dense in $B_\Delta = B(\ell^\infty[(E \setminus \Delta) \cup -(E \setminus \Delta)])$.*

Proof. Suppose otherwise. Then for every $n \geq 1$, there exists an Hadamard set E_n of ratio q such that $D(n, [-n, n])\widehat{\ }$ is not w-ε-dense in B_Δ for all finite sets Δ.

Because $[-n, n]$ is compact, for each finite set Δ, there exists a finite subset $F_{n,\Delta} \subset E_n$ such that $D(n, [-n, n])\widehat{\ }$ is not w-ε-dense in B_Δ. [Otherwise, a limit argument shows that $D(n, [-n, n])\widehat{\ }$ *was* w-ε-dense in B_Δ.]

Let $F_1 \subset (0, \infty)$ be a finite Hadamard set of ratio q such that $D(1, [-1, 1])\widehat{\ }$ is not w-ε-dense in $B(\ell^\infty(F_1 \cup -F_1))$. Assume that $k \geq 1$ and that we have found finite, pairwise disjoint subsets $F_1, \ldots, F_k$ of $\mathbb{R}$ such that $\bigcup_{j=1}^k F_j$ is Hadamard with ratio q and $D(k, [-k, k])\widehat{\ }$ is not w-ε-dense in $B(\ell^\infty(\bigcup_{j=1}^k F_j \cup -F_j))$. Then we can find E_{k+1} as above.

Choose a finite subset $\Delta \subset E_{k+1}$ such that $(E_{k+1} \setminus \Delta) \cup \bigcup_{j=1}^k F_j$ is also Hadamard of ratio q. Now choose a finite subset F_{k+1} of E_{k+1} such that $D(k+1, [-k-1, k+1])\widehat{\ }$ is not w-ε-dense in $B(\ell^\infty(F_{k+1} \cup -F_{k+1}))$. Then $\bigcup_{j=1}^\infty F_j$ is such that for all $1 \leq k$, $D(k, [-k, k])\widehat{\ }$ is not w-ε-dense in

$B(\ell^\infty(\bigcup_{j=1}^\infty F_j \cup -F_j))0$. That contradicts Lemma 3.12 and the I_0 property of the Hadamard set $\bigcup_{j=1}^\infty F_j \cup -F_j$. ∎

Proof of Theorem 3.10. This is now easy. Let U and N be given by Lemma 3.13 for q and let $E \subset \mathbb{R}$ be Hadamard with ratio q. Let $V \subset \mathbb{R}$ be any neighbourhood of 0. Let $t > 1$ be such that $tU \subset V$. Let

$$F = \{t^{-1}\gamma : \gamma \in E \cup -E\}.$$

Then F is symmetric and $F \cap [0,\infty)$ is Hadamard with ratio q so $D(N,U)\widehat{\ }$ is ε-dense in $B(\ell^\infty(F \setminus \Delta))$ for some finite set Δ. Let

$$\Delta' = \{t\gamma : \gamma \in \Delta\}.$$

Then $D(N,V)\widehat{\ }$ is ε-dense in $B(\ell^\infty((E \cup -E) \setminus \Delta'))$. Thus, by Proposition 2.2(5) and Lemma 2.9, $E \cup -E$ is $I_0(U)$ for all open U with a constant that depends only on q. ∎

4. More general $I_0(U)$ sets. We show that if an I_0 set in a discrete abelian group has all finite unions of its translates being I_0 (and another condition necessary when G is not connected), then it is $I_0(U)$ for all open U.

4.1. *Preliminaries for general $I_0(U)$ sets.* The following material is included for completeness.

Definition ([12, 8.2]). Let X_0 be a subgroup of Γ. Then E is *X_0-subtransversal* if each coset of X_0 intersects E in at most one point. The set E is *almost X_0-subtransversal* if there is a finite subset Δ such that $E \setminus \Delta$ is X_0-subtransversal.

Remark 4.1. If the steps of E tend to infinity, then E is almost X_0-subtransversal for every finite subgroup X_0 of Γ. Such is the case when E is an ε-Kronecker subset of a metrizable group and $0 < \varepsilon < \sqrt{2}$ (see [4, Theorem 3·1]). If G is connected, then Γ is torsion-free (and conversely), and therefore every subset of Γ is X_0-subtransversal for the unique finite subgroup of Γ.

Lemma 4.2 ([12, 8.9–8.10]). *Let Γ be a discrete abelian group. If $E \subset \Gamma$ is a Sidon set, then there exists an integer $m \geq 1$ such that for any finite set $\Delta \subset \Gamma$, $E = F \cup \bigcup_i F_i$ where*

(1) *F and the F_i are finite and pairwise disjoint,*
(2) *$\#F_i \leq m$ for all i, and*
(3) *$F_i \cdot F_j^{-1} \cap \Delta = \emptyset$.*

Lemma 4.3 ([12, 8.15]). *Let Γ be a discrete abelian group. Let m be a positive integer and $g \in L^1(G)$, $g \geq 0$. Then there exists $\delta > 0$ and a finite subgroup X_0 of Γ such that if $\chi_1, \ldots, \chi_k$ is an X_0-subtransversal subset of*

Γ with $2 \le k \le m$, then the determinant
$$\det[\widehat{g}(\chi_i\chi_j^{-1})]_{i,j=1}^k \ge \delta.$$

4.2. *Sets that are $I_0(U)$*

THEOREM 4.4. *Suppose that Γ is a discrete abelian group and $E \subset \Gamma$ is an I_0 set such that*

(1) *$E \cdot \Delta$ is I_0 for all finite sets $\Delta \subset \Gamma$, and*
(2) *E is almost X_0-subtransversal for all finite subgroups $X_0 \subset \Gamma$.*

Then for each open set $U \subset G$ there exists a finite set F such that $E \setminus F$ is $I_0(U)$.

Proof. By Lemma 2.1, we may assume that U is a neighbourhood of the identity of G.

Let W be a neighbourhood of the identity of G such that $W^2 \subset U$. We shall show that there is a finite set F such that for every $\phi \in B(\ell^\infty(E \setminus F))$ there exists $\mu \in M_d(W)$ such that

$$|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon/6 \quad \text{for all } \gamma \in E \setminus F. \tag{4.2.1}$$

Then Proposition 2.2(4) implies $E \setminus F$ is $I_0(W^2)$.

Let C_1 be the I_0 constant of E. Let $g \in A(G) \subset L^1(G)$ be such that $g \ge 0$, $\operatorname{Supp} g \subset W$ and $\widehat{g}(\mathbf{1}) = 1$.

Let m be given by Lemma 4.2 for E, which we may apply because I_0 sets are Sidon sets. Let X_0 and δ be given by Lemma 4.3 for E, m and g. Let $C_2 = \max(1, m!/\delta)$.

Let $0 < \varepsilon < 1/2$ be such that $C_1C_2\varepsilon\|g\|_A < 1/24$. Choose a finite symmetric subset $\Delta \subset \Gamma$ such that $\mathbf{1} \in \Delta$ and
$$48C_1\,C_2 \sum_{\chi\in\Gamma\setminus\Delta} |\widehat{g}(\chi)| < 1.$$

Let F_0 be a finite set such that $E\setminus F_0$ is X_0-subtransversal and $(E\setminus F_0)\cap\Delta = \emptyset$. By Lemma 4.2,
$$E \setminus F_0 = F \cup \bigcup F_i,$$
where
$$F, F_i \text{ are finite with } \#F_i \le m \text{ for all } i, \text{ and } F_i \cdot F_j^{-1} \cap \Delta = \emptyset,\ i \ne j.$$

Let $E_0 = F_0 \cup F$, so $E \setminus E_0 = \bigcup F_i$ is X_0-subtransversal and disjoint from Δ.

Let $\phi : E \to \{z : |z| \le 1\}$. It will suffice to find $\omega \in M_d(W)$ such that $|\widehat{\omega} - \phi| \le 1/6$ on $E \setminus E_0$, which easily implies that $E \setminus E_0$ is $I_0(W^2)$.

Fix i, let $F_i = \{\chi_{i,1}, \ldots, \chi_{i,k}\}$, $k = \#F_i \le m$, and consider the linear system

$$\sum_{s=1}^k x_s\widehat{g}(\chi_{i,r}\chi_{i,s}^{-1}) = \phi(\chi_{i,r}), \quad 1 \le r \le k. \tag{4.2.2}$$

If $k = 1$, then (4.2.2) reads $x_1 \widehat{g}(\mathbf{1}) = \phi(\chi_{1,1})$ with

$$x_1 = \phi(\chi_{1,1})/1, \quad \text{so } |x_1| \le 1.$$

If $k \ge 2$ then Lemma 4.3 implies $\det[\widehat{g}(\chi_i \chi_j^{-1})] \ge \delta$. An application of Cramer's rule shows that there is a unique solution to the system (4.2.2) satisfying $|x_s| \le m!/\delta \le C_2$, $1 \le s \le k$. Define $\phi_1(\chi_{i,s}) = x_s$ on F_i for all i. Then, for each i,

$$\sum_s \phi_1(\chi_{i,s}) \widehat{g}(\chi_{i,r}\chi_{i,s}^{-1}) = \phi_1 * \widehat{g}(\chi_{i,r}) = \phi(\chi_{i,r}).$$

Define $\phi_1 = 0$ on E_0. Then $\phi_1 \in \ell^\infty(E)$, $\|\phi_1\|_\infty \le C_2$.

Next, we claim that

$$((F_i \cdot \Delta) \setminus F_i) \cap (E \setminus E_0) = \emptyset. \tag{4.2.3}$$

Indeed, suppose that $\chi \in \chi_0 \cdot \Delta$, $\chi_0 \in F_i$, $\chi \notin F_i$, but $\chi \in E \setminus E_0$. Then $\chi \in F_j$ for some $j \ne i$ and so $(F_j \cdot F_i^{-1}) \cap \Delta \ne \emptyset$, and this is a contradiction, establishing (4.2.3).

Hence, $\bigcup((F_i \cdot \Delta) \setminus F_i) \cap (E \setminus E_0) = \emptyset$. Because $\bigcup_i (F_i \cdot \Delta) \setminus F_i$ and $E \setminus E_0$ are both subsets of $E \cdot \Delta$, which is assumed to be an I_0 set, their closures in $\overline{\Gamma}$ must be disjoint. By the regularity of the topology on $\overline{\Gamma}$, there exists $\mu \in M_d(G)$ such that $|\widehat{\mu}| \le \varepsilon$ on $\bigcup (F_i \cdot \Delta) \setminus F_i$, $|\widehat{\mu} - 1| < \varepsilon$ on $E \setminus E_0$, and $\sup |\widehat{\mu}| \le 2$ everywhere. Also, because E is I_0, there exists $\nu \in M_d(G)$ such that

$$\widehat{\nu} = \phi_1 / \widehat{\mu} \text{ on } E \setminus E_0$$

and

$$\|\nu\| \le C_1 \Big\| \frac{\phi_1}{\widehat{\mu}_{|E\setminus E_0}} \Big\|_\infty \le \frac{C_1 \|\phi_1\|_\infty}{1-\varepsilon} \le 2C_1 \|\phi_1\|_\infty \le 2C_1 C_2.$$

Now consider $\omega = g \cdot (\mu * \nu) \in M_d(W)$. Fix $\chi_0 \in E \setminus E_0$, say $\chi_0 = \chi_{i,r} \in F_i = \{\chi_{i,1}, \ldots, \chi_{i,k}\}$, so i is also fixed. We have already observed that

$$\begin{aligned} \phi(\chi_0) = \phi(\chi_{i,r}) &= \sum_s \phi_1(\chi_{i,s}) \widehat{g}(\chi_{i,r}\chi_{i,s}^{-1}) = \sum_s \widehat{\mu}(\chi_{i,s}) \widehat{\nu}(\chi_{i,s}) \widehat{g}(\chi_0 \chi_{i,s}^{-1}) \\ &= \sum_{\chi \in F_i} \widehat{\mu}(\chi) \widehat{\nu}(\chi) \widehat{g}(\chi_0 \chi^{-1}), \end{aligned}$$

while

$$\widehat{g \cdot \mu * \nu}(\chi_0) = \sum_{\chi \in \Gamma} \widehat{g}(\chi_0 \chi^{-1}) \widehat{\mu}(\chi) \widehat{\nu}(\chi).$$

Thus

$$|\widehat{g \cdot \mu * \nu}(\chi_0) - \phi(\chi_0)| = \Big| \sum_{\chi \in \Gamma \setminus F_i} \widehat{g}(\chi_0 \chi^{-1}) \widehat{\mu}(\chi) \widehat{\nu}(\chi) \Big| \le S_1 + S_2,$$

where

$$S_1 = \Big| \sum_{\chi \in \chi_0 \Delta \setminus F_i} \widehat{g}(\chi_0 \chi^{-1}) \widehat{\mu}(\chi) \widehat{\nu}(\chi) \Big|,$$

$$S_2 = \Big| \sum_{\chi \in \Gamma \setminus (F_i \cup \chi_0 \Delta)} \widehat{g}(\chi_0 \chi^{-1}) \widehat{\mu}(\chi) \widehat{\nu}(\chi) \Big|.$$

Then

$$\begin{aligned} |S_1| &\le \sum_{\chi \in \chi_0 \Delta \setminus F_i} |\widehat{g}(\chi_0 \chi^{-1})| \, |\widehat{\mu}(\chi)| \, \|\nu\| \\ &\le \sum_{\chi \in F_i \cdot \Delta \setminus F_i} |\widehat{g}(\chi_0 \chi^{-1})| \varepsilon \|\nu\| \le \varepsilon \cdot 2C_1 C_2 \|g\|_A \le \frac{1}{12}, \\ |S_2| &\le \sum_{\chi \in \Gamma \setminus (F_i \cup \chi_0 \Delta)} |\widehat{g}(\chi_0 \chi^{-1})| \sup_\lambda |\widehat{\mu}(\lambda)| \, \|\nu\| \le 4C_1 C_2 \sum_{\chi \in \Gamma \setminus \Delta} |\widehat{g}(\chi)| \le \frac{1}{12}. \end{aligned}$$

Therefore,

$$|\phi(\chi_0) - \widehat{g \cdot \mu * \nu}(\chi_0)| \le \frac{1}{6} \quad \text{for all } \chi_0 \in \bigcup F_i = E \setminus E_0.$$

Hence, $E \setminus E_0$ satisfies (4.2.1), as claimed, so $E \setminus E_0$ is $I_0(U)$. ∎

COROLLARY 4.5. *Suppose that G is connected and that $E \subset \Gamma$ is an I_0 set such that $E \cdot \Delta$ is I_0 for all finite sets $\Delta \subset \Gamma$. Then E is $I_0(U)$ for all open sets $U \subset G$.*

Proof. The connectedness of G means all subsets of Γ are X_0-subtransversal for the unique finite subgroup of Γ. Now apply Theorem 4.4. ∎

COROLLARY 4.6. *If G is connected and $0 < \varepsilon < \sqrt{2}$, then any finite union of translates of ε-Kronecker sets is $I_0(U)$ for all open U.*

REMARKS 4.7. (i) If E is $I_0(U)$ for all open U, then E is X_0-subtransversal for all finite subgroups X_0. Indeed, if G_0 is the annihilator of X_0 then G_0 is an open subgroup. If $a, b \in E$ and $ab^{-1} \in X_0$, then a, b coincide on G_0, so cannot be separated by elements of G_0. Hence E is not $I_0(G_0)$.

A similar argument can be made for E almost X_0-subtransversal and $E \setminus F$ being $I_0(U)$.

(ii) An I_0 set can satisfy the finite union of translates condition, yet not be $I_0(U)$ for all U. See Example 5.4.

(iii) Proposition 5.2 (and Example 5.1) shows that hypothesis (1) of Theorem 4.4 is not a necessary condition for $I_0(U)$, but we do not have any ideas on how to weaken that hypothesis.

5. Some examples

5.1. *The set of I_0 sets is not closed under unions of translates.* We have shown that $E \cup E\gamma$ is I_0 when E is either an ε-Kronecker set (Corollary 3.4) or an Hadamard set (Corollary 3.11). That is false for more general I_0 sets, even for sets that are $I_0(U)$ for all open U:

EXAMPLE 5.1. Let $E_1 = \{10^j+2j+1\}$ and $E_2 = \{10^j\}$. Then E_1, E_2 have disjoint closures in $\overline{\mathbb{Z}}$ since $E_1 \subset 2\mathbb{Z}+1$ and $E_2 \subset 2\mathbb{Z}$. Hence $E = E_1 \cup E_2$ is I_0. But $E \cup (E+1)$ is not I_0 because it contains the union $\{10^j+2j+1\} \cup \{10^j+1\}$; see [13, p. 178] or [4, Example 5·1]. Note that E is the union of two ε-Kronecker sets with $\varepsilon < 2\sin(\pi/8)$ and E is not an ε-Kronecker set for any $0 < \varepsilon < 1/2$. Also, $E - E \supset 2\mathbb{N} - 1$.

PROPOSITION 5.2. *The set of Example* 5.1 *is* $I_0(U)$ *for all open sets* $U \subset \mathbb{T}$, *but not with bounded constants.*

Proof. The "not with bounded constants" assertion follows from Theorem 3.1 and the assertion of Example 5.1. Fix an open set $W \subset \mathbb{T}$. Without loss of generality, we may assume that $W = (-\tau, \tau)$ for some $0 < \tau < \pi$, by Lemma 2.1.

Choose N such that $a = \pi/10^N \in (-\tau/4, \tau/4)$. Let $F = E \cap [0, 10^{N+1}]$, so F is a finite subset of E. Then δ_a has Fourier transform

$$\widehat{\delta_a}(m) = \begin{cases} e^{\pi i(10^{j-N}+(2j+1)10^{-N})} & \text{for } m = 10^j + 2j + 1 \in E_1, \\ e^{\pi i 10^{j-N}} & \text{for } m = 10^j \in E_2. \end{cases}$$

In $E \setminus F$,

$$\widehat{\delta_a}(m) = \begin{cases} e^{\pi i(2j+1)10^{-N}} & \text{for } m = 10^j + 2j + 1 \in E_1 \setminus F, \\ e^{2\pi i} & \text{for } m = 10^j \in E_2 \setminus F. \end{cases}$$

Now, $\inf_j |e^{\pi i(2j+1)/10^N} - 1| > 0$. Hence, $\nu = \delta_a - \delta_0$ has transform 0 on $E_2 \setminus F$ and transform bounded away from zero on $E_1 \setminus F$. Because E_1 is $I_0(U)$ for all open U, we can find a discrete measure μ supported on $[-\tau/4, \tau/4]$ such that $\widehat{\mu} = 1/\widehat{\nu}$ on E_1. That is, there exists a discrete measure $\omega = \mu * \nu$ supported on $[-\tau/2, \tau/2]$ such that $\widehat{\omega} = 1$ on $E_1 \setminus F$ and $\widehat{\omega} = 0$ on $E_2 \setminus F$. Because E_1 and E_2 are both $I_0(U)$, it is now easy to show that E is $I_0(W)$. ∎

5.2. *An ε-Kronecker set not $I_0(U)$ for some U.* In disconnected groups, ε-Kronecker does not necessarily imply $I_0(U)$ for all open U. Here is an example.

EXAMPLE 5.3. Let $G = Z_{43} \times \mathbb{T}$, where Z_{43} is the set of 43rd roots of unity. Let $E = \{(1, 1), (0, 1)\} \subset Z_{43} \times \mathbb{Z}$. Then it is easy to see that E is $\frac{1}{5}$-Kronecker. On the other hand, E is not $I_0(\{0\} \times \mathbb{T})$.

5.3. *$E \cdot \Delta$ can be I_0 for all finite Δ but not $I_0(U)$*

EXAMPLE 5.4. Let $G = \mathbb{T} \times F$ where F is a finite group. Let $E' \subset \mathbb{Z}$ be an ε-Kronecker set and put $E = E' \times \{\mathbf{1}\} \cup E' \times \{s\}$, $s \in \widehat{F}$, $s \neq \mathbf{1}_F$. Then $\mathbb{T} \times \{0\}$ is an open subgroup of G and E is not $I_0(\mathbb{T} \times \{0\})$ but $E \cdot \Delta$ is I_0 for all finite sets $\Delta \subset \Gamma = \mathbb{Z} \times \widehat{F}$ since $E \cdot \Delta$ is just a set of the form $E' \cdot \Delta'$ and E' is an ε-Kronecker set in $\mathbb{Z}$. So assumption (1) of Theorem 4.4 is satisfied, but not the conclusion. Of course assumption (2) of Theorem 4.4 fails.

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