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On the Rademacher maximal function

by

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Abstract. This paper studies a new maximal operator introduced by Hytönen, McIntosh and Portal in 2008 for functions taking values in a Banach space. The L^p-boundedness of this operator depends on the range space; certain requirements on type and cotype are present for instance. The original Euclidean definition of the maximal function is generalized to σ-finite measure spaces with filtrations and the L^p-boundedness is shown not to depend on the underlying measure space or the filtration. Martingale techniques are applied to prove that a weak type inequality is sufficient for L^p-boundedness and also to provide a characterization by concave functions.

1. Introduction. The properties of the standard dyadic maximal function

$$Mf(\xi) = \sup_{Q \ni \xi} |\langle f \rangle_Q|, \quad \xi \in \mathbb{R}^n,$$

where $\langle f \rangle_Q$ denotes the average of a locally integrable function f over a dyadic cube Q, are well-known. More precisely, the (sublinear) operator $f \mapsto Mf$ is bounded in L^p for all $p \in (1, \infty]$ and satisfies for all $f \in L^1$ a certain weak type inequality (and is also bounded from the dyadic Hardy space H^1 to L^1). These properties remain unchanged even if one studies functions taking values in a Banach space and replaces absolute values by norms.

In their paper [HMP], Hytönen, McIntosh and Portal needed a new maximal function in order to prove a vector-valued version of Carleson's embedding theorem. Instead of the supremum of (norms of) dyadic averages this maximal function measures their R-bound (see Section 2 for the definition), which in general is not comparable to the supremum. More precisely, they defined the Rademacher maximal function

$$M_R f(\xi) = \mathcal{R}(\langle f \rangle_Q : Q \ni \xi), \quad \xi \in \mathbb{R}^n,$$

for functions f taking values in a Banach space. They proved that the L^p-

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boundedness of $f \mapsto M_R f$ is independent of p in the sense that boundedness for one $p \in (1,\infty)$ implies boundedness for all p in that range, and that for many common range spaces including all UMD function lattices and spaces with type 2, the operator M_R is L^p-bounded. Nevertheless it turned out that the new maximal operator is not bounded for some range spaces, e.g. not for ℓ^1.

The study of the Rademacher maximal operator continues here in a bit more general framework, which was motivated by the need for vector-valued maximal function estimates in the context of non-homogeneous spaces in [H2]. We consider it for operator-valued functions defined on σ-finite measure spaces, where averages are replaced by conditional expectations with respect to filtrations. The boundedness of M_R—the RMF-property (of the range space)—is shown not to depend on these new parameters; instead, it is sufficient to check it for the filtration of dyadic intervals on $[0,1)$ (Theorem 5.1). Here we follow a reduction argument from Maurey [M], originally tailored for the UMD-property. We also show that the RMF-property requires non-trivial type and finite cotype of the Banach spaces involved (Proposition 4.2). The Rademacher maximal function is readily defined for martingales $X = (X_j)_{j=1}^\infty$ of operators by

$$X_R^* = \mathcal{R}(X_j : j \in \mathbb{Z}_+).$$

We will show using ideas from Burkholder [Bur1] that the RMF-property (requiring L^p-boundedness of M_R) is actually equivalent (Theorem 6.7) to the weak type inequality (or the weak RMF-property)

$$\mathbb{P}(X_R^* > \lambda) \lesssim \frac{1}{\lambda}\|X\|_1.$$

Finally, the RMF-property is characterized using concave functions (Theorem 7.3) in the spirit of Burkholder [Bur2].

2. Preliminaries. All *random variables* in Banach spaces (functions from a probability space to the Banach space) are assumed to be $\mathbb{P}$-strongly measurable, by which we mean that they are $\mathbb{P}$-almost everywhere limits of simple functions on the probability space whose measure we denote by $\mathbb{P}$. Their expectation, denoted by $\mathbb{E}$, is given by the Bochner integral. By an L^p-random variable, for $1 \le p < \infty$, we mean a random variable X (in a Banach space) whose pth moment $\mathbb{E}\|X\|^p$ is finite.

Let $(\varepsilon_j)_{j=1}^\infty$ be a sequence of *Rademacher variables*, i.e. independent random variables attaining values -1 and 1 with equal probability $\mathbb{P}(\varepsilon_j = -1) = \mathbb{P}(\varepsilon_j = 1) = 1/2$. By the independence we have $\mathbb{E}(\varepsilon_j\varepsilon_k) = (\mathbb{E}\varepsilon_j)(\mathbb{E}\varepsilon_k) = 0$ whenever $j \ne k$, while (trivially) $\mathbb{E}(\varepsilon_j\varepsilon_k) = 1$ if $j = k$. The equality of a randomized norm and a square sum of norms for vectors $x_1, \dots, x_N$ in a

Hilbert space is thus established by the following calculation:

(2.1)
$$\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|^2 = \mathbb{E}\Big\langle \sum_{j=1}^N \varepsilon_j x_j, \sum_{k=1}^N \varepsilon_k x_k\Big\rangle = \sum_{j,k=1}^N \mathbb{E}(\varepsilon_j\varepsilon_k)\langle x_j, x_k\rangle = \sum_{j=1}^N \|x_j\|^2.$$

The following standard result guarantees the comparability of different randomized norms (see Kahane's book [Kah] for a proof).

THEOREM 2.1 (The Khintchine–Kahane inequality). *For any $1 \le p, q < \infty$, there exists a constant $K_{p,q}$ such that*

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|^p\Big)^{1/p} \le K_{p,q}\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|^q\Big)^{1/q}$$

whenever $x_1, \dots, x_N$ are vectors in a Banach space.

The concepts of type and cotype of a Banach space are intended to measure how far the randomized norms are from ℓ^p sums of norms.

DEFINITION 2.2. A Banach space E is said to have

(i) *type* p for $1 \le p \le 2$ if there exists a constant C such that
$$\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|^2\Big)^{1/2} \le C\Big(\sum_{j=1}^N \|x_j\|^p\Big)^{1/p}$$
for any vectors $x_1, \dots, x_N$ in E, regardless of N;

(ii) *cotype* q for $2 \le q \le \infty$ if there exists a constant C such that
$$\Big(\sum_{j=1}^N \|x_j\|^q\Big)^{1/q} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|^2\Big)^{1/2}$$
for any vectors $x_1, \dots, x_N$ in E, regardless of N. For $q = \infty$ the left hand side in the above inequality is replaced by $\max_{1\le j\le N} \|x_j\|$.

REMARK. A few observations are in order.

(i) As every Banach space has both type 1 and cotype ∞ we say that a Banach space has *non-trivial type* (respectively *finite cotype*) if it has type p for some $p > 1$ (respectively cotype q for some $q < \infty$).

(ii) One can show that L^p-spaces have type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$ when $1 \le p < \infty$. Sequence spaces ℓ^1 and ℓ^∞ are, on the other hand, typical examples of spaces with only trivial type.

(iii) Type and cotype of a Banach space E and its dual E^* are related in a natural way: If E has type p, then E^* has cotype p', where p' is the Hölder conjugate of p.

(iv) The equality (2.1) of randomized norms and square sums of norms in Hilbert spaces means of course that they have both type 2 and cotype 2. A remarkable result of Kwapień's (see the original paper [Kwa], or the new proof by Pisier in [P]) is that a Banach space with both type 2 and cotype 2 is necessarily isomorphic to a Hilbert space.

The geometry of a Banach space can be studied by looking at its finite-dimensional subspaces. We denote by ℓ_N^p, where $p \in [1, \infty]$ and $N \in \mathbb{Z}_+$, the N-dimensional subspace of ℓ^p consisting of sequences for which all but the first N terms are zero. A Banach space E is said to *contain ℓ_N^p's λ-uniformly* for a $\lambda \geq 1$ if there exist for each $N \in \mathbb{Z}_+$ an N-dimensional subspace E_N of E and a bounded isomorphism $\Lambda_N : E_N \to \ell_N^p$ such that $\|\Lambda_N\| \, \|\Lambda_N^{-1}\| \leq \lambda$.

The following theorem of Maurey and Pisier (see [MP] for the original proof, or [DJT, Theorems 13.3 and 14.1]) relates this to the concept of type and cotype:

THEOREM 2.3. *Suppose that E is a Banach space. Then*

(i) *E has a non-trivial type if and only if it does not contain ℓ_N^1's uniformly (i.e. λ-uniformly for some $\lambda \geq 1$).*
(ii) *E has finite cotype if and only if it does not contain ℓ_N^∞'s uniformly.*

PROPOSITION 2.4. *If E^* has non-trivial type, then E has finite cotype.*

Proof. Non-trivial type implies finite cotype for the dual and thus it follows from the assumption that E^{**} has finite cotype. By Theorem 2.3, E^{**} does not contain ℓ_N^∞'s uniformly and the same has to hold for its subspace E. This means that E must have finite cotype. ∎

The proposition above, together with the fact that non-trivial type implies finite cotype, states in other words that if E has only infinite cotype, then both E and E^* have only trivial type.

Evidently, any infinite-dimensional Hilbert space contains ℓ_N^2's 1-uniformly. The following theorem is a variant of Dvoretzky's theorem (see [DJT, Theorems 19.1 and 19.3] or the original paper by Dvoretzky [D]), which says that Banach spaces satisfy almost the same. The definition of K-convexity along with its fundamental properties can likewise be found in [DJT, Chapter 13]. For the purposes of this paper, one can think of K-convexity as the requirement of non-trivial type. Indeed, a Banach space is K-convex if and only if it has non-trivial type ([DJT, Theorem 13.3]). Furthermore, K-convexity is a self-dual property in the sense that a Banach space possesses it if and only if its dual does ([DJT, Corollary 13.7 and Theorem 13.5]).

THEOREM 2.5. *Every infinite-dimensional Banach space contains ℓ_N^2's λ-uniformly for any $\lambda > 1$. If the Banach space is also K-convex, then there*

exists a constant C so that the λ-isomorphic copies of ℓ_N^2's can be chosen to be C-complemented.

We next turn to the study of the type of spaces of operators. Suppose that H and E are Banach spaces. For $y \in E$ and $x^* \in H^*$ we write

$$(y \otimes x^*)x = \langle x, x^* \rangle y, \quad x \in H.$$

Clearly $y \otimes x^* \in \mathcal{L}(H, E)$ and $\|y \otimes x^*\| \leq \|y\|\|x^*\|$. We can also embed H^* and E isometrically into $\mathcal{L}(H, E)$ by fixing respectively a unit vector $y \in E$ or a functional $x^* \in H^*$ with unit norm and writing

$$H^* \simeq y \otimes H^* := \{y \otimes x^* : x^* \in H^*\} \subset \mathcal{L}(H, E),$$
$$E \simeq E \otimes x^* := \{y \otimes x^* : y \in E\} \subset \mathcal{L}(H, E).$$

The following result is most likely well-known but for lack of reference we give a proof:

PROPOSITION 2.6. *If H and E are infinite-dimensional Banach spaces, then $\mathcal{L}(H, E)$ has only trivial type.*

Proof. Suppose first that H is K-convex and let $\lambda > 1$. By Theorem 2.5, both H and E contain ℓ_N^2's λ-uniformly. More precisely, there exist sequences $(H_N)_{N=1}^\infty$ and $(E_N)_{N=1}^\infty$ of subspaces of H and E such that each H_N and E_N is λ-isomorphic to ℓ_N^2. Now, as H is K-convex, we may further assume that for some constant C, each H_N is C-complemented in H so that the projection P_N onto H_N has norm less than or equal to C. We can then embed $\mathcal{L}(H_N, E_N)$ in $\mathcal{L}(H, E)$ by extending an operator $T \in \mathcal{L}(H_N, E_N)$ to $\widetilde{T} = TP_N$ so that $\|\widetilde{T}\| \leq C\|T\|$. Fix an N and denote the isomorphisms from H_N and E_N to ℓ_N^2 by Λ_N^H and Λ_N^E, respectively. Define

$$\Lambda : \mathcal{L}(\ell_N^2, \ell_N^2) \to \mathcal{L}(H_N, E_N)$$

by $\Lambda(T) = (\Lambda_N^E)^{-1} T \Lambda_N^H$. Then $\Lambda^{-1}(S) = \Lambda_N^E S (\Lambda_N^H)^{-1}$ and

$$\|\Lambda\| \, \|\Lambda^{-1}\| \leq \|(\Lambda_N^E)^{-1}\| \, \|\Lambda_N^H\| \, \|\Lambda_N^E\| \, \|(\Lambda_N^H)^{-1}\| \leq \lambda^2.$$

As every sequence in ℓ_N^∞ defines a (diagonal) operator in $\mathcal{L}(\ell_N^2, \ell_N^2)$ with the same operator norm, we have $\ell_N^\infty \hookrightarrow \mathcal{L}(\ell_N^2, \ell_N^2)$ isometrically. Thus $\mathcal{L}(H, E)$ contains ℓ_N^∞'s $C\lambda^2$-uniformly and cannot then by Theorem 2.3 have finite cotype, and thus cannot have non-trivial type either.

Suppose then that H is not K-convex. Then H^* is not K-convex either, has only trivial type and contains ℓ_N^1's uniformly. But $H^* \hookrightarrow \mathcal{L}(H, E)$ isometrically and so $\mathcal{L}(H, E)$ also has only trivial type. ∎

In many questions of vector-valued harmonic analysis the uniform bound of a family of operators has to be replaced by its R-bound (originally defined by Berkson and Gillespie in [BG]).

DEFINITION 2.7. A family $\mathcal{T}$ of operators in $\mathcal{L}(H,E)$ is said to be *R-bounded* if there exists a constant C such that for any $T_1,\dots,T_N \in \mathcal{T}$ and any $x_1,\dots,x_N \in H$, regardless of N, we have

$$\mathbb{E}\Big\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \Big\|^p \leq C^p \mathbb{E}\Big\| \sum_{j=1}^{N} \varepsilon_j x_j \Big\|^p$$

for some $p \in [1,\infty)$. The smallest such constant is denoted by $\mathcal{R}_p(\mathcal{T})$. We denote $\mathcal{R}_2$ by $\mathcal{R}$ for short later on.

Basic properties of R-bounds can be found for instance in [CdPSW]. We wish only to remark that by the Khintchine-Kahane inequality, the R-boundedness of a family does not depend on p, and the constants $\mathcal{R}_p(\mathcal{T})$ are comparable. As a consequence of the inequality $\mathcal{R}_p(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}_p(\mathcal{T})+\mathcal{R}_p(\mathcal{S})$ for any two families $\mathcal{T}$ and $\mathcal{S}$ of operators, every summable sequence of operators is also R-bounded:

$$\mathcal{R}_p(\{T_j\}_{j=1}^{\infty}) \leq \sum_{j=1}^{\infty} \|T_j\|.$$

We will then compare R-boundedness and uniform boundedness. Any R-bounded set is seen to be uniformly bounded:

$$\sup_{T\in\mathcal{T}} \|T\|_{\mathcal{L}(H,E)} \leq \mathcal{R}_p(\mathcal{T})$$

for any $1 \leq p < \infty$. In Hilbert spaces also the converse holds. More generally, the following result is proven by Arendt and Bu in [AB] (while the authors credit Pisier with the proof):

PROPOSITION 2.8. *Suppose that H and E are Banach spaces. The following are equivalent:*

(i) *H has cotype 2 and E has type 2.*
(ii) *Every uniformly bounded family of linear operators in $\mathcal{L}(H,E)$ is R-bounded.*

REMARK. It is clear from the above that if H and E have cotype 2 and type 2, respectively, and if $\mathcal{X} \subset \mathcal{L}(H,E)$ is a Banach space whose norm dominates the operator norm, then all uniformly ($\mathcal{X}$-) bounded sets are also R-bounded.

There are at least two natural ways to use R-boundedness for sets of vectors in E. One can fix a functional x^* with unit norm on a Banach space H and use the embedding $E \simeq E \otimes x^* \subset \mathcal{L}(H,E)$. Doing so, a set $\mathcal{S}$ of vectors in E is R-bounded if there exists a constant C such that

$$\mathbb{E}\Big\| \sum_{j=1}^{N} \varepsilon_j (y_j \otimes x^*) x_j \Big\|^p \leq C^p \mathbb{E}\Big\| \sum_{j=1}^{N} \varepsilon_j x_j \Big\|^p$$

for any choice of vectors $y_1,\dots y_N \in \mathcal{S}$ and $x_1,\dots,x_N \in H$.

In particular, one can choose the scalar field for H. As linear operators from the scalars to E are of the form $\lambda \mapsto \lambda y$ for some $y \in E$, it makes sense to call a set $\mathcal{S}$ of vectors in E R-bounded if there exists a constant C such that

$$\mathbb{E}\Big\| \sum_{j=1}^{N} \varepsilon_j \lambda_j y_j \Big\|^p \le C^p \mathbb{E} \Big| \sum_{j=1}^{N} \varepsilon_j \lambda_j \Big|^p$$

for all vectors $y_1, \dots, y_N$ in $\mathcal{S}$ and all scalars $\lambda_1, \dots, \lambda_N$. These two conditions are easily seen to be equivalent.

3. The Rademacher maximal function. Suppose that H and E are Banach spaces and that $\mathcal{X} \subset \mathcal{L}(H, E)$ is a Banach space whose norm dominates the operator norm. We are mostly interested in the case $\mathcal{X} \simeq E$, i.e. $\mathcal{X} = E \otimes x^*$ for some $x^* \in H^*$ or H is the scalar field. Another typical choice for $\mathcal{X}$ is $\mathcal{L}(H, E)$ itself. Further, when H is a Hilbert space, we can take the so-called γ-radonifying operators for $\mathcal{X}$ (for the definition, see Linde and Pietsch [LP], van Neerven [vN] or [DJT, Chapter 12]). Their natural norm is not equivalent to the operator norm, thus giving us a non-trivial example of an interesting $\mathcal{X}$. Finally, for Hilbert spaces H_1 and H_2 one can consider the Schatten–von Neumann classes $S_p(H_1, H_2)$ with $1 \le p < \infty$ (see [DJT, Chapter 4]).

We will now set out to define the Rademacher maximal function. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space and denote the corresponding Lebesgue–Bochner space of $\mathcal{F}$-measurable $\mathcal{X}$-valued functions by $L^p(\mathcal{F}; \mathcal{X})$ (or $L^p(\mathcal{X})$), $1 \le p \le \infty$. The space of strongly measurable functions f for which $1_A f$ is integrable for every set $A \in \mathcal{F}$ with finite measure, is denoted by $L^1_\sigma(\mathcal{F}; \mathcal{X})$.

If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ such that $(\Omega, \mathcal{G}, \mu)$ is σ-finite, there exists for every function $f \in L^1_\sigma(\mathcal{F}; \mathcal{X})$ a *conditional expectation* $\mathbb{E}(f \,|\, \mathcal{G}) \in L^1_\sigma(\mathcal{G}; \mathcal{X})$ with respect to $\mathcal{G}$ which is the (almost everywhere) unique strongly $\mathcal{G}$-measurable function satisfying

$$\int_A \mathbb{E}(f \,|\, \mathcal{G})\, d\mu = \int_A f \, d\mu$$

for every $A \in \mathcal{G}$ with finite measure. The operator $\mathbb{E}(\cdot \,|\, \mathcal{G})$ is a contractive projection from $L^p(\mathcal{F}; \mathcal{X})$ onto $L^p(\mathcal{G}; \mathcal{X})$ for any $p \in [1, \infty]$. This follows immediately if the vector-valued conditional expectation is constructed as the tensor extension of the scalar-valued conditional expectation, which is a positive operator (see Stein [S] for the scalar-valued case).

Conditional expectations satisfy Jensen's inequality: If $\phi : \mathcal{X} \to \mathbb{R}$ is a convex function and $f \in L^1_\sigma(\mathcal{X})$ is such that $\phi \circ f \in L^1_\sigma$, then

$$\phi \circ \mathbb{E}(f \,|\, \mathcal{G}) \le \mathbb{E}(\phi \circ f \,|\, \mathcal{G})$$

for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ (for which $(\Omega, \mathcal{G}, \mu)$ is σ-finite). The proof in the case of a finite measure space can be found in [H1].

Suppose then that $(\mathcal{F}_j)_{j\in\mathbb{Z}}$ is a *filtration*, that is, an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that each $(\Omega, \mathcal{F}_j, \mu)$ is σ-finite. For a function $f \in L^1_\sigma(\mathcal{F}; \mathcal{X})$, we denote the conditional expectations with respect to this filtration by

$$E_j f := \mathbb{E}(f \,|\, \mathcal{F}_j), \qquad j \in \mathbb{Z}.$$

The standard maximal function (with respect to $(\mathcal{F}_j)_{j\in\mathbb{Z}}$) is given by

$$Mf(\xi) = \sup_{j\in\mathbb{Z}} \|E_j f(\xi)\|, \qquad \xi \in \Omega,$$

for functions f in $L^1_\sigma(\mathcal{X})$. The operator $f \mapsto Mf$ is known to be bounded from $L^p(\mathcal{X})$ to L^p whenever $1 < p \le \infty$, regardless of $\mathcal{X}$.

Definition 3.1. The *Rademacher maximal function* of a function $f \in L^1_\sigma(\mathcal{F}; \mathcal{X})$ is defined by

$$M_R f(\xi) = \mathcal{R}(E_j f(\xi) : j \in \mathbb{Z}), \qquad \xi \in \Omega.$$

Remark. Two immediate observations are listed below.

(i) The μ-measurability of $M_R f$ can be seen by studying it as the supremum over N of the truncated versions

$$M_R^{(N)} f(\xi) = \mathcal{R}(E_j f(\xi) : |j| \le N), \qquad \xi \in \Omega.$$

Indeed, every $M_R^{(N)} f$ is a composition of a strongly μ-measurable function

$$\Omega \to \mathcal{X}^{2N+1} : \ \xi \mapsto (E_j f(\xi))_{j=-N}^{N}$$

and a continuous function (we have assumed that the norm of $\mathcal{X}$ dominates the operator norm)

$$\mathcal{X}^{2N+1} \to \mathbb{R} : \ (T_j)_{j=-N}^{N} \mapsto \mathcal{R}(T_j : |j| \le N).$$

(ii) By the properties of R-bounds we obtain the pointwise relation $Mf \le M_R f$. If H has cotype 2 and E has type 2 it follows from Proposition 2.8 (and the following remark) that $M_R f \lesssim Mf$. This is the case in particular when $H = L^q$ for $1 \le q \le 2$ and $E = L^p$ for $2 \le p < \infty$ over some measure spaces.

Example 3.2. Equip the Euclidean space $\mathbb{R}^n$ with the Borel σ-algebra and the Lebesgue measure. For each integer j, let $\mathcal{D}_j$ denote a partition of $\mathbb{R}^n$ into *dyadic cubes* with edges of length 2^{-j}. Suppose in addition that every cube in $\mathcal{D}_j$ is a union of 2^n cubes in $\mathcal{D}_{j+1}$. For instance, one can take the "standard" dyadic cubes $\mathcal{D}_j = \{2^{-j}([0,1)^n + m) : m \in \mathbb{Z}^n\}$. A filtration $(\mathcal{F}_j)_{j\in\mathbb{Z}}$ is then obtained by defining $\mathcal{F}_j$ as the σ-algebra generated by $\mathcal{D}_j$.

We write $\langle f\rangle_Q$ for the average of an $\mathcal{X}$-valued function f over a dyadic cube Q, that is,

$$\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f(\eta)\,d\eta.$$

Our maximal functions are now given by

$$Mf(\xi) = \sup_{Q\ni\xi}\|\langle f\rangle_Q\| \quad\text{and}\quad M_Rf(\xi) = \mathcal{R}(\langle f\rangle_Q : Q\ni\xi), \qquad \xi\in\mathbb{R}^n.$$

The Euclidean version of the Rademacher maximal function was originally studied by Hytönen, McIntosh and Portal [HMP] via the identification $\mathcal{L}(\mathbb{C},E)\simeq E$. They showed using interpolation that the L^p-boundedness of $f\mapsto M_Rf$ for one $p\in(1,\infty)$ implies boundedness for all p in that range. They also provided an example of a space, namely ℓ^1, for which the Rademacher maximal operator is not bounded.

Definition 3.3. Let $1<p<\infty$. A Banach space $\mathcal{X}\subset\mathcal{L}(H,E)$ is said to have RMF_p with respect to a given filtration on a given σ-finite measure space if the corresponding Rademacher maximal operator is bounded from $L^p(\mathcal{X})$ to L^p.

The smallest constant for which the boundedness holds will be called the RMF_p-constant for the given filtration on the given measure space. When dealing with the Euclidean case, we occasionally drop the subscript p and refer to the property as RMF with respect to $\mathbb{R}^n$. Note that the RMF_p-property is inherited by closed subspaces. In particular, if $\mathcal{L}(H,E)$ has RMF_p, then both E and H^* have it.

We will show that if $\mathcal{X}$ has RMF_p with respect to the filtration of dyadic intervals on $[0,1)$, then it has RMF_p with respect to any filtration on any σ-finite measure space. Supporting evidence is found in the Euclidean case: If one restricts to the unit cube $[0,1)^n$ with the filtration of dyadic cubes contained in $[0,1)^n$, it is not difficult to show that RMF_p with respect to this filtration on $[0,1)^n$ is equivalent to RMF_p with respect to the filtration of standard dyadic cubes on $\mathbb{R}^n$.

Martingales are later on used to study a weak type inequality for the maximal operator. In the Euclidean case, a similar inequality can be proven with the aid of Calderón–Zygmund decomposition: Suppose that $\mathcal{X}\subset\mathcal{L}(H,E)$ has RMF_p with respect to the filtration of dyadic cubes on $\mathbb{R}^n$ for some $p\in(1,\infty)$, i.e. M_R is bounded from $L^p(\mathcal{X})$ to L^p. Then there exists a constant C such that for all $f\in L^1(\mathcal{X})$,

$$|\{\xi\in\mathbb{R}^n : M_Rf(\xi)>\lambda\}| \le \frac{C}{\lambda}\|f\|_{L^1(\mathcal{X})}$$

whenever $\lambda>0$. The crucial part of the proof is to observe that M_Ra vanishes outside a dyadic cube containing the support of an atom a (whose average is zero).

4. RMF-property, type and cotype. We will now study what kind of restrictions the boundedness of the Rademacher maximal operator puts on the type and cotype of the spaces involved.

Unlike many other maximal operators, M_R is not in general bounded from $L^\infty(\mathcal{L}(H,E))$ to L^∞. We actually have the following:

PROPOSITION 4.1. *The Rademacher maximal operator is bounded from* $L^\infty(0,1;\mathcal{L}(H,E))$ *to* $L^\infty(0,1)$ *if and only if* H *has cotype* 2 *and* E *has type* 2.

Proof. If H has cotype 2 and E has type 2, all the uniformly bounded sets are R-bounded and $M_R f \le CMf$ for all f in $L^\infty(0,1;\mathcal{L}(H,E))$. Suppose on the contrary that H does not have cotype 2 or E does not have type 2, and fix a $C>0$. Now there exists a positive integer N and operators $T_1,\dots,T_N$ in $\mathcal{L}(H,E)$ with at most unit norm such that the R-bound of $\{T_1,\dots,T_N\}$ is greater than C. We then construct an operator-valued L^∞-function on $[0,1)$ that has the operators T_j as dyadic averages on an interval. Let us write $I_j=[0,2^{j-N})$, $j=1,\dots,N$, so that $I_1=[0,2^{1-N})$ is the smallest interval and $I_N=[0,1)$. We set $S_1=T_1$ and

$$S_j = 2T_j - T_{j-1}, \qquad j=2,\dots,N.$$

Now $\|S_j\|\le 3$ for all $j=1,\dots,N$, so that if we define $f(\xi)=S_1$ for $\xi\in I_1$ and $f(\xi)=S_j$ for $\xi\in I_j\setminus I_{j-1}$, $j=2,\dots,N$, we have $f\in L^\infty(0,1;\mathcal{L}(H,E))$.

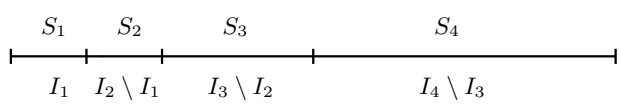

Fig. 1. The construction of f with $N=4$

We then look at the averages of f over the intervals I_j. Obviously

$$\begin{aligned}
\langle f\rangle_{I_1} &= S_1 = T_1,\\
\langle f\rangle_{I_2} &= \frac{S_1+S_2}{2} = \frac{T_1+2T_2-T_1}{2} = T_2,\\
\langle f\rangle_{I_3} &= \frac{S_1+S_2+2S_3}{4} = \frac{2T_2+4T_3-2T_2}{4} = T_3.
\end{aligned}$$

More generally, observing the telescopic behaviour we calculate

$$\langle f\rangle_{I_j} = \frac{1}{2^{j-1}}\Big(S_1+\sum_{k=1}^{j}2^{k-1}S_k\Big) = \frac{1}{2^{j-1}}(T_1+2^{j-1}T_j-T_1) = T_j$$

for $j=2,\dots,N$, as was desired. Thus $M_R f>C$ on I_1, where C was chosen arbitrarily large and the bound 3 for the norm of f does not depend on C. The operator M_R cannot therefore be bounded from $L^\infty(0,1;\mathcal{L}(H,E))$ to $L^\infty(0,1)$. ∎

Based on the counterexample from [HMP] that the sequence space ℓ^1 does not have RMF we prove the following statement.

PROPOSITION 4.2. *If for some $p \in (1,\infty)$, $\mathcal{L}(H,E)$ has RMF$_p$ with respect to the usual dyadic filtration on $\mathbb{R}$, then H has finite cotype and E has non-trivial type.*

Proof. Suppose on the contrary that E has only trivial type. By Theorem 2.3 it follows that for some $\lambda \geq 1$ there exists a sequence $(E_N)_{N=1}^{\infty}$ of subspaces and a sequence $(\Lambda_N^E)_{N=1}^{\infty}$ of isomorphisms between each E_N and ℓ_N^1 such that $\|\Lambda_N^E\|\,\|(\Lambda_N^E)^{-1}\| \leq \lambda$. Let us then fix an N. It is shown in [HMP] that there exists a function $f \in L^p(0,1;\ell^1)$ for any $p \in (1,\infty)$ with the following properties:

(i) $f(\xi) \in \ell_{2^N}^1$ for all $\xi \in [0,1)$,
(ii) $\|f(\xi)\| = 1$ for all $\xi \in [0,1)$ so that $\|f\|_{L^p(0,1;\ell^1)} = 1$,
(iii) $\|M_R f\|_{L^p(0,1)} \geq C_1 \log\log N$, where the constant C_1 does not depend on N.

Define then a function $g : [0,1) \to E$ by $g(\xi) = (\Lambda_{2^N}^E)^{-1}(f(\xi))$ and note that $\|g\|_{L^p(0,1;E)} \leq \|(\Lambda_{2^N}^E)^{-1}\|$. Since M_R is bounded from $L^p(0,1;E)$ to $L^p(0,1)$ there exists a constant C_2 such that $\|M_R g\|_{L^p(0,1)} \leq C_2\|g\|_{L^p(0,1;E)}$. But now, since $f(\xi) = \Lambda_{2^N}^E(g(\xi))$ we have $\|M_R f(\xi)\| \leq \|\Lambda_{2^N}^E\|\,\|M_R g(\xi)\|$. Thus

$$\|M_R f\|_{L^p(0,1)} \leq \|\Lambda_{2^N}^E\|\,\|M_R g\|_{L^p(0,1)} \leq C_2\|\Lambda_{2^N}^E\|\,\|g\|_{L^p(0,1;E)} \leq C_2\lambda,$$

which gives a contradiction whenever N is chosen so large that $C_1 \log\log N \geq C_2\lambda$.

The claim on finite cotype is proven similarly. Suppose on the contrary that H has only infinite cotype. Then H^* has only trivial type and one can proceed as above by defining a function $h : [0,1) \to H^*$ by $h(\xi) = \Lambda_{2^N}^{H^*}(f(\xi))$. ∎

Recall that $\mathcal{L}(H,E)$ has only trivial type whenever H and E are infinite-dimensional Banach spaces. Therefore it cannot have RMF via the identification $\mathcal{L}(H,E) \simeq \mathcal{L}(\mathbb{C},\mathcal{L}(H,E))$.

Since L^p-spaces have type 2 whenever $2 \leq p < \infty$, they have the RMF-property. We show next that they have RMF also when $1 < p < 2$. This is implied by the following heredity property of RMF.

PROPOSITION 4.3. *Let $1 < p < \infty$. Suppose that (Σ,ν) is a σ-finite measure space and that $\mathcal{X} \subset \mathcal{L}(H,E)$ has RMF$_p$ with respect to the usual dyadic filtration on $\mathbb{R}^n$. Then $L^p(\Sigma;\mathcal{X})$ has RMF$_p$ with respect to the usual dyadic filtration on $\mathbb{R}^n$.*

Proof. We use the identification $L^p(\mathbb{R}^n; L^p(\Sigma; \mathcal{X})) \simeq L^p(\mathbb{R}^n \times \Sigma; \mathcal{X})$ and write

$$\widetilde{M}_R f(\xi, \eta) = \mathcal{R}\bigg(\frac{1}{|Q|} \int_Q f(\zeta, \eta)\, d\zeta : Q \ni \xi \bigg), \qquad (\xi, \eta) \in \mathbb{R}^n \times \Sigma,$$

for the Rademacher maximal function in the first variable. By the RMF_p-property of $\mathcal{X}$ we have, for ν-almost every η,

$$\int_{\mathbb{R}^n} \widetilde{M}_R f(\xi, \eta)^p\, d\xi \lesssim \int_{\mathbb{R}^n} \|f(\xi, \eta)\|^p\, d\xi.$$

We then calculate

$$\begin{aligned} \mathbb{E}\Big\| \sum_{Q \ni \xi} \varepsilon_Q \lambda_Q \langle f \rangle_Q \Big\|^p_{L^p(\Sigma; \mathcal{X})} &= \int_\Sigma \mathbb{E}\bigg| \sum_{Q \ni \xi} \varepsilon_Q \lambda_Q \frac{1}{|Q|} \int_Q f(\zeta, \eta)\, d\zeta \bigg|^p d\nu(\eta) \\ &\lesssim \int_\Sigma \widetilde{M}_R f(\xi, \eta)^p\, d\nu(\eta)\, \mathbb{E}\bigg| \sum_{Q \ni \xi} \varepsilon_Q \lambda_Q \bigg|^p \end{aligned}$$

and so

$$\mathcal{R}(\langle f \rangle_Q : Q \ni \xi)^p \lesssim \int_\Sigma \widetilde{M}_R f(\xi, \eta)^p\, d\nu(\eta).$$

Therefore,

$$\int_{\mathbb{R}^n} M_R f(\xi)^p\, d\xi \lesssim \int_\Sigma \int_{\mathbb{R}^n} \widetilde{M}_R f(\xi, \eta)^p\, d\xi\, d\nu(\eta) \lesssim \int_\Sigma \int_{\mathbb{R}^n} \|f(\xi, \eta)\|^p\, d\xi\, d\nu(\eta),$$

so that M_R is bounded from $L^p(L^p(\Sigma; \mathcal{X}))$ to L^p. ∎

Remark. The previous proposition also follows from the more general results proven in [HMP], namely that both non-commutative L^p-spaces and all UMD function lattices have RMF.

5. Reduction to Haar filtrations. We will show that the RMF-property is independent of the filtration and the underlying measure space in the following sense:

Theorem 5.1. *Let $1 < p < \infty$. If $\mathcal{X}$ has RMF_p with respect to the filtration of dyadic intervals on $[0, 1)$, then it has RMF_p with respect to any filtration on any σ-finite measure space.*

When this is the case, we simply say that $\mathcal{X}$ has RMF_p. The proof of Theorem 5.1 uses ideas from Maurey [M], where a similar result is proven for the UMD-property. We begin with the simplest possible case of filtrations of finite algebras on finite measure spaces and proceed gradually toward more general situations. In order to do so, we first work on measure spaces $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) = 1$ that are *divisible* in the sense that any set $A \in \mathcal{F}$

with positive measure has for all $c \in (0,1)$ a (measurable) subset with measure $c\mu(A)$.

By a *basis* of a finite subalgebra $\mathcal{G}$ of $\mathcal{F}$ we mean a partition of Ω into disjoint non-empty sets $A_1, \dots, A_m \in \mathcal{G}$ that generate the subalgebra so that each $A \in \mathcal{G}$ can be expressed as a union of some of these A_k's. Such a partition, denoted by $\operatorname{bs} \mathcal{G}$, always exists and is unique. Observe that functions measurable with respect to a finite algebra can be identified with functions defined on the basis of this algebra (or any finer algebra).

A filtration $(\mathcal{F}_j)_{j=1}^{\infty}$ of finite subalgebras of $\mathcal{F}$ is called a *Haar filtration* if $\operatorname{bs} \mathcal{F}_j$ consists of $j+1$ sets of positive measure. We also write $\mathcal{F}_0 = \{\emptyset, \Omega\}$ so that $\operatorname{bs} \mathcal{F}_0 = \{\Omega\}$. Furthermore, every $\mathcal{F}_j$ is obtained from $\mathcal{F}_{j-1}$ by splitting a set $B \in \operatorname{bs} \mathcal{F}_{j-1}$ into two sets B' and B'' of positive measure. A Haar filtration is said to be *dyadic* if in each splitting, $\mu(B')$ (and hence also $\mu(B'')$) is a dyadic fraction of $\mu(B)$; and the filtration is *standard* if each B splits into sets of equal measure.

A typical example of a filtration of finite algebras is of course the filtration of dyadic intervals on $[0,1)$. We denote by $\mathcal{D}_j$ the finite algebra of dyadic intervals of length 2^{-j} on $[0,1)$, and so

$$\operatorname{bs} \mathcal{D}_j = \{[(k-1)2^{-j}, k2^{-j}) : k = 1, \dots, 2^j\}.$$

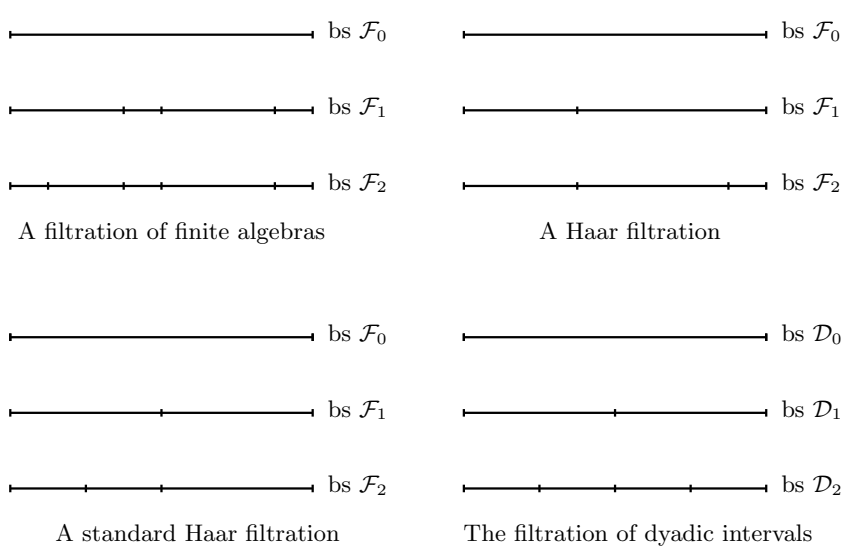

Fig. 2. Different filtrations of finite algebras

Suppose that $(\mathcal{F}_j)_{j=1}^{N}$ is a filtration of finite algebras. By adding one set at a time (to the basis), one can construct a Haar filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^{K_N}$ such

that

$$\widetilde{\mathcal{F}}_1 \subset \widetilde{\mathcal{F}}_2 \subset \cdots \subset \widetilde{\mathcal{F}}_{K_1} = \mathcal{F}_1 \subset \widetilde{\mathcal{F}}_{K_1+1} \subset \cdots \subset \widetilde{\mathcal{F}}_{K_N} = \mathcal{F}_N,$$

where K_j+1 is the number of sets in $\operatorname{bs}\mathcal{F}_j$. Likewise, the filtration of dyadic intervals on $[0,1)$ can be "embedded" in a standard Haar filtration on $[0,1)$.

Note that the RMF_p-constant of $\mathcal{X}$ with respect to a filtration $(\mathcal{F}_j)_{j=1}^N$ of finite algebras is at least the RMF_p-constant with respect to any "subfiltration" $(\mathcal{F}_{j_k})_{k=1}^M$, where $1 \le j_{k_1} \le \cdots \le j_{k_M} \le N$. Indeed, for any $\mathcal{F}_N$-measurable f we have

$$\mathcal{R}(\mathbb{E}(f \,|\, \mathcal{F}_{j_k})(A) : 1 \le k \le M) \le \mathcal{R}(\mathbb{E}(f \,|\, \mathcal{F}_j)(A) : 1 \le j \le N), \qquad A \in \operatorname{bs}\mathcal{F}_N,$$

and the claim follows.

Two filtrations $(\mathcal{F}_j)_{j=1}^\infty$ and $(\widetilde{\mathcal{F}}_j)_{j=1}^\infty$ of finite algebras (possibly on different measure spaces) are said to be *equivalent* if there exists for every $j \in \mathbb{Z}_+$ a measure preserving bijection between $\operatorname{bs}\mathcal{F}_j$ and $\operatorname{bs}\widetilde{\mathcal{F}}_j$. Observe that if b is such a bijection from $\operatorname{bs}\mathcal{F}_N$ to $\operatorname{bs}\widetilde{\mathcal{F}}_N$, then for every $\mathcal{F}_N$-measurable f we have

$$\mathbb{E}(f \,|\, \mathcal{F}_j) = \mathbb{E}(f \circ b^{-1} \,|\, \widetilde{\mathcal{F}}_j) \circ b$$

for any $j = 1, \ldots, N$. It is a matter of calculation that the RMF_p-constant of $\mathcal{X}$ (if finite) is the same with respect to equivalent filtrations of finite algebras.

Evidently, every filtration of finite algebras on any measure space (of total measure one) is equivalent to a filtration on the unit interval. The next lemma shows that when dealing with dyadic Haar filtrations, we can choose an equivalent filtration on the unit interval that very much resembles the filtration of dyadic intervals. The result goes back to Maurey [M] and a detailed proof can be found in Hytönen [H1].

LEMMA 5.2. *Every dyadic Haar filtration on any measure space with total measure one is equivalent to a dyadic Haar filtration $(\mathcal{F}_j)_{j=1}^N$ on the unit interval such that $\mathcal{F}_j \subset \mathcal{D}_{K_j}$ for some integers K_j and*

$$\mathbb{E}(f \,|\, \mathcal{F}_j) = \mathbb{E}(f \,|\, \mathcal{D}_{K_j}), \qquad 1 \le j \le N,$$

for any $\mathcal{F}_N$-measurable f.

Hence, if $\mathcal{X}$ has RMF_p with respect to the filtration of dyadic intervals on $[0,1)$, then it has RMF_p with respect to any dyadic Haar filtration on any measure space with total measure one and the RMF_p-constant is at most the RMF_p-constant with respect to the filtration of dyadic intervals.

We say that $\mathcal{X}$ has RMF_p *uniformly* with respect to a class of filtrations on a class of measure spaces if the RMF_p-constants in question are uniformly bounded.

For the next three lemmas, fix a divisible measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) = 1$. In each of the lemmas we start with a filtration $(\mathcal{F}_j)_{j=1}^{\infty}$, truncate it at a positive integer N and construct a related more "regular" one, whose σ-algebras we denote by $\widetilde{\mathcal{F}}_j$. Objects corresponding to these are denoted likewise, for instance, conditional expectations are denoted by E_j and $\widetilde{E}_j$, respectively.

LEMMA 5.3. *If $\mathcal{X}$ has RMF_p uniformly with respect to dyadic Haar filtrations on $(\Omega, \mathcal{F}, \mu)$, then it has RMF_p uniformly with respect to all Haar filtrations on $(\Omega, \mathcal{F}, \mu)$.*

Proof. Suppose that $(\mathcal{F}_j)_{j=1}^{\infty}$ is a Haar filtration. Let f be an $\mathcal{F}_N$-measurable $\mathcal{X}$-valued function for a fixed positive integer N and let $\varepsilon > 0$. We aim to show that

$$\|M_R^{(N)} f\|_{L^p} \le r(\varepsilon, N, f) + C\|f\|_{L^p(\mathcal{X})},$$

where $r(\varepsilon, N, f) \to 0$ as $\varepsilon \to 0$ and C depends only on $\mathcal{X}$ and p.

To construct a dyadic Haar filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^{N}$ that approximates $(\mathcal{F}_j)_{j=1}^{N}$, we proceed inductively. Assume that we have constructed $\widetilde{\mathcal{F}}_{j-1}$ in our desired dyadic Haar filtration so that $\mu(B \triangle \widetilde{B}) < 2^{j-1-N}\varepsilon$ whenever $\widetilde{B}$ in $\operatorname{bs} \widetilde{\mathcal{F}}_{j-1}$ corresponds to a B in $\operatorname{bs} \mathcal{F}_{j-1}$. If B in $\operatorname{bs} \mathcal{F}_{j-1}$ splits into B' and B'' in $\operatorname{bs} \mathcal{F}_j$, then using divisibility we choose $\widetilde{B}' \subset \widetilde{B}$ whose measure is a dyadic fraction of $\mu(\widetilde{B})$ and which contains $\widetilde{B} \cap B'$ while $\mu(\widetilde{B}' \setminus B') < 2^{j-1-N}\varepsilon$. Now, as $B' \setminus \widetilde{B}' = B' \setminus \widetilde{B} \subset B \setminus \widetilde{B}$, we see that

$$\mu(B' \triangle \widetilde{B}') \le \mu(\widetilde{B}' \setminus B') + \mu(B \setminus \widetilde{B}) < \frac{\varepsilon}{2^{N-j}}.$$

Also for $\widetilde{B}'' = \widetilde{B} \setminus \widetilde{B}'$ we have

$$\mu(B'' \triangle \widetilde{B}'') < \frac{\varepsilon}{2^{N-j}}.$$

We have now constructed a dyadic Haar filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^{N}$ for which $\mu(B \triangle \widetilde{B}) < \varepsilon$ whenever $\widetilde{B}$ corresponds to a B in some $\mathcal{F}_j$.

Now

$$\begin{aligned}\|M_R^{(N)} f\|_{L^p} &= \Big(\int_{\Omega} \mathcal{R}(E_j f(\xi) : 1 \le j \le N)^p \, d\mu(\xi) \Big)^{1/p} \\ &\le \Big(\int_{\Omega} \mathcal{R}(E_j f(\xi) - \widetilde{E}_j f(\xi) : 1 \le j \le N)^p \, d\mu(\xi) \Big)^{1/p} + \|\widetilde{M}_R f\|_{L^p},\end{aligned}$$

where the maximal operator $\widetilde{M}_R$ satisfies by assumption $\|\widetilde{M}_R f\|_{L^p} \le C\|f\|_{L^p(\mathcal{X})}$ for a constant C independent of the filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^{N}$. Estimat-

ing the R-bound in the first term by summing the norms we get

$$\Big(\int_\Omega \mathcal{R}(E_j f(\xi) - \widetilde{E}_j f(\xi) : 1 \le j \le N)^p \, d\mu(\xi)\Big)^{1/p}$$
$$\le \Big(\int_\Omega \Big(\sum_{j=1}^N \|E_j f(\xi) - \widetilde{E}_j f(\xi)\|\Big)^p \, d\mu(\xi)\Big)^{1/p} \le \sum_{j=1}^N \|E_j f - \widetilde{E}_j f\|_{L^p(\mathcal{X})}.$$

To estimate $\|E_j f - \widetilde{E}_j f\|_{L^p(\mathcal{X})}$ for a fixed j, recall that $\mu(B \triangle \widetilde{B}) < \varepsilon$ when $\widetilde{B}$ in $\operatorname{bs}\widetilde{\mathcal{F}}_j$ corresponds to a B in $\operatorname{bs}\mathcal{F}_j$. Decomposing Ω as

$$\Omega = \bigcup_{B \in \operatorname{bs}\mathcal{F}_j} B = \bigcup_{B \in \operatorname{bs}\mathcal{F}_j} ((B \cap \widetilde{B}) \cup (B \setminus \widetilde{B}))$$

gives us

$$\|E_j f - \widetilde{E}_j f\|_{L^p(\mathcal{X})}$$
$$\le \sum_{B \in \operatorname{bs}\mathcal{F}_j} \big(\|1_{B\cap\widetilde{B}}(E_j f - \widetilde{E}_j f)\|_{L^p(\mathcal{X})} + \|1_{B\setminus\widetilde{B}}(E_j f - \widetilde{E}_j f)\|_{L^p(\mathcal{X})}\big).$$

For $\xi \in B \cap \widetilde{B}$ we have

$$E_j f(\xi) = \frac{1}{\mu(B)} \int_B f \, d\mu \quad \text{and} \quad \widetilde{E}_j f(\xi) = \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} f \, d\mu$$

and thus using Hölder's inequality we see that

$$\|1_{B\cap\widetilde{B}}(E_j f - \widetilde{E}_j f)\|_{L^p(\mathcal{X})}$$
$$= \mu(B \cap \widetilde{B})^{1/p} \Big\| \frac{1}{\mu(B)} \int_B f \, d\mu - \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} f \, d\mu \Big\|$$
$$\le \frac{1}{\mu(B)} \Big\| \int_B f \, d\mu - \int_{\widetilde{B}} f \, d\mu \Big\| + \frac{|\mu(B) - \mu(\widetilde{B})|}{\mu(B)\mu(\widetilde{B})} \Big\| \int_{\widetilde{B}} f \, d\mu \Big\|$$
$$\le \frac{1}{\mu(B)} \int_{B \triangle \widetilde{B}} \|f(\xi)\| \, d\mu(\xi) + \frac{|\mu(B) - \mu(\widetilde{B})|}{\mu(B)\mu(\widetilde{B})} \int_\Omega \|f(\xi)\| \, d\mu(\xi)$$
$$\le \frac{\mu(B \triangle \widetilde{B})^{1-1/p}}{\mu(B)} \|f\|_{L^p(\mathcal{X})} + \frac{\mu(B \triangle \widetilde{B})}{\mu(B)\mu(\widetilde{B})} \|f\|_{L^p(\mathcal{X})}$$
$$\le \left(\varepsilon^{1-1/p} + \frac{\varepsilon}{\mu(\widetilde{B})}\right) \frac{\|f\|_{L^p(\mathcal{X})}}{\mu(B)}$$
$$\le \left(\varepsilon^{1-1/p} + \frac{\varepsilon}{M - \varepsilon}\right) \frac{\|f\|_{L^p(\mathcal{X})}}{M}$$

whenever $\varepsilon < M$, where $M = \min\{\mu(B) : B \in \operatorname{bs}\mathcal{F}_N\}$ so that $\mu(B) \ge M$

and $\mu(\widetilde{B}) \geq M - \varepsilon$ for each B in $\operatorname{bs}\mathcal{F}_j$. On the other hand,

$$\|1_{B\setminus\widetilde{B}}(E_jf - \widetilde{E}_jf)\|_{L^p(\mathcal{X})} \leq \mu(B\triangle\widetilde{B})^{1/p}\|E_jf - \widetilde{E}_jf\|_{L^\infty(\mathcal{X})} \leq \varepsilon^{1/p}2\|f\|_{L^\infty(\mathcal{X})}$$

for each B in $\operatorname{bs}\mathcal{F}_j$.

All in all, as every $\operatorname{bs}\mathcal{F}_j$ contains at most $N+1$ sets, we have established that

$$\begin{aligned}\sum_{j=1}^{N}\|E_jf - \widetilde{E}_jf\|_{L^p(\mathcal{X})} &\leq N(N+1)\bigg(\varepsilon^{1-1/p} + \frac{\varepsilon}{M-\varepsilon}\bigg)\frac{\|f\|_{L^p(\mathcal{X})}}{M}\\ &\quad + N(N+1)\varepsilon^{1/p}2\|f\|_{L^\infty(\mathcal{X})}\\ &= r(\varepsilon, N, f)\end{aligned}$$

and clearly $r(\varepsilon, N, f) \to 0$ as $\varepsilon \to 0$. ∎

LEMMA 5.4. *If $\mathcal{X}$ has RMF_p uniformly with respect to Haar filtrations on $(\Omega, \mathcal{F}, \mu)$, then it has RMF_p uniformly with respect to filtrations of finite algebras on $(\Omega, \mathcal{F}, \mu)$.*

Proof. This follows immediately from our earlier observations: Given a filtration $(\mathcal{F}_j)_{j=1}^{\infty}$ of finite algebras and a positive integer N, we can construct a Haar filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^{K_N}$ so that

$$\widetilde{\mathcal{F}}_1 \subset \widetilde{\mathcal{F}}_2 \subset \cdots \subset \widetilde{\mathcal{F}}_{K_1} = \mathcal{F}_1 \subset \widetilde{\mathcal{F}}_{K_1+1} \subset \cdots \subset \widetilde{\mathcal{F}}_{K_N} = \mathcal{F}_N.$$

For any $\mathcal{F}_N$-measurable f we have

$$\mathcal{R}(E_jf(A) : 1 \leq j \leq N) \leq \mathcal{R}(\widetilde{E}_jf(A) : 1 \leq j \leq K_N), \qquad A \in \operatorname{bs}\mathcal{F}_N,$$

and the claim follows. ∎

LEMMA 5.5. *If $\mathcal{X}$ has RMF_p uniformly with respect to filtrations of finite algebras on $(\Omega, \mathcal{F}, \mu)$, then it has RMF_p uniformly with respect to all filtrations on $(\Omega, \mathcal{F}, \mu)$.*

Proof. Suppose that $(\mathcal{F}_j)_{j=1}^{\infty}$ is a filtration, N a positive integer, f a function in $L^p(\mathcal{F}_N; \mathcal{X})$ and $\varepsilon > 0$. We begin by choosing simple functions $s_j \in L^p(\mathcal{F}_j; \mathcal{X})$, $j = 1, \ldots, N$, so that

$$\|E_jf - s_j\|_{L^p(\mathcal{X})} < \frac{\varepsilon}{2^{j+1}}.$$

For $j = 1, \ldots, N$, let $\widetilde{\mathcal{F}}_j$ be the finite algebra generated by $s_1, \ldots, s_j$ and observe that $\widetilde{\mathcal{F}}_j \subset \widetilde{\mathcal{F}}_{j+1}$, i.e. $(\widetilde{\mathcal{F}}_j)_{j=1}^{N}$ is a filtration. Now

$$\begin{aligned}\|M_R^{(N)}f\|_{L^p} &= \Big(\int_\Omega \mathcal{R}(E_jf(\xi) : 1 \leq j \leq N)^p\, d\mu(\xi)\Big)^{1/p}\\ &\leq \Big(\int_\Omega \mathcal{R}(E_jf(\xi) - \widetilde{E}_jf(\xi) : 1 \leq j \leq N)^p\, d\mu(\xi)\Big)^{1/p} + \|\widetilde{M}_Rf\|_{L^p},\end{aligned}$$

where the maximal operator $\widetilde{M}_R$ satisfies $\|\widetilde{M}_R f\|_{L^p} \le C\|f\|_{L^p(\mathcal{X})}$ for a constant C independent of the filtration $(\widetilde{\mathcal{F}}_j)_{j=1}^N$. This independence is crucial, as $\widetilde{\mathcal{F}}_j$'s arose from f.

We then estimate

$$\begin{aligned}\Big(\int_\Omega \mathcal{R}(E_j f(\xi) - \widetilde{E}_j f(\xi) : 1 \le j \le N)^p \, d\mu(\xi)\Big)^{1/p} &\\ \le \Big(\int_\Omega \Big(\sum_{j=1}^N \|E_j f(\xi) - \widetilde{E}_j f(\xi)\|\Big)^p \, d\mu(\xi)\Big)^{1/p} &\\ \le \sum_{j=1}^N \|E_j f - \widetilde{E}_j f\|_{L^p(\mathcal{X})} &\\ \le \sum_{j=1}^N (\|E_j f - s_j\|_{L^p(\mathcal{X})} + \|\widetilde{E}_j f - s_j\|_{L^p(\mathcal{X})}).&\end{aligned}$$

Furthermore, since

$$\begin{aligned}\|\widetilde{E}_j f - s_j\|_{L^p(\mathcal{X})} &= \|\widetilde{E}_j f - \widetilde{E}_j s_j\|_{L^p(\mathcal{X})} = \|\widetilde{E}_j (E_j f - s_j)\|_{L^p(\mathcal{X})} \\ &\le \|E_j f - s_j\|_{L^p(\mathcal{X})}\end{aligned}$$

we get

$$\begin{aligned}\Big(\int_\Omega \mathcal{R}(E_j f(\xi) - \widetilde{E}_j f(\xi) : 1 \le j \le N)^p \, d\mu(\xi)\Big)^{1/p} &\\ \le 2\sum_{j=1}^N \|E_j f - s_j\|_{L^p(\mathcal{X})} < \sum_{j=1}^N \frac{\varepsilon}{2^j} < \varepsilon. \ \blacksquare&\end{aligned}$$

We then show that the assumption on divisibility can be dropped.

LEMMA 5.6. *If $\mathcal{X}$ has RMF$_p$ with respect to any filtration on any divisible measure space with total measure one, then it has RMF$_p$ with respect to any filtration on any measure space with total measure one.*

Proof. Suppose that $(\mathcal{F}_j)_{j=1}^\infty$ is a filtration on a not necessarily divisible measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) = 1$. Now the σ-algebras $\widetilde{\mathcal{F}}_j = \{F \times [0,1] : F \in \mathcal{F}_j\}$ form a filtration on the product of $(\Omega, \mathcal{F}, \mu)$ and the unit interval with Lebesgue measure, which is obviously a divisible measure space. For $f \in L^p(\Omega; \mathcal{X})$ we put $\widetilde{f}(\xi, t) = f(\xi), (\xi, t) \in \Omega \times [0,1]$, and observe that $\|\widetilde{f}\|_{L^p(\mathcal{X})} = \|f\|_{L^p(\mathcal{X})}$. Also $\widetilde{E}_j \widetilde{f}(\xi, t) = E_j f(\xi)$ for all $(\xi, t) \in \Omega \times [0,1]$, and so $\|\widetilde{M}_R \widetilde{f}\|_{L^p(\mathcal{X})} = \|M_R f\|_{L^p(\mathcal{X})}$. ∎

The results follow immediately for finite measure spaces. Suppose that $(\Omega, \mathcal{F}, \mu)$ is such. Then the above argument applies to the measure $\mu(\Omega)^{-1}\mu$

on $(\Omega, \mathcal{F})$ and evidently the conditional expectations are the same in these two measure spaces. Thus the Rademacher maximal operator remains unaltered and the inequality stating the boundedness is only a matter of scaling by $\mu(\Omega)^{-1}$.

Suppose then that $\mathcal{X}$ has RMF_p uniformly with respect to any filtration on any finite measure space and let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space with a filtration $(\mathcal{F}_j)_{j=1}^{\infty}$. Since $\mathcal{F}_1$ is σ-finite (by definition), we can write Ω as a union of disjoint sets $A_k \in \mathcal{F}_1$, $k \in \mathbb{Z}_+$, each with finite measure. Let us define for positive integers k the finite measures $\mu_k(A) = \mu(A \cap A_k)$ on $\mathcal{F}$. The conditional expectation of a function $f \in L^p(\Omega; \mathcal{X})$ with respect to $\mathcal{F}_j$ and μ_k is simply the conditional expectation of $1_{A_k} f$ with respect to $\mathcal{F}_j$, which further equals $1_{A_k} E_j f$. In symbols

$$E_j^{(k)} f = 1_{A_k} E_j f,$$

where $E_j^{(k)} f$ denotes the conditional expectation of f with respect to $\mathcal{F}_j$ and μ_k. Thus

$$\begin{aligned}
\|M_R f\|_{L^p}^p &= \sum_{k=1}^{\infty} \int_{A_k} \mathcal{R}(E_j f(\xi) : j \in \mathbb{Z}_+)^p \, d\mu(\xi) \\
&= \sum_{k=1}^{\infty} \int_{A_k} \mathcal{R}(E_j^{(k)} f(\xi) : j \in \mathbb{Z}_+)^p \, d\mu_k(\xi) \\
&\le \sum_{k=1}^{\infty} C^p \int_{A_k} \|f(\xi)\|^p \, d\mu_k(\xi) = C^p \|f\|_{L^p(\mathcal{X})}^p.
\end{aligned}$$

So far we have only considered filtrations indexed by positive integers. Suppose that $\mathcal{X}$ has RMF_p with respect to any filtration indexed by $\mathbb{Z}_+$ on any σ-finite measure space and let $(\mathcal{F}_j)_{j \in \mathbb{Z}}$ be a filtration on $(\Omega, \mathcal{F}, \mu)$. Then $\mathcal{X}$ has RMF_p with respect to $(\mathcal{F}_j)_{j=-N}^{\infty}$ with a constant independent of N and thus by the monotone convergence theorem with respect to $(\mathcal{F}_j)_{j \in \mathbb{Z}}$.

This concludes the proof of Theorem 5.1.

6. The weak RMF-property. We start by recalling some terminology. A *stochastic process* (a sequence of random variables on some probability space) $X = (X_j)_{j=1}^{\infty}$ is always *adapted* to the filtration $(\mathcal{F}_j)_{j=1}^{\infty}$, where $\mathcal{F}_j$ is the σ-algebra $\sigma(X_1, \ldots, X_j)$ generated by $X_1, \ldots, X_j$, in the sense that each X_j is $\mathcal{F}_j$-measurable. We call a sequence of L^1-random variables a *martingale* if $\mathbb{E}(X_k \,|\, \mathcal{F}_j) = X_j$ whenever $j \le k$.

Note that for any martingale $X = (X_j)_{j=1}^{\infty}$ we have $\mathbb{E}X_j = \mathbb{E}X_k$ for all $j, k \in \mathbb{Z}_+$. It is customary to write $\mathcal{F}_0$ for the trivial σ-algebra and X_0 for the common expectation of X_j's. By defining $Y_j = X_j - X_0$ one can restrict to martingales $Y = (Y_j)_{j=1}^{\infty}$ for which $Y_0 = \mathbb{E}Y_j = 0$.

We say that a stochastic process $X = (X_j)_{j=1}^{\infty}$ is *L^p-bounded* for $p \in [1,\infty)$ if $\|X\|_p^p := \sup_{j\in\mathbb{Z}_+} \mathbb{E}\|X_j\|^p < \infty$ and for $p = \infty$ if the infimum $\|X\|_\infty$ of all C for which every $\|X_j\| \le C$ almost surely, is finite. A stochastic process $X = (X_j)_{j=1}^{\infty}$ is said to be *simple* if the algebras $\mathcal{F}_j$ are finite (i.e. the random variables X_j are simple). A simple martingale is called a *(dyadic/standard) Haar martingale* if the algebras $\mathcal{F}_j$ form a (dyadic/standard) Haar filtration.

Given a martingale $(X_j)_{j=1}^N$ we define its *difference sequence* $(D_j)_{j=1}^N$ by $D_j = X_j - X_{j-1}$ for $j \ge 1$. Furthermore, if $v = (v_j)_{j=1}^{\infty}$ is a real L^∞-bounded stochastic process (on the same probability space), we define

$$(v \star X)_j = \sum_{k=1}^{j} v_k D_k, \qquad j \in \mathbb{Z}_+.$$

If v is *predictable* with respect to X in the sense that each v_j is $\mathcal{F}_{j-1}$-measurable (and v_1 is constant almost surely), then the *martingale transform* $v \star X = ((v \star X)_j)_{j=1}^{\infty}$ is itself a martingale.

Definition 6.1. Let $1 < p < \infty$. A Banach space E is said to have UMD_p if there exists a constant C such that for every L^p-martingale $X = (X_j)_{j=1}^N$ in E we have

$$\mathbb{E}\|(\varepsilon \star X)_N\|^p \le C^p \mathbb{E}\|X_N\|^p$$

whenever $\varepsilon = (\varepsilon_j)_{j=1}^N$ is a sequence of signs $\{-1, 1\}$.

This property is independent of p in the sense that if a Banach space has UMD_p for one $p \in (1,\infty)$ then it has UMD_p for all $p \in (1,\infty)$ (see Maurey [M]). Thus the parameter p can be omitted from the definition.

One can ask how the RMF-property relates to the UMD-property. First of all, every UMD-space can be shown to be reflexive (see for instance [M]). Our typical example $\mathcal{L}(H,E)$ is usually non-reflexive, but has RMF at least when H has cotype 2 and E has type 2. More interestingly, James constructed in [J] a non-reflexive Banach space E with type 2. Thus $E \hookrightarrow \mathcal{L}(H,E)$ can have RMF without being a UMD-space. Bourgain showed in [Bou] that the Schatten–von Neumann class $S_p(H_1,H_2)$ is UMD for $1 < p < \infty$. As H_1 and H_2 are spaces of type and cotype 2, it follows from our earlier observations that $S_p(H_1,H_2)$ has RMF as a subspace of $\mathcal{L}(H_1,H_2)$. It has also been shown in [HMP] that $S_p(H_1,H_2)$ has RMF as $\mathcal{L}(\mathbb{C}, S_p(H_1,H_2))$.

Let $\mathcal{X} \subset \mathcal{L}(H,E)$ be a Banach space whose norm dominates the operator norm. For a stochastic process $X = (X_j)_{j=1}^{\infty}$ in $\mathcal{X}$ we define the *Doob* and *Rademacher maximal functions* by

$$X^* = \sup_{j\in\mathbb{Z}_+} \|X_j\| \qquad \text{and} \qquad X_R^* = \mathcal{R}(X_j : j \in \mathbb{Z}_+),$$

respectively.

The boundedness properties of Doob's maximal function are well-known: Every L^p-bounded martingale X satisfies

$$\mathbb{E}|X^*|^p \le (p')^p \|X\|_p^p,$$

where p' is the Hölder conjugate of p and $1 < p < \infty$. Furthermore, for every L^1-bounded martingale X we have

$$\mathbb{P}(X^* > \lambda) \le \frac{1}{\lambda}\|X\|_1$$

whenever $\lambda > 0$.

Recall that the RMF_p-property is independent of the filtration and of the underlying measure space in the sense of the previous section (Theorem 5.1). Regarding the unit interval as a probability space on which the conditional expectations with respect to dyadic intervals define martingales, we see that $\mathcal{X}$ has RMF_p if and only if there exists a constant C such that

$$\mathbb{E}|X_R^*|^p \le C^p \|X\|_p^p$$

for any L^p-bounded martingale X in $\mathcal{X}$.

Applying ideas from Burkholder [Bur1] we will show that $\mathcal{X}$ has RMF_p for some $p \in (1,\infty)$ if and only if it has *weak RMF*, i.e. there exists a constant C such that all L^1-bounded martingales X in $\mathcal{X}$ satisfy

$$\mathbb{P}(X_R^* > \lambda) \le \frac{C}{\lambda}\|X\|_1 \tag{6.1}$$

whenever $\lambda > 0$.

To show the necessity of the weak type inequality (6.1) we invoke the Gundy decomposition (see Gundy [G] for the original proof).

THEOREM 6.2 (Gundy decomposition). *Suppose that X is an L^1-bounded martingale in $\mathcal{X}$ and that $\lambda > 0$. There exists a decomposition $X = G+H+B$ of X into martingales G, H and B which satisfy*

(i) $\|G\|_1 \le 4\|X\|_1$ *and* $\|G\|_\infty \le 2\lambda$,
(ii) $\mathbb{E}\|H_1\| + \sum_{j=2}^\infty \mathbb{E}\|H_j - H_{j-1}\| \le 4\|X\|_1$ $(H = (H_j)_{j=1}^\infty)$,
(iii) $\mathbb{P}(B^* > 0) \le \frac{3}{\lambda}\|X\|_1$.

PROPOSITION 6.3. *If $\mathcal{X}$ has RMF_p for some $p \in (1,\infty)$, then it has weak RMF.*

Proof. Taking the Gundy decomposition of X at height λ we may write

$$\mathbb{P}(X_R^* > \lambda) \le \mathbb{P}(B_R^* > \lambda/3) + \mathbb{P}(H_R^* > \lambda/3) + \mathbb{P}(G_R^* > \lambda/3),$$

and estimate each term separately. Firstly $\mathbb{P}(B_R^* > 0) = \mathbb{P}(B^* > 0)$, since $B_R^* = 0$ if and only if $B^* = 0$. Thus

$$\mathbb{P}(B_R^* > \lambda/3) \le \mathbb{P}(B_R^* > 0) = \mathbb{P}(B^* > 0) \le \frac{3}{\lambda}\|X\|_1.$$

Secondly,

$$H_R^* = \mathcal{R}(H_j : j \in \mathbb{Z}_+) = \mathcal{R}\Big(\sum_{k=1}^{j}(H_k - H_{k-1}) : j \in \mathbb{Z}_+\Big) \leq \sum_{j=1}^{\infty}\|H_j - H_{j-1}\|,$$

where the last inequality follows from a simple rearrangement of sums. Hence

$$\begin{aligned}\mathbb{P}(H_R^* > \lambda/3) &\leq \mathbb{P}\Big(\sum_{j=1}^{\infty}\|H_j - H_{j-1}\| > \frac{\lambda}{3}\Big) \leq \frac{3}{\lambda}\mathbb{E}\sum_{j=1}^{\infty}\|H_j - H_{j-1}\| \\ &= \frac{3}{\lambda}\sum_{j=1}^{\infty}\mathbb{E}\|H_j - H_{j-1}\| \leq \frac{12}{\lambda}\|X\|_1.\end{aligned}$$

Thirdly,

$$\begin{aligned}\mathbb{P}(G_R^* > \lambda/3) &\leq \Big(\frac{3}{\lambda}\Big)^p \mathbb{E}|G_R^*|^p \leq C\Big(\frac{3}{\lambda}\Big)^p\|G\|_p^p \\ &\leq C\frac{3^p 2^{p-1}}{\lambda}\|G\|_1 \leq C\frac{3^p 2^{p+1}}{\lambda}\|X\|_1,\end{aligned}$$

where the property $\|G\|_\infty \leq 2\lambda$ was used to deduce that

$$\|G\|_p^p = \sup_{j\in\mathbb{Z}_+}\mathbb{E}\|G_j\|^p \leq \|G\|_\infty^{p-1}\sup_{j\in\mathbb{Z}_+}\mathbb{E}\|G_j\| \leq (2\lambda)^{p-1}\|G\|_1. \blacksquare$$

We then turn to the converse. We obtain the desired results for standard Haar martingales, but recalling the earlier reduction, this will not be a restriction. The argument is based on a "good-λ inequality" (Lemma 6.5) which says roughly that the chance of X_R^* being large while X^* diminishes is vanishingly small.

LEMMA 6.4. *If $X = (X_j)_{j=1}^{\infty}$ is a standard Haar martingale, then $(\|D_j\|)_{j=1}^{\infty}$ is predictable with respect to X.*

Proof. For every $j \geq 1$ there is exactly one event $B \in \operatorname{bs}\mathcal{F}_{j-1}$ on which $X_j - X_{j-1}$ is non-zero. As $B = B_1 \cup B_2$ for some $B_1, B_2 \in \operatorname{bs}\mathcal{F}_j$ with $\mathbb{P}(B_1) = \mathbb{P}(B_2)$ and $\mathbb{E}(X_j - X_{j-1} \,|\, \mathcal{F}_{j-1}) = 0$, there exists a $T \in \mathcal{X}$ such that $X_j - X_{j-1} = 1_{B_1}T - 1_{B_2}T$. Consequently,

$$\|D_j\| = \|X_j - X_{j-1}\| = 1_{B_1}\|T\| + 1_{B_2}\|T\| = 1_B\|T\|$$

and so $\|D_j\|$ is $\mathcal{F}_{j-1}$-measurable. $\blacksquare$

We will need the concept of a stopping time: We say that a random variable τ in $\mathbb{Z}_+ \cup \{\infty\}$ is a *stopping time* with respect to a stochastic process X if $\{\tau = j\}$ is in $\mathcal{F}_j$ for every positive integer j. In this case we define

$$X_\tau = \sum_{j=1}^{\infty} 1_{\{\tau=j\}}X_j.$$

Observe that $X_\tau = 0$ when $\tau = \infty$. An easy calculation shows that if τ is a stopping time with respect to an L^1-bounded martingale X, then $\mathbb{E}\|X_\tau\| \le \|X\|_1$.

LEMMA 6.5. *Suppose that $\mathcal{X}$ has weak RMF. Then for all $\delta \in (0,1)$ and $\beta > 2\delta + 1$ there exists an $\alpha(\delta) > 0$ which tends to zero as $\delta \searrow 0$ and which is such that for all L^p-bounded standard Haar martingales X in $\mathcal{X}$ we have*

$$\mathbb{P}(X_R^* > \beta\lambda,\, X^* \le \delta\lambda) \le \alpha(\delta)\mathbb{P}(X_R^* > \lambda)$$

whenever $\lambda > 0$.

Proof. Let $X = (X_j)_{j=1}^\infty$ be an L^p-bounded standard Haar martingale in $\mathcal{X}$. Define the stopping times

$$\begin{aligned}\tau_1 &= \min\{j \in \mathbb{Z}_+ : \mathcal{R}(X_k : 1 \le k \le j) > \lambda\}\\ \tau_2 &= \min\{j \in \mathbb{Z}_+ : \mathcal{R}(X_k : 1 \le k \le j) > \beta\lambda\}\\ \sigma &= \min\{j \in \mathbb{Z}_+ : \|X_j\| > \delta\lambda \text{ or } \|D_{j+1}\| > 2\delta\lambda\}\end{aligned}$$

where Lemma 6.4 guarantees that $\{\sigma = j\} \in \mathcal{F}_j$ for each $j \in \mathbb{Z}_+$. Define

$$v_j = 1_{\{\tau_1 < j \le \tau_2 \wedge \sigma\}},$$

and note that $\{\tau_1 < j \le \tau_2 \wedge \sigma\}$ is the intersection of the complements of $\{\tau_2 \wedge \sigma > j\}$ and $\{\tau_1 > j-1\}$, both of which lie in $\mathcal{F}_{j-1}$. Hence $v = (v_j)_{j=1}^\infty$ is predictable and so $v \star X$ is a martingale. When $\tau_1 < \tau_2 \wedge \sigma$ we calculate

$$(v\star X)_j = \sum_{k=1}^{j} v_k D_k = \sum_{k=\tau_1+1}^{\tau_2\wedge\sigma\wedge j} (X_k - X_{k-1}) = \begin{cases} 0, & 1 \le j \le \tau_1,\\ X_j - X_{\tau_1}, & \tau_1 < j \le \tau_2 \wedge \sigma,\\ X_{\tau_2\wedge\sigma} - X_{\tau_1}, & j > \tau_2 \wedge \sigma.\end{cases}$$

We first show that

$$\{X_R^* > \beta\lambda,\, X^* \le \delta\lambda\} \subset \{(v \star X)_R^* > (\beta - 2\delta - 1)\lambda\}.$$

Suppose that $X_R^* > \beta\lambda$ and $X^* \le \delta\lambda$. Now $\tau_2 < \infty$ and as $\|D_{j+1}\| \le \|X_{j+1}\| + \|X_j\| \le 2\delta\lambda$ for all j, we also have $\sigma = \infty$. Since for every j,

$$\mathcal{R}(X_k : 1 \le k \le j) \le \mathcal{R}(X_k : 1 \le k \le j-1) + \|D_j\|,$$

we have

$$\mathcal{R}(X_k : 1 \le k \le \tau_2 - 1) \ge \mathcal{R}(X_k : 1 \le k \le \tau_2) - \|D_{\tau_2}\| > (\beta - 2\delta)\lambda > \lambda.$$

Thus $\tau_1 < \tau_2$ and

$$(v \star X)_j = \begin{cases} 0, & 1 \le j \le \tau_1,\\ X_j - X_{\tau_1}, & \tau_1 < j \le \tau_2,\\ X_{\tau_2} - X_{\tau_1}, & j > \tau_2.\end{cases}$$

Hence

$$\begin{aligned}(v\star X)_R^* &= \mathcal{R}(X_j - X_{\tau_1} : \tau_1 < j \le \tau_2)\\ &\ge \mathcal{R}(X_j : \tau_1 < j \le \tau_2) - \|X_{\tau_1}\|\\ &\ge \mathcal{R}(X_j : 1 \le j \le \tau_2) - \mathcal{R}(X_j : 1 \le j \le \tau_1) - \|X_{\tau_1}\|\\ &\ge \mathcal{R}(X_j : 1 \le j \le \tau_2) - \mathcal{R}(X_j : 1 \le j < \tau_1) - 2\|X_{\tau_1}\|\\ &> \beta\lambda - \lambda - 2\delta\lambda > (\beta - 2\delta - 1)\lambda,\end{aligned}$$

as required.

We then aim to find a suitable upper bound for $\|v\star X\|_1$. To do this, consider the cases $\{\tau_1 < \tau_2 \wedge \sigma\}$ and $\{\tau_1 \ge \tau_2 \wedge \sigma\}$ separately. In the former case, an earlier calculation gives

$$\|(v\star X)_j\| \le \|X_{\tau_2\wedge\sigma\wedge j}\| + \|X_{\tau_1}\|,$$

where $\|X_{\tau_1}\| \le \delta\lambda$. Furthermore

$$\|X_{\tau_2\wedge\sigma\wedge j}\| \le \|X_{\tau_2\wedge\sigma\wedge j-1}\| + \|D_{\tau_2\wedge\sigma\wedge j}\| \le \delta\lambda + 2\delta\lambda$$

and so $\|(v\star X)_j\| \le 4\delta\lambda$ for all $j \in \mathbb{Z}_+$. In the latter case each v_j is 0 and so $(v\star X)_j = 0$. This happens in particular in the event $\{\tau_1 = \infty\} = \{X_R^* \le \lambda\}$. Thus in conclusion

$$(v\star X)^* \le 4\delta\lambda 1_{\{\tau_1<\infty\}}$$

and so

$$\|v\star X\|_1 \le \mathbb{E}(v\star X)^* \le 4\delta\lambda\mathbb{P}(X_R^* > \lambda).$$

Putting all these estimates together we get

$$\begin{aligned}\mathbb{P}(X_R^* > \beta\lambda,\, X^* \le \delta\lambda) &\le \mathbb{P}((v\star X)_R^* > (\beta - 2\delta - 1)\lambda)\\ &\le \frac{C}{(\beta - 2\delta - 1)\lambda}\|v\star X\|_1\\ &\le \frac{4C\delta}{\beta - 2\delta - 1}\mathbb{P}(X_R^* > \lambda).\end{aligned}$$

Fixing a $\beta > 2\delta + 1$ we may take

$$\alpha(\delta) = \frac{4C\delta}{\beta - 2\delta - 1}. \ \blacksquare$$

The previous lemma allows us to deduce the strong type inequality from the weak type inequality:

PROPOSITION 6.6. *Suppose that $\mathcal{X}$ has weak RMF and let $1 < p < \infty$. Then there exists a constant C such that for any L^p-bounded standard Haar martingale X in $\mathcal{X}$ we have $\mathbb{E}|X_R^*|^p \le C^p\|X\|_p^p$.*

Proof. Let $X = (X_j)_{j=1}^N$ be a standard Haar martingale in $\mathcal{X}$ (note that it suffices to prove the claim for finite martingales independently of N). We

apply the good-λ inequality and write

$$\begin{aligned}\mathbb{E}|X_R^*|^p &= \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}(X_R^* > \beta\lambda)\,d\lambda \\ &\le \beta^p\alpha(\delta)\int_0^\infty p\lambda^{p-1}\mathbb{P}(X_R^* > \lambda)\,d\lambda + \beta^p\int_0^\infty p\lambda^{p-1}\mathbb{P}(X^* > \delta\lambda)\,d\lambda \\ &= \beta^p\alpha(\delta)\mathbb{E}|X_R^*|^p + \frac{\beta^p}{\delta^p}\mathbb{E}|X^*|^p,\end{aligned}$$

where $\mathbb{E}|X^*|^p \le C^p\|X\|_p^p$ and $\mathbb{E}|X_R^*|^p$ is finite. Choosing δ so small that $\beta^p\alpha(\delta) < 1$ we get

$$\mathbb{E}|X_R^*|^p \le \frac{\beta^p C^p}{(1-\beta^p\alpha(\delta))\delta^p}\|X\|_p^p. \ \blacksquare$$

We collect our results as follows:

THEOREM 6.7. *The following conditions are equivalent:*

(i) $\mathcal{X}$ *has RMF$_p$ for all* $p \in (1,\infty)$.
(ii) $\mathcal{X}$ *has RMF$_p$ for some* $p \in (1,\infty)$.
(iii) $\mathcal{X}$ *has weak RMF.*

Proof. Trivially the first condition implies the second. That the third follows from the second was Proposition 6.3. In Proposition 6.6 we showed that the weak RMF-property implies that for any $p \in (1,\infty)$, $\mathbb{E}|X_R^*|^p \lesssim \|X\|_p^p$ whenever X is an L^p-bounded standard Haar martingale in $\mathcal{X}$. As was noted before, the filtration of dyadic intervals on $[0,1)$ can be "embedded" in a standard Haar filtration. Thus the weak RMF-property is sufficient for the L^p-boundedness, $1 < p < \infty$, of the Rademacher maximal operator on the unit interval. By Theorem 5.1 this implies RMF$_p$ for all $p \in (1,\infty)$. ∎

7. RMF-property and concave functions. The existence of a biconcave function $v : E \times E \to \mathbb{R}$ for which

$$v(x,y) \ge \left\|\frac{x+y}{2}\right\|^p - C^p\left\|\frac{x-y}{2}\right\|^p$$

can be shown to be equivalent to E being a UMD-space (see [Bur2]). These ideas have been applied (again in [Bur2]) to prove the boundedness of Doob's maximal operator and we will now use them to study the Rademacher maximal function. More precisely, we will show that for a fixed $p \in (1,\infty)$, a constant C satisfies $\mathbb{E}|X_R^*|^p \le C\|X\|_p^p$ for all finite simple martingales $X = (X_j)_{j=1}^N$ in $\mathcal{X}$ if and only if there exists a suitable majorant for the real-valued function

$$u(\mathcal{T},T) = \mathcal{R}(\mathcal{T})^p - C\|T\|^p,$$

defined for finite subsets $\mathcal{T}$ of operators in $\mathcal{X}$ and $T \in \mathcal{X}$. Observe that $\mathbb{E}|X_R^*|^p - C\|X\|_p^p \leq 0$ can equivalently be written as

$$\mathbb{E}u(\{X_j\}_{j=1}^N, X_N) \leq 0,$$

since $\|X\|_p^p = \mathbb{E}\|X_N\|^p$.

PROPOSITION 7.1. *The estimate*

$$\mathbb{E}u(\{X_j\}_{j=1}^N, X_N) \leq 0$$

holds for all finite simple martingales $X = (X_j)_{j=1}^N$ in $\mathcal{X}$ if and only if there exists a function v satisfying

(i) $v(\mathcal{T}, T) \geq u(\mathcal{T}, T)$,
(ii) $v(\{T\}, T) \leq 0$,
(iii) $v(\mathcal{T} \cup \{T\}, T) = v(\mathcal{T}, T)$,
(iv) $v(\mathcal{T}, \cdot)$ *is concave,*

for all finite subsets $\mathcal{T}$ of $\mathcal{X}$ and all $T \in \mathcal{X}$.

The proof of sufficiency is based on the following lemma.

LEMMA 7.2. *Suppose that v is as in Proposition* 7.1 *and $(X_j)_{j=1}^N$ is a simple martingale in $\mathcal{X}$. Then, for all $2 \leq k \leq N$, we have*

$$\mathbb{E}v(\{X_j\}_{j=1}^k, X_k) \leq \mathbb{E}v(\{X_j\}_{j=1}^{k-1}, X_{k-1}).$$

Proof. Let us fix a k and write $\mathcal{F}_j$ for the σ-algebra generated by $X_1, \ldots, X_j$. By the simplicity of $(X_j)_{j=1}^N$, the set $\{X_j\}_{j=1}^{k-1}$ has a finite number s of different possibilities $\mathcal{T}_1, \ldots, \mathcal{T}_s \subset \mathcal{X}$ so that the event A_r of $\mathcal{T}_r$ happening is in $\mathcal{F}_{k-1}$. Now, using the third property of v we get

$$\begin{aligned} v(\{X_j\}_{j=1}^k, X_k) = v(\{X_j\}_{j=1}^{k-1} \cup \{X_k\}, X_k) &= v(\{X_j\}_{j=1}^{k-1}, X_k) \\ &= \sum_{r=1}^s 1_{A_r} v(\mathcal{T}_r, X_k) \end{aligned}$$

and so the fourth property with the aid of Jensen's inequality implies

$$\mathbb{E}(v(\mathcal{T}_r, X_k) \,|\, \mathcal{F}_{k-1}) \leq v(\mathcal{T}_r, \mathbb{E}(X_k \,|\, \mathcal{F}_{k-1})) = v(\mathcal{T}_r, X_{k-1}).$$

Thus

$$\begin{aligned} \mathbb{E}v(\{X_j\}_{j=1}^k, X_k) &= \sum_{r=1}^s \mathbb{E}(1_{A_r} v(\mathcal{T}_r, X_k)) = \sum_{r=1}^s \mathbb{E}(1_{A_r} \mathbb{E}(v(\mathcal{T}_r, X_k) \,|\, \mathcal{F}_{k-1})) \\ &\leq \sum_{r=1}^s \mathbb{E}(1_{A_r} v(\mathcal{T}_r, X_{k-1})) = \mathbb{E}v(\{X_j\}_{j=1}^{k-1}, X_{k-1}), \end{aligned}$$

where the second equality relies on A_r's belonging to $\mathcal{F}_{k-1}$. ∎

Proof of Proposition 7.1. With the aid of Lemma 7.2, the existence of a desired v is now readily seen to imply that

$$\begin{aligned}\mathbb{E}u(\{X_j\}_{j=1}^N, X_N) &\le \mathbb{E}v(\{X_j\}_{j=1}^N, X_N) \le \mathbb{E}v(\{X_j\}_{j=1}^{N-1}, X_{N-1})\\ &\le \cdots \le \mathbb{E}v(\{X_1\}, X_1) \le 0.\end{aligned}$$

On the other hand, the validity of $\mathbb{E}u(\{X_j\}_{j=1}^N, X_N) \le 0$ for finite simple martingales enables us to construct the auxiliary function v with the desired properties by defining

$$v(\mathcal{T}, T) = \sup \mathbb{E}u(\{X_j\}_{j=1}^N \cup \mathcal{T}, X_N),$$

where the supremum is taken over all finite and simple martingales $(X_j)_{j=1}^N$ (where N is allowed to vary) for which $X_1 = T$ almost surely. Let us check that the required properties are satisfied. For the first property, take $N = 1$ and $X_1 = T$ almost surely to see that

$$\begin{aligned}u(\mathcal{T}, T) = \mathcal{R}(\mathcal{T})^p - C\|T\|^p &\le \mathcal{R}(\mathcal{T} \cup \{T\})^p - C\|T\|^p\\ &= \mathbb{E}(\mathcal{R}(\mathcal{T} \cup \{X_1\})^p - C\|X_1\|^p) \le v(\mathcal{T}, T).\end{aligned}$$

For the third one, it suffices to note that if $X_1 = T$ almost surely, then $\{T\} \subset \{X_j\}_{j=1}^N$ almost surely and so $v(\mathcal{T} \cup \{T\}, T) = v(\mathcal{T}, T)$. The second property follows from the assumption and the third property: Let $X = (X_j)_{j=1}^N$ be a simple martingale with $X_1 = T$ almost surely. Now

$$\mathbb{E}u(\{X_j\}_{j=1}^N \cup \emptyset, X_N) \le 0$$

and so $v(\emptyset, T) \le 0$. By the third property,

$$v(\{T\}, T) = v(\emptyset, T) \le 0.$$

To see that $v(\mathcal{T}, \cdot)$ is concave, take operators T_1 and T_2 and put $T = \alpha T_1 + (1-\alpha)T_2$ for some $0 < \alpha < 1$. We need to show that $v(\mathcal{T}, T) \ge \alpha v(\mathcal{T}, T_1) + (1-\alpha)v(\mathcal{T}, T_2)$. To do this, take m_1 and m_2 such that $m_i < v(\mathcal{T}, T_i)$. Now there exist finite simple martingales $(X_j^{(i)})_{j=1}^N$ (defined on the unit interval) such that $X_1^{(i)} = T_i$ almost surely and

$$\mathbb{E}u(\{X_j^{(i)}\}_{j=1}^N \cup \mathcal{T}, X_N^{(i)}) > m_i.$$

Let $X_1 = T$ almost surely and define

$$X_j(t) = \begin{cases} X_{j-1}^{(1)}\left(\dfrac{t}{\alpha}\right), & t \in [0, \alpha),\\[2ex] X_{j-1}^{(2)}\left(\dfrac{t-\alpha}{1-\alpha}\right), & t \in [\alpha, 1),\end{cases}$$

for $j = 2, \ldots, N+1$.

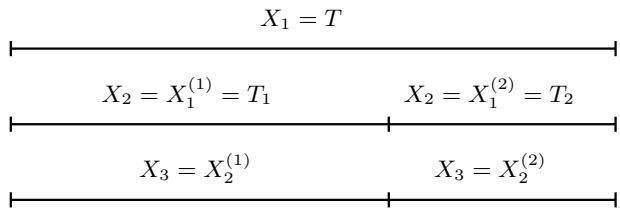

Fig. 3. The construction of X_1, X_2 and X_3

A moment's reflection assures us that $(X_j)_{j=1}^{N+1}$ is also a simple martingale. Now

$$\begin{aligned}
v(\mathcal{T},T) &> \mathbb{E}u(\{X_j\}_{j=1}^{N+1} \cup \mathcal{T}, X_{N+1}) \geq \mathbb{E}u(\{X_j\}_{j=2}^{N+1} \cup \mathcal{T}, X_{N+1}) \\
&= \int_0^\alpha u\left(\left\{X_j^{(1)}\left(\frac{t}{\alpha}\right)\right\}_{j=1}^{N} \cup \mathcal{T}, X_N^{(1)}\left(\frac{t}{\alpha}\right)\right) dt \\
&\quad + \int_\alpha^1 u\left(\left\{X_j^{(2)}\left(\frac{t-\alpha}{1-\alpha}\right)\right\}_{j=1}^{N} \cup \mathcal{T}, X_N^{(2)}\left(\frac{t-\alpha}{1-\alpha}\right)\right) dt \\
&= \alpha \int_0^1 u(\{X_j^{(1)}(s)\}_{j=1}^{N} \cup \mathcal{T}, X_N^{(1)}(s))\, ds \\
&\quad + (1-\alpha) \int_0^1 u(\{X_j^{(2)}(s)\}_{j=1}^{N} \cup \mathcal{T}, X_N^{(2)}(s))\, ds \\
&> \alpha m_1 + (1-\alpha) m_2.
\end{aligned}$$

Letting $m_i \to v(\mathcal{T}, T_i)$ we get concavity. The proof of Proposition 7.1 is now complete. ∎

Remark. Had we assumed in Proposition 7.1 that $\mathbb{E}u(\{X_j\}_{j=1}^{N}, X_N) \leq 0$ holds only for standard Haar martingales, we would have obtained a function v satisfying properties (1)–(3) but for which $v(\mathcal{T}, \cdot)$ is only midpoint concave. Indeed, suppose that the supremum in the definition of v is taken over finite standard Haar martingales and observe that properties other than concavity follow exactly as above. In the proof of midpoint concavity, let $T = (T_1 + T_2)/2$ and define $(X_j)_{j=1}^{2N+1}$ as follows:

$$\begin{aligned}
X_1 &= T \quad \text{almost surely,} \\
X_2(t) &= \begin{cases} X_1^{(1)}(2t) = T_1, & t \in [0, 1/2), \\ X_1^{(2)}(2t-1) = T_2, & t \in [1/2, 1), \end{cases} \\
X_{2j-1}(t) &= \begin{cases} X_j^{(1)}(2t), & t \in [0, 1/2), \\ X_{2j-2}(t), & t \in [1/2, 1), \end{cases}
\end{aligned}$$

$$X_{2j}(t) = \begin{cases} X_{2j-1}(t), & t \in [0, 1/2), \\ X_j^{(2)}(2t-1), & t \in [1/2, 1). \end{cases}$$

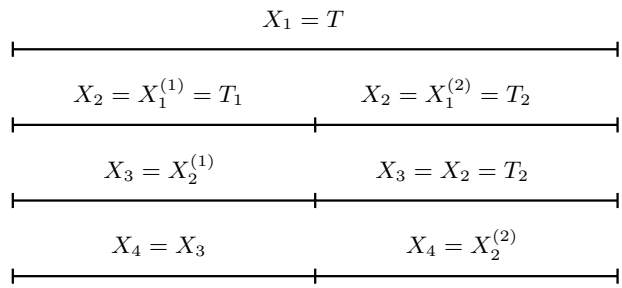

Fig. 4. The construction of X_1, X_2, X_3 and X_4

This way $(X_j)_{j=1}^{2N+1}$ becomes a standard Haar martingale and calculations as in the proof of Proposition 7.1 give us $v(\mathcal{T}, T) \geq v(\mathcal{T}, T_1)/2 + v(\mathcal{T}, T_2)/2$.

In conclusion, we state:

THEOREM 7.3. *Let $1 < p < \infty$. Then $\mathcal{X}$ has RMF$_p$ if and only if there exists a function v such that for some constant C,*

(i) $v(\mathcal{T}, T) \geq \mathcal{R}(\mathcal{T})^p - C\|T\|^p$,
(ii) $v(\{T\}, T) \leq 0$,
(iii) $v(\mathcal{T} \cup \{T\}, T) = v(\mathcal{T}, T)$,
(iv) $v(\mathcal{T}, \cdot)$ *is midpoint concave,*

for all finite subsets $\mathcal{T}$ of $\mathcal{X}$ and all $T \in \mathcal{X}$.

Proof. If $\mathcal{X}$ has RMF$_p$, there exists a constant C is such that $\mathbb{E}|X_R^*|^p \leq C\|X\|_p^p$ for all standard Haar martingales $X = (X_j)_{j=1}^N$ in $\mathcal{X}$. Equivalently,

$$\mathbb{E}(\mathcal{R}(X_j : 1 \leq j \leq N)^p - C\|X_N\|^p) \leq 0$$

for standard Haar martingales $X = (X_j)_{j=1}^N$, which by Proposition 7.1 enables us to construct a desired v.

To show the converse, we first sketch a proof of the known fact that midpoint concave functions that are locally bounded from below are actually concave. Suppose that a function $f : \mathcal{X} \to \mathbb{R}$ is midpoint concave but not concave. Then there exist $T_0, T_1 \in \mathcal{X}$ such that even though

$$f((1-\alpha)T_0 + \alpha T_1) \geq (1-\alpha)f(T_0) + \alpha f(T_1)$$

holds (by induction from midpoint concavity) for all α of the dyadic form $m2^{-k}$ with $k \geq 1$ and $m \in \{1, \ldots, 2^k\}$, it does not hold for some $\alpha' \in (0, 1)$. Assuming with no loss of generality that $f(T_1) \geq f(T_0)$, we claim that such an f cannot be locally bounded from below. Let us write $T_\alpha = (1-\alpha)T_0 + \alpha T_1$

and $c_\alpha = (1-\alpha)f(T_0) + \alpha f(T_1)$ for $\alpha \in (0,1)$ so that $\delta = c_{\alpha'} - f(T_{\alpha'}) > 0$. One can now express α' as the midpoint of an interval $[\alpha, \alpha''] \subset (0,1)$, where α is dyadic and so close to α' that

$$f(T_\alpha) - f(T_{\alpha'}) \geq (1-\alpha)f(T_0) + \alpha f(T_1) - f(T_{\alpha'}) \geq \delta/2.$$

By midpoint concavity of f we have $f(T_{\alpha'}) \geq (f(T_\alpha) + f(T_{\alpha''}))/2$ and so

$$f(T_{\alpha''}) \leq f(T_{\alpha'}) - (f(T_\alpha) - f(T_{\alpha'})) \leq c_{\alpha'} - 3\delta/2 \leq c_{\alpha''} - 3\delta/2,$$

where the last inequality follows from $\alpha'' \geq \alpha'$ by our assumption $f(T_1) \geq f(T_0)$. Hence, starting with an $\alpha' \in (0,1)$ such that $f(T_{\alpha'}) \leq c_{\alpha'} - \delta$ we find an $\alpha'' \in (0,1)$ for which $f(T_{\alpha''}) \leq c_{\alpha''} - 3\delta/2$. Continuing this way we see that f cannot be locally bounded from below since the numbers $c(\alpha)$ are bounded from above by $f(T_1)$.

Suppose then that there exists a function v with the listed properties. By the first property, $v(\mathcal{T}, \cdot)$ is locally bounded from below. Hence v is concave and by Proposition 7.1 we have $\mathbb{E}|X_R^*|^p \leq C\|X\|_p^p$ for all finite simple martingales $X = (X_j)_{j=1}^N$. By Theorem 5.1 (or just by Lemma 5.5), $\mathcal{X}$ has RMF_p. ∎

Observe that this is another way to see that to have the condition $\mathbb{E}|X_R^*|^p \leq C\|X\|_p^p$ for finite simple martingales it suffices to check it for standard Haar martingales.

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