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L_1-uniqueness of degenerate elliptic operators

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Abstract. Let Ω be an open subset of $\mathbb{R}^d$ with $0 \in \Omega$. Furthermore, let $H_\Omega = -\sum_{i,j=1}^d \partial_i c_{ij} \partial_j$ be a second-order partial differential operator with domain $C_c^\infty(\Omega)$ where the coefficients $c_{ij} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ are real, $c_{ij} = c_{ji}$ and the coefficient matrix $C = (c_{ij})$ satisfies bounds $0 < C(x) \le c(|x|)I$ for all $x \in \Omega$. If

$$\int_0^\infty ds\, s^{d/2} e^{-\lambda\mu(s)^2} < \infty$$

for some $\lambda > 0$ where $\mu(s) = \int_0^s dt\, c(t)^{-1/2}$ then we establish that H_Ω is L_1-unique, i.e. it has a unique L_1-extension which generates a continuous semigroup, if and only if it is Markov unique, i.e. it has a unique L_2-extension which generates a submarkovian semigroup. Moreover these uniqueness conditions are equivalent to the capacity of the boundary of Ω, measured with respect to H_Ω, being zero. We also demonstrate that the capacity depends on two gross features, the Hausdorff dimension of subsets A of the boundary of the set and the order of degeneracy of H_Ω at A.

1. Introduction. In a recent paper [RS11] we established that Markov uniqueness and L_1-uniqueness are equivalent properties for a second-order, symmetric, elliptic operator with bounded Lipschitz continuous coefficients c_{ij} on an open subset Ω of $\mathbb{R}^d$. Moreover, these properties hold if and only if the corresponding capacity of the boundary $\partial\Omega$ of Ω is zero. In this note we extend these results to operators with locally bounded coefficients with a possible growth at infinity. As an illustration of our results we establish that Markov uniqueness, L_1-uniqueness and the capacity condition are equivalent if the matrix $C = (c_{ij})$ satisfies $\|C(x)\| \sim |x|^2(\log|x|)^\alpha$ as $|x| \to \infty$ with $\alpha \in [0,1]$. In addition we give an example with $\|C(x)\| \sim |x|^2(\log|x|)^{1+\varepsilon}$ as $|x| \to \infty$, where $\varepsilon > 0$ is arbitrarily small, which is Markov unique but not L_1-unique. Our results extend uniqueness criteria previously established for the special case $\Omega = \mathbb{R}^d$ (see [Dav85], [Ebe99, Chapter 2], [Sta99, Section 2], and references therein).

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Let Ω be an open subset of $\mathbb{R}^d$. Define H_Ω as the positive symmetric operator on $L_2(\Omega)$ with domain $D(H_\Omega) = C_c^\infty(\Omega)$ and action

$$H_\Omega\varphi = -\sum_{i,j=1}^{d} \partial_i c_{ij} \partial_j \varphi$$

where $\partial_i = \partial/\partial x_i$ and the coefficients c_{ij} satisfy

1. $c_{ij} = c_{ji} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ are real,
2. $C(x) = (c_{ij}(x))$ is a strictly positive-definite matrix for all $x \in \Omega$.

Here $W^{s,p}_{\mathrm{loc}}(\Omega)$ denotes the local version of the usual Sobolev spaces and $W^{s,p}_{\mathrm{loc}}(\overline{\Omega})$ denotes the restriction to Ω of functions in $W^{s,p}_{\mathrm{loc}}(\mathbb{R}^d)$. The class of operators defined above will be denoted by $\mathcal{E}_\Omega$.

It follows that each $H_\Omega \in \mathcal{E}_\Omega$ is locally strongly elliptic, i.e. for each relatively compact $V \subset \Omega$ there are $\mu_V, \lambda_V > 0$ such that $\mu_V I \le C(x) \le \lambda_V I$ for all $x \in V$. There are, however, two potential sources of degeneracy. It is possible that $c_{ij}(x) \to 0$ as $x \to \partial\Omega$ or that $c_{ij}(x) \to \infty$ as $|x| \to \infty$.

In order to control the possible growth of the coefficients at infinity we first choose coordinates such that $0 \in \Omega$ and then introduce the strictly positive non-decreasing function c by

$$r \in \langle 0, \infty\rangle \mapsto c(r) = \sup\{\|C(x)\| : x \in \Omega,\ |x| < r\} \tag{1.1}$$

where $\|C(x)\|$ denotes the norm of the matrix $C(x) = (c_{ij}(x))$. It follows that $\|C(x)\| \le c(|x|)$ and $c(0_+) > 0$. The growth conditions will be expressed either explicitly or implicitly in terms of the asymptotic properties of the positive increasing function μ given by

$$s \in \langle 0, \infty\rangle \mapsto \mu(s) = \int_0^s dt\, c(t)^{-1/2}. \tag{1.2}$$

This function is a lower bound on the Riemannian distance to infinity measured with respect to the metric C^{-1}. If, for example, $c(s) \sim s^2(\log s)^\alpha$ as $s \to \infty$ with $\alpha \in [0, 2\rangle$ then $\mu(s) \sim (\log s)^{1-\alpha/2} \to \infty$ as $s \to \infty$.

We are interested in criteria for various uniqueness properties of H_Ω and adopt the terminology of [Ebe99]. In particular H_Ω, viewed as an operator on $L_p(\Omega)$ for $p \in [1, \infty]$, is defined to be L_p-unique if it has a unique extension which generates an L_p-continuous semigroup. Moreover, it is defined to be Markov unique if it has a unique self-adjoint extension on $L_2(\Omega)$ which generates a submarkovian semigroup, i.e. an L_2-continuous contraction semigroup S with the property that $0 \le S_t\varphi \le \mathbb{1}$ whenever $0 \le \varphi \le \mathbb{1}$. It follows that H_Ω is L_2-unique if and only if it is essentially self-adjoint (see [Ebe99, Corollary 1.1.2]). Then the self-adjoint closure is automatically submarkovian and H_Ω is Markov unique. Moreover, if H_Ω is L_1-unique then it is Markov unique ([Ebe99, Lemma 1.1.6]).

First, introduce the positive quadratic form h_Ω associated with H_Ω by

$$h_\Omega(\varphi) = \sum_{i,j=1}^{d} \int_\Omega dx\, c_{ij}(x)(\partial_i\varphi)(x)(\partial_j\varphi)(x) = (\varphi, H_\Omega\varphi)$$

with domain $D(h_\Omega) = D(H_\Omega) = C_c^\infty(\Omega)$. Since h_Ω is the form of the symmetric operator H_Ω it is closable with respect to the graph norm $\varphi \mapsto \|\varphi\|_{D(h_\Omega)} = (h_\Omega(\varphi) + \|\varphi\|_2^2)^{1/2}$. In what follows we use the well known relationship between positive closed quadratic forms and positive self-adjoint operators (see [Kat80, Chapter 6]) together with the corresponding theory of Dirichlet forms and submarkovian operators (see [BH91], [MR92], [FOT94]). The closure $\overline{h}_\Omega$ of h_Ω is automatically a Dirichlet form and the corresponding positive self-adjoint operator, the Friedrichs extension H_Ω^F of H_Ω, is submarkovian. Formally H_Ω^F corresponds to the self-adjoint extension of H_Ω with Dirichlet conditions on the boundary $\partial\Omega$ of Ω. In order to emphasize this interpretation we adopt the alternative notation $H_{\Omega,D} = H_\Omega^F$ and $h_{\Omega,D} = \overline{h}_\Omega$.

Secondly, we introduce a positive self-adjoint extension of H_Ω related to Neumann boundary conditions. Let $\chi \in C_c^\infty(\Omega)$ with $0 \le \chi \le \mathbb{1}_\Omega$ and define $h_{\Omega,\chi}$ as the form of the symmetric operator on $L_2(\Omega)$ with coefficients χc_{ij}. Then $h_{\Omega,\chi}$ is closable, its closure $\overline{h}_{\Omega,\chi}$ is a Dirichlet form and $\overline{h}_{\Omega,\chi} \le h_{\Omega,D}$. Next set $\mathcal{C}_\Omega = \{\chi \in C_c^\infty(\Omega) : 0 \le \chi \le \mathbb{1}_\Omega\}$. It follows that $\mathcal{C}_\Omega$ is a convex set which is directed with respect to the natural order, and if $\chi, \eta \in \mathcal{C}_\Omega$ with $\chi \le \eta$ then $\overline{h}_{\Omega,\chi} \le \overline{h}_{\Omega,\eta}$. Now we define $h_{\Omega,N}$ by

$$h_{\Omega,N}(\varphi) = \lim\{\overline{h}_{\Omega,\chi}(\varphi) : \chi \in \mathcal{C}_\Omega\} = \sup\{\overline{h}_{\Omega,\chi}(\varphi) : \chi \in \mathcal{C}_\Omega\}. \tag{1.3}$$

Since $h_{\Omega,N}$ is the limit of quadratic forms it is a quadratic form, and since it is the supremum of a family of closed forms it is a closed form. It is automatically a Dirichlet form satisfying $h_{\Omega,N} \le h_{\Omega,D}$. If $H_{\Omega,N}$ is the positive self-adjoint operator associated with $h_{\Omega,N}$ it readily follows that $H_{\Omega,N}$ is a submarkovian extension of H_Ω and $H_{\Omega,N} \le H_{\Omega,D}$. If $\partial\Omega$ is smooth, or even Lipschitz continuous, then $H_{\Omega,N}$ corresponds to the extension of H_Ω with Neumann boundary conditions but we adopt this definition for general open Ω.

In order to formulate our main result on uniqueness properties we need two extra definitions.

The operator $H_\Omega \in \mathcal{E}_\Omega$ is defined to be *conservative* if the submarkovian semigroup $S^{\Omega,D}$ generated by $H_{\Omega,D}$ is conservative, i.e. if $S_t^{\Omega,D}\mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $t > 0$. Moreover, the *capacity* of the measurable subset $A \subset \overline{\Omega}$ relative to the operator H_Ω is defined by

$$\begin{aligned}\operatorname{cap}_\Omega(A) = \inf\{\|\psi\|^2_{D(h_{\Omega,N})} : \psi \in D(h_{\Omega,N}) \text{ and there exists an open set}\\ U \subset \mathbb{R}^d \text{ such that } U \supseteq A \text{ and } \psi \ge 1 \text{ a.e. on } U \cap \Omega\}.\end{aligned}$$

Thus cap_Ω corresponds to the capacity relative to the Dirichlet form $h_{\Omega,N}$ as defined in [BH91] or [FOT94].

THEOREM 1.1. *Assume $H_\Omega \in \mathcal{E}_\Omega$. Consider the following conditions:*

I. *H_Ω is conservative,*
II. *H_Ω is L_1-unique,*
III. *H_Ω is Markov unique,*
IV. $\mathrm{cap}_\Omega(\partial\Omega) = 0$.

Then I⇔II⇒III⇒IV.

Conversely, if $\mu(s) \to \infty$ as $s \to \infty$ then IV⇒III. *If in turn*

$$\int_0^\infty ds\, s^{d/2} e^{-\lambda\mu(s)^2} < \infty \tag{1.4}$$

for one $\lambda > 0$ then IV⇒III⇒I *and all four conditions are equivalent.*

Since μ is a positive increasing function with $\mu(0_+) = 0$ the finiteness restriction (1.4) is a condition on the growth μ at infinity, i.e. an implicit condition on the possible growth of the coefficients of H_Ω. If the coefficients are uniformly bounded then $\mu(s) = O(s)$ as $s \to \infty$ and (1.4) is satisfied. Then the four conditions of Theorem 1.1 are equivalent. This retrieves the results of Theorems 1.2 and 1.3 of [RS11].

The theorem is in part a restatement of standard results. The equivalence of conditions I and II was established by Davies [Dav85, Theorem 2.2], whose arguments were based on earlier results of Azencott [Aze74]. Although Davies' assumptions were somewhat different his arguments apply with little modification to the current setting. The implication II⇒III is a straightforward result which is established, for example, in [Ebe99, Lemma 1.6]. The implication III⇒IV follows as in the proof of Theorem 1.2 in [RS11] for operators with $c_{ij} \in W^{1,\infty}(\Omega)$.

In the special case $c(s) \sim s^2(\log s)^\alpha$ for large s it follows that $\mu(s) \sim (\log s)^{1-\alpha/2}$ and $\mu(s) \to \infty$ as $s \to \infty$ for $\alpha \in [0, 2\rangle$. On the other hand if $\alpha \in [0, 1]$ then (1.4) is satisfied for all sufficiently large $\lambda > 0$. Thus if $\alpha \in [0, 2\rangle$ then Markov uniqueness of H_Ω is equivalent to the capacity of the boundary being zero, and if $\alpha \in [0, 1]$ then it is also equivalent to L_1-uniqueness of H_Ω.

Note that if $\Omega = \mathbb{R}^d$, the capacity condition is clearly satisfied and one concludes that $H_{\mathbb{R}^d}$ is L_1-unique whenever (1.4) is satisfied for one large $\lambda > 0$. More generally we establish in Section 4.2 that the capacity condition depends on the Hausdorff dimension of bounded subsets $A \subset \partial\Omega$ and the order of degeneracy of H_Ω at A.

2. Submarkovian extensions. The Friedrichs extension $H_{\Omega,D}$ of H_Ω is well known to be the largest submarkovian extension, i.e. the extension with the minimal form domain. In this section we examine some basic properties of the smallest submarkovian extension, i.e. the extension with the maximal form domain (see [FOT94, Section 3.3.3], [Ebe99, Section 3c] or [RS11, Section 3]). In particular we identify $H_{\Omega,N}$ as the smallest submarkovian extension.

We begin by discussing the imposition of Neumann boundary conditions on a general submarkovian extension K_Ω of $H_\Omega \in \mathcal{E}_\Omega$. Let k_Ω be the Dirichlet form corresponding to K_Ω. Then $D(k_\Omega) \cap L_\infty(\Omega)$ is an algebra. Clearly $C_c^\infty(\Omega) \subseteq D(k_\Omega) \cap L_\infty(\Omega)$. Thus one can define the truncated form $k_{\Omega,\chi}$ for each $\chi \in C_c^\infty(\Omega)$ by $D(k_{\Omega,\chi}) = D(k_\Omega) \cap L_\infty(\Omega)$ and

$$k_{\Omega,\chi}(\varphi) = k_\Omega(\varphi, \chi\varphi) - 2^{-1} k_\Omega(\chi, \varphi^2) \tag{2.1}$$

for $\varphi \in D(k_{\Omega,\chi})$. The $k_{\Omega,\chi}$ have many properties similar to those of the forms $h_{\Omega,\chi}$. In particular they are Markovian forms satisfying $0 \le k_{\Omega,\chi} \le k_\Omega$. Moreover, if $\chi_1, \chi_2 \in \mathcal{C}_\Omega$ and $\chi_1 \le \chi_2$ then $k_{\Omega,\chi_1} \le k_{\Omega,\chi_2}$ (see [BH91, Proposition I.4.1.1]). But it is not evident that the $k_{\Omega,\chi}$ are closable. This, however, is part of our first result.

THEOREM 2.1. *Let $H_\Omega \in \mathcal{E}_\Omega$. Further let K_Ω be a submarkovian extension of H_Ω, and k_Ω the corresponding Dirichlet form. If $\chi \in C_c^\infty(\Omega)$ then the truncated form $k_{\Omega,\chi}$ defined by* (2.1) *is closable and the closure $\overline{k}_{\Omega,\chi}$ satisfies $\overline{k}_{\Omega,\chi} = \overline{h}_{\Omega,\chi}$. Therefore*

$$h_{\Omega,N} \le k_\Omega \le h_{\Omega,D}.$$

In particular H_Ω is Markov unique if and only if $h_{\Omega,N} = h_{\Omega,D}$.

Proof. The first step in the proof is a regularity property which extends a similar result for operators with bounded coefficients given by Theorem 1.1.IV in [RS11].

LEMMA 2.2. *Let K_Ω be a positive, self-adjoint extension of H_Ω. Then*

$$C_c^\infty(\Omega) D(K_\Omega) \subseteq D(\overline{H}_\Omega).$$

Proof. First, if K_Ω is a self-adjoint extension of H_Ω then $H_\Omega \subseteq K_\Omega \subseteq H_\Omega^*$. Therefore it suffices to establish that $C_c^\infty(\Omega) D(H_\Omega^*) \subseteq D(\overline{H}_\Omega)$. This property was proved for operators with bounded Lipschitz coefficients in Theorem 2.1 of [RS11] but the proof is also valid for operators with coefficients which are only locally bounded. For example, if $\eta \in C_c^\infty(\Omega)$ with $\operatorname{supp} \eta = K$ and V is a relatively compact subset of Ω with $K \subset V$ then to deduce that $\eta D(H_\Omega^*) \subseteq D(\overline{H}_\Omega)$ it suffices to prove that $\eta D(H_V^*) \subseteq D(\overline{H}_V)$ where H_V is the restriction of H_Ω to $C_c^\infty(V)$. Since the coefficients of H_V are uniformly bounded the result follows from Theorem 2.1 of [RS11]. ■

Next we prove the first statement of Theorem 2.1.

LEMMA 2.3. *If* $\chi \in C_c^\infty(\Omega)$ *then* $k_{\Omega,\chi}$ *is closable and the closure* $\overline{k}_{\Omega,\chi}$ *satisfies* $\overline{k}_{\Omega,\chi} = \overline{h}_{\Omega,\chi}$.

Proof. First $C_c^\infty(\Omega)D(K_\Omega) \subseteq D(\overline{H}_\Omega)$ by Lemma 2.2. Now fix $\varphi \in D(K_\Omega) \cap L_\infty(\Omega)$. Then for each $\chi \in \mathcal{C}_\Omega$ one has $\chi\varphi \in D(\overline{H}_\Omega)$. Moreover,

$$k_\Omega(\varphi, \chi\varphi) = (K_\Omega\varphi, \chi\varphi) = (\varphi, \overline{H}_\Omega\chi\varphi),$$
$$k_\Omega(\chi, \varphi^2) = (K_\Omega\chi, \varphi^2) = (H_\Omega\chi, \varphi^2).$$

Therefore

$$k_{\Omega,\chi}(\varphi) = (\varphi, \overline{H}_\Omega\chi\varphi) - 2^{-1}(H_\Omega\chi, \varphi^2).$$

Next choose a $\chi_1 \in \mathcal{C}_\Omega$ with $\chi_1 = 1$ on $\operatorname{supp}\chi$ and set $\varphi_1 = \chi_1\varphi$. It follows from Lemma 2.2 that $\varphi_1 \in D(\overline{H}_\Omega) \cap L_\infty(\Omega)$. Moreover,

$$\begin{aligned} k_{\Omega,\chi}(\varphi) &= (\varphi, \overline{H}_\Omega\chi\varphi_1) - 2^{-1}(H_\Omega\chi, \varphi_1^2) \\ &= (\varphi_1, \overline{H}_\Omega\chi\varphi_1) - 2^{-1}(H_\Omega\chi, \varphi_1^2) = \overline{h}_{\Omega,\chi}(\varphi_1). \end{aligned}$$

The first equality is obvious since $\operatorname{supp}\overline{H}_\Omega\chi \subset \operatorname{supp}\chi$. The second equality follows by approximating φ in $L_2(\Omega)$ by a sequence $\varphi_n \in C_c^\infty(\Omega)$ and noting that

$$(\varphi_n, \overline{H}_\Omega\chi\varphi_1) = (H_\Omega\varphi_n, \chi\varphi_1) = (H_\Omega\chi_1\varphi_n, \chi\varphi_1) = (\chi_1\varphi_n, \overline{H}_\Omega\chi\varphi_1).$$

The third equality is also obvious. But for χ and φ fixed, $\overline{h}_{\Omega,\chi}(\chi_1\varphi)$ is independent of the choice of χ_1. Moreover, if χ_2 is a second choice with $\chi_2 = 1$ on $\operatorname{supp}\chi$ then $\chi_1 - \chi_2 = 0$ on $\operatorname{supp}\chi$ and $\overline{h}_{\Omega,\chi}((\chi_1 - \chi_2)\varphi) = 0$. Therefore if $\chi_1 \le \chi_2 \le \cdots \le \mathbb{1}_\Omega$ is an increasing family of C_c^∞-functions with $\chi_n = 1$ on $\operatorname{supp}\chi$ then $\overline{h}_{\Omega,\chi}((\chi_n - \chi_m)\varphi) = 0$ but $\|\chi_n\varphi - \varphi\|_2 \to 0$. This establishes that $\varphi \in D(\overline{h}_{\Omega,\chi})$ and $\overline{h}_{\Omega,\chi}(\varphi) = k_{\Omega,\chi}(\varphi)$. Then, however,

$$\overline{h}_{\Omega,\chi}(\varphi) = k_{\Omega,\chi}(\varphi) \le k_\Omega(\varphi)$$

for all $\varphi \in D(K_\Omega) \cap L_\infty(\Omega)$. Since $D(K_\Omega)$ is a core of k_Ω it follows by continuity that $\overline{h}_{\Omega,\chi}(\varphi) = k_{\Omega,\chi}(\varphi)$ for all $\varphi \in D(k_\Omega) \cap L_\infty(\Omega) = D(k_{\Omega,\chi})$. Therefore $\overline{h}_{\Omega,\chi}$ is a closed extension of $k_{\Omega,\chi}$. Thus $k_{\Omega,\chi}$ is closable and its closure satisfies $\overline{k}_{\Omega,\chi} \subseteq \overline{h}_{\Omega,\chi}$.

But $h_{\Omega,\chi}(\psi) = k_{\Omega,\chi}(\psi)$ for all $\psi \in C_c^\infty(\Omega)$ and $C_c^\infty(\Omega)$ is a core of $\overline{h}_{\Omega,\chi}$ by definition. Therefore $\overline{k}_{\Omega,\chi} \supseteq \overline{h}_{\Omega,\chi}$. Combination of these conclusions gives $\overline{k}_{\Omega,\chi} = \overline{h}_{\Omega,\chi}$. ∎

One can now immediately deduce Theorem 2.1. The first statement has been established by Lemma 2.3. Hence

$$h_{\Omega,N} = \sup_{\chi\in\mathcal{C}_\Omega} \overline{k}_{\Omega,\chi}.$$

But $k_{\Omega,\chi} \le k_\Omega$ for all $\chi \in \mathcal{C}_\Omega$. Therefore $h_{\Omega,N} \le k_\Omega$. Finally $k_\Omega \supseteq h_\Omega$. Hence $k_\Omega \le \overline{h}_\Omega = h_{\Omega,D}$. ∎

The form $h_{\Omega,N}$ has a *carré du champ* in the sense of [BH91, Section I.4]. This is initially defined as the bilinear form from $W^{1,2}_{\rm loc}(\Omega) \times W^{1,2}_{\rm loc}(\Omega)$ into $L_{1,{\rm loc}}(\Omega)$ given by

$$\Gamma(\varphi;\psi)(x) = \sum_{i,j=1}^{d} c_{ij}(x)(\partial_i\varphi)(x)(\partial_j\psi)(x)$$

and $\Gamma(\varphi) = \Gamma(\varphi;\varphi)$. Then

$$D(h_{\Omega,N}) = \Big\{\varphi \in W^{1,2}_{\rm loc}(\Omega) : \sup_V \int_V dx\, \Gamma(\varphi)(x) < \infty\Big\},$$

where the supremum is over all relatively compact subsets V of Ω and

$$h_{\Omega,N}(\varphi) = \sup_V \int_V dx\, \Gamma(\varphi)(x)$$

for all $\varphi \in D(h_{\Omega,N})$. It follows readily that if $\varphi \in D(h_{\Omega,N})$ then $\Gamma(\varphi)$ is a positive $L_1(\Omega)$-function with $\|\Gamma(\varphi)\|_1 = h_{\Omega,N}(\varphi)$. The foregoing explicit identification of the form of the minimal extension has been used in previous discussions of Markov uniqueness: [FOT94, Section 3.3.3], [Ebe99, Section 3c] or [RS11, Section 3].

A number of properties of general submarkovian extensions follows from the identification of the minimal extension. If k_Ω is the form of the submarkovian extension K_Ω of H_Ω it follows from Theorem 2.1 that $D(k_\Omega) \subseteq D(h_{\Omega,N})$. Therefore k_Ω has a *carré du champ* since $\Gamma(\varphi) \in L_1(\Omega)$ for all $\varphi \in D(k_\Omega)$. Moreover, $k_\Omega(\varphi) = \|\Gamma(\varphi)\|_1$ for all $\varphi \in D(k_\Omega)$. Further the form $h_{\Omega,N}$ is strongly local in the sense of [FOT94] and hence the restriction k_Ω is also strongly local.

Subsequently we need two Dirichlet form implications of the elliptic regularity property.

COROLLARY 2.4. *Let K_Ω be a submarkovian extension of $H_\Omega \in \mathcal{E}_\Omega$ and k_Ω the corresponding Dirichlet form. Then*

$$C_c^\infty(\Omega)D(k_\Omega) \subseteq D(\overline{h}_\Omega).$$

Proof. If $\eta \in C_c^\infty(\Omega)$ and $\varphi \in D(K_\Omega)$ then it follows from Lemma 2.2 that $\eta\varphi \in D(\overline{H}_\Omega) \subset D(K_\Omega) \subseteq D(k_\Omega) \subseteq D(h_{\Omega,N})$. Moreover,

$$\begin{aligned}\overline{h}_\Omega(\eta\varphi) = h_{\Omega,N}(\eta\varphi) &\leq 2\int_\Omega \Gamma(\eta)\varphi^2 + 2\int_\Omega \eta^2\Gamma(\varphi)\\ &\leq 2(\|\Gamma(\eta)\|_\infty + \|\eta\|_\infty^2)\|\varphi\|^2_{D(k_\Omega)}.\end{aligned}$$

Since $D(K_\Omega)$ is a core of k_Ω with respect to the $D(k_\Omega)$-graph norm, this estimate extends to all $\varphi \in D(k_\Omega)$ by continuity. The statement of the corollary follows immediately. ∎

COROLLARY 2.5. *If $H_\Omega \in \mathcal{E}_\Omega$ then $C_c^\infty(\mathbb{R}^d)D(h_{\Omega,N}) \subseteq D(h_{\Omega,N})$.*

Proof. Fix $\rho \in C_c^\infty(\mathbb{R}^d)$ and $\chi \in \mathcal{C}_\Omega$. If $\varphi \in C_c^\infty(\Omega)$, then $\rho\varphi \in C_c^\infty(\Omega)$ and $\|\rho\varphi\|_2 \le \|\rho\|_\infty \|\varphi\|_2$. Moreover,

$$h_{\Omega,\chi}(\rho\varphi) \le 2\|\rho\|_\infty^2 h_{\Omega,\chi}(\varphi) + 2a_\rho \|\nabla\rho\|_\infty^2 \|\varphi\|_2^2$$

where $a_\rho = \sup_{x\in\operatorname{supp}\rho} \|C(x)\|$. Therefore, by continuity, $\rho D(\overline{h}_{\Omega,\chi}) \subseteq D(\overline{h}_{\Omega,\chi})$ and

$$\|\rho\varphi\|_{D(\overline{h}_{\Omega,\chi})} \le a(\rho)\|\varphi\|_{D(\overline{h}_{\Omega,\chi})}$$

for all $\varphi \in D(\overline{h}_{\Omega,\chi})$ with $a(\rho) = 2(a_\rho\|\nabla\rho\|_\infty^2 + \|\rho\|_\infty^2)$. Since this estimate is uniform for $\chi \in \mathcal{C}_\Omega$ it follows that $\rho D(h_{\Omega,N}) \subseteq D(h_{\Omega,N})$ and $\|\rho\varphi\|_{D(h_{\Omega,N})} \le a(\rho)\|\varphi\|_{D(h_{\Omega,N})}$. ∎

3. L_1-uniqueness. In this section we prove Theorem 1.1. Much of the proof consists of refinements of previous arguments.

I⇔II. This equivalence was established by Davies [Dav85, Theorem 2.2] for a large class of second-order elliptic operators with smooth coefficients. But his arguments extend to the current situation with only minor modifications. We omit further details.

II⇒III. This is a general structural result which is proved, for example, in Lemma 1.1.6 of [Ebe99].

III⇒IV. First note that Markov uniqueness of H_Ω is equivalent to the identity $h_{\Omega,N} = h_{\Omega,D}$ by Theorem 2.1. But in general $h_{\Omega,N} \supseteq h_{\Omega,D}$ and $C_c^\infty(\Omega)$ is a core of $h_{\Omega,D}$. Therefore Markov uniqueness of H_Ω implies that $C_c^\infty(\Omega)$ is a core of $h_{\Omega,N}$.

Secondly, let $\psi \in D(h_{\Omega,N}) \cap L_\infty(\Omega)$ with $\psi = 1$ on $U \cap \Omega$ where U is an open subset containing $\partial\Omega$. Then since $C_c^\infty(\Omega)$ is a core of $h_{\Omega,N}$ there is a sequence $\psi_n \in C_c^\infty(\Omega)$ such that $\|\psi - \psi_n\|_{D(h_{\Omega,N})} \to 0$ as $n \to \infty$. Set $\varphi_n = \psi - \psi_n$. Then $\varphi_n \in D(h_{\Omega,N})$, $\|\varphi_n\|_{D(h_{\Omega,N})} \to 0$ and since ψ_n has compact support there is an open subset U_n containing $\partial\Omega$ such that $\varphi_n = 1$ on $(U \cap U_n) \cap \Omega$. Therefore $\operatorname{cap}_\Omega(\partial\Omega) = 0$.

Combination of the foregoing observations establishes the first statement of Theorem 1.1. Now we turn to the proof of the second statement.

IV⇒III. Assume $\mu(s) \to \infty$ as $s \to \infty$ where μ is defined by (1.1) and (1.2). Then H_Ω is Markov unique if and only if $C_c^\infty(\Omega)$ is a core of $h_{\Omega,N}$. Thus it is necessary to demonstrate that each $\varphi \in D(h_{\Omega,N}) \cap L_\infty(\Omega)$ can be approximated in the $D(h_{\Omega,N})$-graph norm by a sequence $\varphi_n \in C_c^\infty(\Omega)$.

Fix $\varphi \in D(h_{\Omega,N}) \cap L_\infty(\Omega)$. Next fix $\rho \in C_c^\infty(\mathbb{R})$ with $0 \le \rho \le 1$, $\rho(s) = 1$ if $s \le 1$ and $\rho(s) = 0$ if $s \ge 2$. Then define ρ_n by $\rho_n(x) = \rho(n^{-1}\mu(|x|))$. It follows that $\rho_n(x) = 1$ if $\mu(|x|) \le n$ and $\rho(x) = 0$ if $\mu(|x|) \ge 2n$. Moreover, $\|\Gamma(\rho_n)\|_\infty \le b^2 n^{-2}$ with $b = \|\rho'\|_\infty$. Then $\rho_n\varphi \in D(h_{\Omega,N}) \cap L_\infty(\Omega)$ by

Corollary 2.5 and

$$\begin{aligned}\|\varphi-\rho_n\varphi\|^2_{D(h_{\Omega,N})}&\le 2\int_\Omega \varphi^2\Gamma(\rho_n)+2\int_\Omega(\mathbb{1}_\Omega-\rho_n)^2\Gamma(\varphi)+\|(\mathbb{1}_\Omega-\rho_n)\varphi\|_2^2\\&\le 2b^2n^{-2}\|\varphi\|_2^2+\int_\Omega(\mathbb{1}_\Omega-\rho_n)^2(2\Gamma(\varphi)+\varphi^2).\end{aligned}$$

The first term on the right hand side clearly tends to zero as $n\to\infty$. But it follows by construction that $(\mathbb{1}_\Omega-\rho_n)^2\to 0$ pointwise on Ω. Therefore the second term also tends to zero by the Lebesgue dominated convergence theorem. Thus φ is approximated by the sequence $\rho_n\varphi$ in the $D(h_{\Omega,N})$-graph norm.

Next since $\mathrm{cap}_\Omega(\partial\Omega)=0$ one may choose $\chi_n\in D(h_{\Omega,N})$ and open subsets $U_n\supset\partial\Omega$ such that $0\le\chi_n\le 1$, $\|\chi_n\|_{D(h_{\Omega,N})}\le n^{-1}$ and $\chi_n\ge 1$ on $U_n\cap\Omega$. But since $h_{\Omega,N}$ is a Dirichlet form one may assume $\chi_n=1$ on $U_n\cap\Omega$. Then with $\varphi_n=(\mathbb{1}_\Omega-\chi_n)\rho_n\varphi$ one has

$$\lim_{n\to\infty}\|\varphi-\varphi_n\|_{D(h_{\Omega,N})}\le\lim_{n\to\infty}\|\chi_n\rho_n\varphi\|_{D(h_{\Omega,N})}$$

by the Cauchy–Schwarz estimate and the conclusion of the previous paragraph. But

$$\|\chi_n\rho_n\varphi\|^2_{D(h_{\Omega,N})}=h_{\Omega,N}(\chi_n\rho_n\varphi)+\|\chi_n\rho_n\varphi\|_2^2$$

and the second term on the right hand side tends to zero as $n\to\infty$ because $\|\chi_n\rho_n\varphi\|_2\le\|\chi_n\|_2\|\varphi\|_\infty$. The first term on the right can be estimated by

$$\begin{aligned}h_{\Omega,N}(\chi_n\rho_n\varphi)&\le 2\int_\Omega\varphi^2\Gamma(\chi_n)+4\int_\Omega\varphi^2\Gamma(\rho_n)+4\int_\Omega\chi_n^2\Gamma(\varphi)\\&\le 2\|\varphi\|_\infty^2h_{\Omega,N}(\chi_n)+4b^2n^{-2}\|\varphi\|_2^2+4\int_\Omega\chi_n^2\Gamma(\varphi)\end{aligned}$$

since $\|\Gamma(\rho_n)\|_\infty\le b^2n^{-2}$. The first term on the right hand side tends to zero because $h_{\Omega,N}(\chi_n)\le n^{-1}$ and the second obviously tends to zero. Finally the third term tends to zero by an equicontinuity estimate because $\chi_n^2\le 1$ and $\Gamma(\varphi)\in L_1(\Omega)$. Thus one now concludes that φ is approximated by the sequence φ_n in the $D(h_{\Omega,N})$-graph norm.

Finally, $\operatorname{supp}\varphi_n\subseteq\Omega_n=((\operatorname{supp}\rho_n)\cap\Omega)\cap(\Omega\setminus(U_n\cap\Omega))$ and Ω_n is a relatively compact subset of Ω. Therefore $H_{\Omega,N}$ is strongly elliptic in restriction to Ω_n. Consequently φ_n, and hence φ, can be approximated by a sequence of $C_c^\infty(\Omega_n)$-functions in the $D(h_{\Omega,N})$-graph norm.

This completes the proof of the second statement of Theorem 1.1. Now we turn to the proof of the third statement. By the foregoing it suffices to prove the following.

III$\Rightarrow$I. The proof is an elaboration of the argument used to demonstrate the comparable implication in Theorem 1.3 in [RS11].

Proposition 3.1. *Assume $H_\Omega \in \mathcal{E}_\Omega$ is Markov unique. Further assume that*

$$\int_0^\infty ds\, s^{d/2} e^{-\lambda\mu(s)^2} < \infty$$

for one $\lambda > 0$ where $\mu(s) = \int_0^s c^{-1/2}$ with c defined by (1.1). *Then H_Ω is conservative.*

Proof. The proof is in several steps.

Step 1: *Ω bounded.* If Ω is bounded then H_Ω is conservative by Step 1 in the proof of Theorem 1.3 in [RS11]. Therefore we now assume that Ω is unbounded.

Step 2: *Bounded approximation.* The second step consists of introducing an increasing sequence of bounded sets Ω_n and conservative operators $H_{\Omega;n} \in \mathcal{E}_{\Omega_n}$ which approximate H_Ω in a suitable manner.

First, fix $\rho \in C_c^\infty(\mathbb{R})$ with $0 \le \rho \le 1$, $\rho(s) = 1$ if $|s| \le 1$ and $\rho(s) = 0$ if $|s| \ge 2$. Then introduce the sequence ρ_n by $\rho_n(x) = \rho(n^{-1}|x|)$. Thus $\rho_n(x) = 1$ if $|x| \le n$ and $\rho(x) = 0$ if $|x| \ge 2n$. The family of functions ρ_n is monotonically increasing. Set $B_n = \{x \in \mathbb{R}^d : |x| < n\}$ and $\Omega_n = \Omega \cap B_{2n}$. Note that Ω_n is bounded.

Secondly, define $H_{\Omega;n} \in \mathcal{E}_{\Omega_n}$ as the operator with coefficients $\rho_n c_{ij}$ acting on $L_2(\Omega_n)$. Then it follows that $H_{\Omega;n}$ is Markov unique since the capacity of $\partial\Omega_n$ with respect to the Neumann form associated with $H_{\Omega;n}$ is zero. Therefore $H_{\Omega;n}$ is conservative by Step 1. Then if H_n is the extension to $L_2(\Omega)$ of the unique submarkovian extension $H_{\Omega;n,N}$ $(= H_{\Omega;n,D})$ of $H_{\Omega;n}$ acting on $L_2(\Omega_n)$, i.e. if $H_n = H_{\Omega;n,N} \oplus 0$ with $L_2(\Omega) = L_2(\Omega_n) \oplus L_2(\Omega_n)^\perp$, then H_n is conservative.

Step 3: *L_2-convergence.* The third step is to establish strong convergence on $L_2(\Omega)$ of the semigroups $S^{(n)}$ generated by the H_n to the semigroup S generated by the unique submarkovian extension $H_{\Omega,N}$ $(= H_{\Omega,D})$ of H_Ω. This follows by a monotone convergence argument. The closed form h_n corresponding to H_n on $L_2(\Omega)$ is given by $h_n(\varphi) = h_{\Omega;n,N}(\varphi)$ for all $\varphi \in C_c^\infty(\Omega)$ and then by closure for all $\varphi \in D(h_n)$. Since the ρ_n are a monotonically increasing family of functions on $\mathbb{R}^d$ the forms h_n are a monotonically increasing family of Dirichlet forms. If $h = \sup_{n\ge1} h_n$ then h is a Dirichlet form.

It follows from the monotonic increase of the forms h_n that the operators H_n converge in the strong resolvent sense on $L_2(\Omega)$ to the operator H corresponding to h (see, for example, [Kat80, Section VIII.4], or [MR92, Section I.3]). Moreover, the semigroups $S^{(n)}$ converge strongly on $L_2(\Omega)$ to the submarkovian semigroup S generated by H. It also follows readily that H

is a submarkovian extension of H_Ω. Therefore $H = H_{\Omega,N}$ $(= H_{\Omega,D})$, by Markov uniqueness.

Our next aim is to prove that the semigroups $S^{(n)}$ converge strongly to S on $L_1(\Omega)$. Following a tactic used in [RS08], [RS11], we convert the L_2-convergence of the semigroups into L_1-convergence by the use of suitable off-diagonal bounds.

Step 4: *L_2-off-diagonal bounds.* Let

$$D_n = \Big\{\psi \in W^{1,\infty}(\Omega) : \sum_{i,j=1}^{d} \rho_n c_{ij}(\partial_i\psi)(\partial_j\psi) \le 1\Big\}. \tag{3.1}$$

The corresponding Riemannian (pseudo-)distance is defined by

$$d_n(x;y) = \sup_{\psi\in D_n} (\psi(x) - \psi(y)) \tag{3.2}$$

for all $x, y \in \Omega$. This function has the metric properties of a distance but it takes the value infinity if either x or y is not in Ω_n. Secondly, introduce the corresponding set-theoretic distance by

$$d_n(A;B) = \inf_{x\in A,\, y\in B} d_n(x;y)$$

where A and B are general measurable subsets of Ω. Finally define D by setting $\rho_n = \mathbb{1}_\Omega$ in (3.1). Then $D \subseteq D_n$ and the corresponding Riemannian distance $d(\cdot;\cdot)$, defined in analogy with (3.2), satisfies $d(x;y) \le d_n(x;y)$.

Lemma 3.2. *If A, B are open subsets of Ω then*

$$\sup_{n\ge1} |(\varphi_A, S_t^{(n)}\varphi_B)| \vee |(\varphi_A, S_t\varphi_B)| \le e^{-d(A;B)^2(4t)^{-1}} \|\varphi_A\|_2\|\varphi_B\|_2$$

for all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$ and $t > 0$, with the convention $e^{-\infty} = 0$.

Bounds of this type have now been derived by many authors (see, for example, [Aus07], [CGT82], [Dav92], [Gri99], [Stu95], [Stu98]) under a variety of ellipticity assumptions. A proof applicable in the current context can be found in [RS08, Section 4]. The bounds for $S^{(n)}$ are initially in terms of $d_n(A;B)$, but $d_n(A;B) \le d(A;B)$; since the $S_t^{(n)}$ are L_2-convergent to S_t, the bounds also hold for S.

Next $C(x) \le c(|x|)I$ for all $x \in \Omega$. Therefore

$$D_n \supseteq \widehat{D}_n = \{\psi \in W^{1,\infty}(\Omega) : \rho(n^{-1}|x|)c(|x|)|(\nabla\psi)(x)|^2 \le 1\}. \tag{3.3}$$

Consequently,

$$\widehat{d}_n(x;y) = \sup_{\psi\in\widehat{D}_n} (\psi(x) - \psi(y)) \le d_n(x;y) \tag{3.4}$$

for all $x, y \in \Omega$. Moreover, if $\widehat{D}$ is defined by setting $\rho = \mathbb{1}_\Omega$ in (3.3) and $\widehat{d}(\cdot;\cdot)$ is defined in analogy with (3.4) then $\widehat{d}(x;y) \le d(x;y)$ for all $x, y \in \Omega$.

Thus the bounds of Lemma 3.2 are also valid with $d(A;B)$ replaced by $\widehat{d}(A;B)$.

If A,B are bounded open sets with $A \subset \Omega \cap B_m$ and $B \subset \Omega \cap (B_M)^{\mathrm{c}}$ where $M > m \ge 1$, define

$$\rho_m = \sup_{x\in\Omega\cap B_m} \widehat{d}(x;0) \quad \text{and} \quad \nu_M = \inf_{x\in\Omega\cap(B_M)^{\mathrm{c}}} \widehat{d}(x;0).$$

Then it follows from the triangle inequality $\widehat{d}(x;0) \le \widehat{d}(x;y) + \widehat{d}(y;0)$ that

$$\nu_M \le \inf_{x\in B} \widehat{d}(x;0) \le \inf_{x\in B} \widehat{d}(x;y) + \widehat{d}(y;0) \le \inf_{x\in B} \widehat{d}(x;y) + \rho_m$$

for all $y \in A$. Therefore

$$\widehat{d}(A;B) \ge \nu_M - \rho_m \ge 0$$

where the last inequality follows because $M > m$. But it can be seen directly from the definition of $\widehat{d}(\cdot;\cdot)$ that

$$\widehat{d}(x;0) = \int_0^{|x|} ds\, c(s)^{-1/2}$$

for all $x \in \Omega$. Therefore

$$\rho_m = \int_0^m ds\, c(s)^{-1/2} = \mu(m) \quad \text{and} \quad \widehat{\nu}_M = \int_0^M ds\, c(s)^{-1/2} = \mu(M).$$

Hence

$$\widehat{d}(A;B) \ge \mu(M) - \mu(m) \ge 0.$$

Consequently, one has the following variation of Lemma 3.2.

LEMMA 3.3. *If $M > m \ge 1$ and A,B are bounded open sets with $A \subset \Omega \cap B_m$ and $B \subset \Omega \cap (B_M)^{\mathrm{c}}$ then*

$$\sup_{n\ge1} |(\varphi_A, S_t^{(n)}\varphi_B)| \vee |(\varphi_A, S_t\varphi_B)| \le e^{\mu(m)^2(4t)^{-1}} e^{-\mu(M)^2(8t)^{-1}} \|\varphi_A\|_2 \|\varphi_B\|_2$$

for all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$ and $t > 0$.

Proof. The bounds on $|(\varphi_A, S_t\varphi_B)|$ follow directly from the bounds of Lemma 3.2, the foregoing observation that $\widehat{d}(A;B) \ge \mu(M) - \mu(m) \ge 0$ and the estimate

$$(\mu(M) - \mu(m))^2 \ge 2^{-1}\mu(M)^2 - \mu(m)^2.$$

Since $\widehat{d}_n(A;B) \ge \widehat{d}(A;B)$ the bounds on $\sup_{n\ge1} |(\varphi_A, S_t^{(n)}\varphi_B)|$ follow by similar reasoning. We omit further details. ∎

Now we are prepared for the key estimate.

LEMMA 3.4. *There is a $b > 0$ such that if $M > m \geq 1$ then*

$$\sup_{n\geq 1} |(\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t^{(n)}\varphi)| \vee |(\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t\varphi)| \leq b e^{\mu(m)^2(4t)^{-1}} \int_M^\infty ds\, s^{d/2} e^{-\mu(s)^2(8t)^{-1}} \|\varphi\|_2$$

for all $\varphi \in L_2(\Omega \cap B_m)$ and $t > 0$.

Proof. The proof is a variation of an argument of [ERSZ07]. Let $C_p = B_{p+1} \setminus B_p$. It follows that $(B_M)^{\mathrm{c}} = \bigcup_{p\geq M} C_p$. If A is a bounded open set with $\operatorname{supp}\varphi \subset A \subseteq \Omega \cap B_m$ then by Lemma 3.3:

$$\begin{aligned}|(\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t\varphi)| &= \Big|\sum_{p\geq M} (\mathbb{1}_{C_p}, S_t\varphi)\Big| \\ &\leq e^{\mu(m)^2(4t)^{-1}} \sum_{p\geq M} e^{-\mu(p)^2(8t)^{-1}} |B_{p+1}|^{1/2} \|\varphi\|_2.\end{aligned}$$

But the sum is a Riemann approximation to the integral occurring in the statement of the lemma. Therefore the bounds for $|(\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t\varphi)|$ follow immediately. The bounds for $|(\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t^{(n)}\varphi)|$ follow by almost identical reasoning. ∎

STEP 5: *L_1-convergence.* The fifth step consists of proving that the semigroups $S_t^{(n)}$ are strongly convergent on $L_1(\Omega)$ to S_t (see [RS08, Proposition 6.2] for a similar result).

Since the semigroups $S_t^{(n)}$ and S_t are all submarkovian it suffices to prove convergence on a subset of $L_1(\Omega)$ whose span is dense. In particular it suffices to prove convergence on positive functions in $L_1(A) \cap L_2(A)$ for each bounded open subset A of Ω.

Fix $A \subset \Omega \cap B_m$ and $\varphi \in L_1(A) \cap L_2(A)$. Assume φ is positive. Then

$$\begin{aligned}\|(S_t^{(n)} - S_t)\varphi\|_1 &\leq \|\mathbb{1}_{B_M}(S_t^{(n)} - S_t)\varphi\|_1 + \|\mathbb{1}_{(B_M)^{\mathrm{c}}} S_t^{(n)}\varphi\|_1 + \|\mathbb{1}_{(B_M)^{\mathrm{c}}} S_t\varphi\|_1 \\ &\leq |B_M|^{1/2} \|(S_t^{(n)} - S_t)\varphi\|_2 + (\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t^{(n)}\varphi) + (\mathbb{1}_{(B_M)^{\mathrm{c}}}, S_t\varphi) \\ &\leq |B_M|^{1/2} \|(S_t^{(n)} - S_t)\varphi\|_2 \\ &\qquad + 2b e^{\mu(m)^2(4t)^{-1}} \int_M^\infty ds\, s^{d/2} e^{-\mu(s)^2(8t)^{-1}} \|\varphi\|_2\end{aligned}$$

where we have used the positivity of the semigroups and the functions to express the L_1-norms as pairings between L_1 and L_∞. The last step uses Lemma 3.4. But the integral is convergent for one $t = t_0 > 0$, by assumption. Therefore it is convergent for all $t \in \langle 0, t_0]$. Then since $S_t^{(n)}$ is L_2-convergent to S_t for all $t > 0$ and since the last term on the right hand side converges to zero as $M \to \infty$ for each $t \in \langle 0, t_0]$ it follows that $S_t^{(n)}$ is L_1-convergent to S_t for all $t \in \langle 0, t_0]$. Finally, the semigroup property and contractivity imply that $S_t^{(n)}$ is L_1-convergent to S_t for all $t > 0$.

STEP 6: *Conservation.* The conservation property for S now follows because the approximating semigroups $S^{(n)}$ are conservative, by Step 2, and are L_1-convergent to S, by Step 5. Therefore

$$(\mathbb{1}_\Omega, S_t\varphi) = \lim_{n\to\infty} (\mathbb{1}_\Omega, S_t^{(n)}\varphi) = \lim_{n\to\infty} (S_t^{(n)}\mathbb{1}_\Omega, \varphi) = (\mathbb{1}_\Omega, \varphi)$$

for all $\varphi \in L_1(\Omega)$. Hence $S_t\mathbb{1}_\Omega = \mathbb{1}_\Omega$. ∎

This completes the proof of Proposition 3.1 and the third statement of Theorem 1.1. ∎

4. Illustrations and examples. In this section we illustrate the foregoing results with some applications and examples.

Theorem 1.1 established that L_1-uniqueness of $H_\Omega \in \mathcal{E}_\Omega$ is a consequence of two distinct properties, a capacity condition on the boundary and a growth condition on the coefficients. Therefore we separate the initial discussion into two parts each concentrating on one of these conditions.

4.1. Growth properties. If $\Omega = \mathbb{R}^d$ then the capacity condition plays no role and so we begin by considering this case. We continue to use the function c and the corresponding function μ defined by (1.1) and (1.2), respectively, as measures of the coefficient growth. The following statement combines the L_1-properties which follow from the foregoing with the comparable L_2-properties established earlier by Davies et al. (see [Dav85] and references therein).

PROPOSITION 4.1. *Let $H \in \mathcal{E}_{\mathbb{R}^d}$. Then the following are valid:*

I. *If $\mu(s) \to \infty$ as $s \to \infty$ then H is L_2-unique.*

II. *If $\int_0^\infty ds\, s^{d/2} e^{-\lambda\mu(s)^2} < \infty$ for one $\lambda > 0$ then H is L_1-unique.*

The second statement is a direct consequence of the second statement of Theorem 1.1 since one automatically has $\operatorname{cap}_\Omega(\partial\Omega) = 0$. The first statement follows from [Dav85, Theorem 3.2]. This theorem asserts that H is essentially self-adjoint, i.e. L_2-unique, if there exists a strictly positive differentiable function η over $\mathbb{R}^d$ with $\|\Gamma(\eta)\|_\infty < \infty$ such that $\eta(x) \to \infty$ as $x \to \infty$. But it follows by assumption that $\eta(x) = \int_0^{|x|} dt\, c(t)^{-1/2}$ has these properties.

Next we consider a special case of the growth property examined earlier by Davies [Dav85] and Eberle [Ebe99].

EXAMPLE 4.2. Assume that $c(s) \le as^2(\log s)^\alpha$ for some $a > 0$, $\alpha \ge 0$ and all large s. In this case $\mu(s) \ge b(\log s)^{1-\alpha/2}$ with $b > 0$ for all large s. Therefore $\mu(s) \to \infty$ as $s \to \infty$ if $\alpha < 2$ and H is L_2-unique by the first statement of Proposition 4.1. But Davies has demonstrated by specific example, [Dav85, Example 3.5], that if $d \geq 2$ and $\alpha > 2$ then L_2-uniqueness can fail (see also [Ebe99, Chapter 2, Section c]). Finally Theorem 2.3 in [Ebe99] treats the borderline case $\alpha = 2$. This theorem establishes that if $c(s) \leq as^2(\log s)^2$ for all large s then H is not only L_2-unique but also L_p-unique for all $p \in \langle 1, 2]$.

Next if $\alpha \leq 1$ then the second statement of Proposition 4.1 establishes that H is L_1-unique. Indeed if $\alpha \leq 1$ then $\mu(s) \geq b(\log s)^{1-\alpha/2}$ for s large and the integral (1.4) is finite for large λ. But L_1-uniqueness can fail if $c(s) \sim s^2(\log s)^\alpha$ with $\alpha > 1$ for large s. To verify this let $d = 1$. Then there is a strictly positive even $\psi \in W^{2,\infty}(\mathbb{R})$ such that $x \geq 0 \mapsto \psi(x)$ is monotonically increasing and

$$\psi(x) = 1 - (\log(\log|x|))^{-1} \tag{4.1}$$

for $|x| \geq 3$. In particular $\psi(x) \to 1$ as $|x| \to \infty$. Now define c by $c(x) = \psi'(x)^{-1}\int_0^x ds\, \psi(s)$. Then c is strictly positive and $c \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. Moreover, $c(x) \sim |x|^2(\log|x|)(\log(\log|x|))$ as $|x| \to \infty$. But if H is the corresponding operator on $C_c^\infty(\mathbb{R})$, i.e. if $H\varphi = -(c\varphi')'$, then $(I + H)\psi = \psi - (c\psi')' = 0$. Therefore the range of $I + H$ is not L_1-dense and H is not L_1-unique.

Thus within this class of examples the growth bound $c(s) \leq as^2(\log s)$ is optimal for L_1-uniqueness and the bound $c(s) \leq as^2(\log s)^2$ is optimal for L_2-uniqueness.

Note that if $d = 1$, $\Omega = \langle 0, \infty\rangle$ and one repeats the foregoing construction with ψ positive, strictly increasing, $\psi(x) = O(x)$ as $|x| \to 0$ and satisfying (4.1) for $x \geq 3$, then c is strictly positive and $c(x) = O(x)$ as $x \to 0$. Moreover, $c(x) \sim x^2(\log x)(\log(\log x))$ for all large x. Therefore the corresponding operator H is Markov unique, by [RS10, Theorem 2.7], but again it is not L_1-unique. In fact it is not L_p-unique for any $p \in [1, \infty\rangle$.

The function μ is a lower bound on the Riemannian distance to infinity measured with respect to the metric C^{-1} associated with the operator H_Ω. If one has more detailed information on the geometry one can obtain a stronger conclusion by the same general reasoning. This is illustrated by the following example of a Grušin-type operator.

Let $d = 2$ and $\Omega = \Omega_+ \cup \Omega_-$ with $\Omega_\pm = \{x = (x_1, x_2) : \pm x_1 > 0\}$. Define the *Grušin operator* H by $D(H) = C_c^\infty(\Omega)$ and

$$(H\varphi)(x) = -\partial_1(c_1(x_1)\partial_1\varphi)(x) - c_2(x_1)(\partial_2^2\varphi)(x),$$

where $c_1, c_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ are strictly positive and $c_i(x) \sim |x|^{(2\delta_i, 2\delta_i')}$ with $\delta_i, \delta_i' \geq 0$. Here we use the notation of [RS06], [RS08]. Specifically $s^{(\alpha,\alpha')} = s^\alpha$ if $\alpha \leq 1$ and $s^{(\alpha,\alpha')} = s^{\alpha'}$ if $\alpha \geq 1$, and functions f, g satisfy the relation $f \sim g$ if there are $a, a' > 0$ such that $af \leq g \leq a'f$. We assume that

$\delta_1, \delta'_1 < 1$ but there are no upper bounds on δ_2 and δ'_2. Thus $H \in \mathcal{E}_\Omega$ with

$$C(x) = \begin{pmatrix} c_1(x_1) & 0 \\ 0 & c_2(x_1) \end{pmatrix}.$$

Therefore $\|C(x)\| = c_1(x_1) \vee c_2(x_1) \le a|x_1|^{2(\delta'_1 \vee \delta'_2)} \le a|x_1|^{2(1 \vee \delta'_2)}$ for all $|x_1| \ge 1$. Although the asymptotic growth of C is dictated by c_2, which behaves asymptotically like $|x_1|^{2\delta'_2}$, the uniqueness properties are independent of the magnitude of δ'_2.

Proposition 4.3. *Let H denote the Grušin operator defined above.*

I. *If $\delta_1 \in [0, 1/2\rangle$ then H is not Markov unique and consequently not L_1-unique.*
II. *If $\delta_1 \in [1/2, 1\rangle$ then H is L_1-unique and consequently Markov unique.*

Proof. The first statement of the proposition follows from the observations at the end of Section 6 of [RS08] and in particular from Proposition 6.10.

Some care has to be taken in comparing the current statements with those of [RS08]. The operator H is defined on $C_c^\infty(\mathbb{R}^2 \setminus \{x_1 = 0\})$ but the operator H_δ studied in [RS08] corresponds to the extension of H to $C_c^\infty(\mathbb{R}^2)$. The Friedrichs extension $H_{\delta,D}$ of H_δ is the self-adjoint extension H_N of H which satisfies the Neumann-type boundary condition $(c_1\partial_1)(0_+, x_2) = (c_1\partial_1)(0_-, x_2)$ on the line of degeneracy $x_1 = 0$. The Friedrichs extension H_D of H is, however, the self-adjoint extension with the Dirichlet-type boundary condition $\varphi(0_+, x_2) = \varphi(0_-, x_2)$. If $\delta_1 \in [0, 1/2\rangle$ then these extensions are distinct, and in addition there are extensions with analogous Robin boundary conditions sandwiched between the minimal extension H_N $(= H_{\delta,D})$ and the maximal extension H_D. But if $\delta_1 \in [1/2, 1\rangle$ then $H_N = H_D$ and all the operators coincide (see [RS08, Proposition 6.10]).

The proof of L_1-uniqueness for $\delta_1 \in [1/2, 1\rangle$ is by reasoning similar to that used to prove Proposition 3.1 and it does not require an upper bound on δ'_2. The argument follows the lines of the proof of Theorem 6.1 of [RS08], details of which are given in [RS06]. First, Markov uniqueness follows from Proposition 6.10 of [RS08]. Secondly, one deduces that H is conservative by the arguments given in [RS06]. The semigroup S generated by H_N $(= H_D)$ is approximated on $L_2(\Omega)$ by semigroups $S^{(N,\varepsilon)}$ generated by the Grušin operators with coefficients $(C \wedge NI) + \varepsilon I$. Then S and $S^{N,\varepsilon}$ satisfy L_2-off-diagonal bounds with respect to the corresponding Riemannian distances by [RS08, Proposition 4.1]. But if $N \ge 1 \ge \varepsilon > 0$ then these distances are all larger than the Riemannian distance $d_1(\cdot\,;\cdot)$ corresponding to the Grušin operator with coefficients $(c_1 + 1, c_2 + 1)$. Therefore S and the approximants $S^{(N,\varepsilon)}$ all satisfy L_2-off-diagonal bounds with respect to $d_1(\cdot\,;\cdot)$. Since the operator with coefficients (c_1+1, c_2+1) has $\delta_1 = 0 = \delta_2$ it follows that $d_1(\cdot;\cdot)$ is independent of δ_1 and δ_2.

Next let $B_{1,r} = \{x \in \mathbb{R}^2 : d_1(0;x) < r\}$. Then if $\varphi \in L_2(B_{1,m})$ it follows by L_2-off-diagonal bounds, similar to those of Lemma 3.2, as in the proof of Lemma 3.4, that

$$|(\mathbb{1}_{(B_M)^c}, S_t^{(N,\varepsilon)}\varphi)| \vee |(\mathbb{1}_{(B_M)^c}, S_t\varphi)| \\ \le \sum_{p\ge M} |B_{1,p+1}|^{1/2} e^{-d_1(C_p;B_{1,m})^2(4t)^{-1}} \|\varphi\|_2,$$

where $C_p = B_{1,p+1}\backslash B_{1,p}$. But $d_1(C_p;B_{1,m}) \ge p-m$ by the triangle inequality. Moreover, it follows from Proposition 5.1 of [RS08] that there is an $a>0$ such that $|B_{1,p}| \le a^2 p^{D'}$ with $D' = 1 + (1+\delta_2'-\delta_1')(1-\delta_1')^{-1}$. Therefore

$$|(\mathbb{1}_{(B_M)^c}, S_t^{(N,\varepsilon)}\varphi)| \vee |(\mathbb{1}_{(B_M)^c}, S_t\varphi)| \le a \sum_{p\ge M} p^{D'/2} e^{-(p-m)^2(4t)^{-1}} \|\varphi\|_2$$

and the estimate is uniform for all $N \ge 1$ and $\varepsilon \in \langle 0,1]$. Finally the $S^{(N,\varepsilon)}$ are conservative, since their generators are strongly elliptic, and they are L_2-convergent to S. But

$$\|(S_t^{(N,\varepsilon)} - S_t)\varphi\|_1 \le |B_{1,M}|^{1/2} \|(S_t^{(N,\varepsilon)} - S_t)\varphi\|_2 \\ + (\mathbb{1}_{(B_M)^c}, S_t^{(N,\varepsilon)}\varphi) + (\mathbb{1}_{(B_M)^c}, S_t\varphi)$$

for all positive $\varphi \in L_2(B_{1,m})$. Therefore taking the limits $N\to\infty$, $\varepsilon \to 0$ and $M\to\infty$ one deduces that the $S^{(N,\varepsilon)}$ are L_1-convergent to S. Hence S is conservative and H is L_1-unique by Theorem 1.1. ∎

4.2. Capacity estimates. In this subsection we suppose that Ω is a proper subset of $\mathbb{R}^d$ and examine the capacity condition $\operatorname{cap}_\Omega(\partial\Omega) = 0$. It follows from the general additivity properties of the capacity (see [BH91] or [FOT94]) that $\operatorname{cap}_\Omega(\partial\Omega) = 0$ if and only if $\operatorname{cap}_\Omega(A) = 0$ for all bounded measurable subsets A of $\partial\Omega$. The capacity $\operatorname{cap}_\Omega(A)$ depends on two gross features of A and H_Ω, the dimension of the set and the order of degeneracy of H_Ω at A.

First, let $A \subset \overline{\Omega}$ be a measurable subset with $|A| = 0$. Further, let $\mathcal{H}^s$ denote the Hausdorff measure on $\mathbb{R}^d$ and $d_{\mathrm{H}}(A)$ the Hausdorff dimension of the set A. It follows that $\mathcal{H}^s(A) = \infty$ if $s \in [0, d_{\mathrm{H}}(A)\rangle$, $\mathcal{H}^s(A) = 0$ if $s \in \langle d_{\mathrm{H}}(A), \infty\rangle$, but at the critical point $s = d_{\mathrm{H}}(A)$ the measure can take any value in $[0,\infty]$.

Secondly, the operator $H_\Omega \in \mathcal{E}_\Omega$ is defined to be *degenerate of order* $\gamma(A)$ if for each bounded measurable subset $B \subset A$ there is an open subset U containing B and a number $b > 0$ such that $0 < C(x) \le b d(x;B)^{\gamma(A)} I$ for all $x \in U \cap \Omega$, where $d(x;B)$ denotes the Euclidean distance of x from B.

The next proposition is an extension of Proposition 4.2 in [RS11].

PROPOSITION 4.4. *Let $A \subset \partial\Omega$ be a measurable subset with $|A| = 0$. Assume $H_\Omega \in \mathcal{E}_\Omega$ is degenerate of order $\gamma(A)$ on A. If* either $\gamma(A) \geq 2$, or $d_{\rm H}(A) < d + \gamma(A) - 2$, or $d_{\rm H}(A) = d + \gamma(A) - 2$ *and* $\mathcal{H}^{d+\gamma(A)-2}(A) < \infty$, *then* $\operatorname{cap}_\Omega(A) = 0$.

Proof. If $\gamma(A) \geq 2$ then $\operatorname{cap}_\Omega(A) = 0$ by the proof of Proposition 4.2 of [RS11]. Therefore in the proof of the second and third statements we may assume $\gamma(A) \leq 2$.

The proofs of these latter statements are by adaptation of well known results for the Laplacian (see, for example, [EvG92, Theorem 2 in Section 4.7.1 and Theorem 3 in Section 4.7.2] or [MZ97, Section 2.1.7]). Consider the second statement of the proposition.

It suffices to establish that $\operatorname{cap}_\Omega(B) = 0$ for each bounded measurable subset B of A. Let $B_r = B(y;r)$ be the Euclidean ball of radius r centred at y with $B \cap B_r \neq \emptyset$. Then $d(x;B) \leq 4r$ for all $x \in B_{2r}$. Therefore there is a number $b > 0$ which is independent of y such that $\|C(x)\| \leq b r^{\gamma(A)}$ for all $x \in B_{2r} \cap \Omega$. Next fix $\eta \in C_c^\infty(B_{2r})$ with $0 \leq \eta \leq 1$, $\eta = 1$ on B_r and $|\nabla\eta| \leq 2r^{-1}$. Then

$$\|\eta\|^2_{D(h_\Omega)} \leq \int\limits_{B_{2r}\cap\Omega} (\|C(x)\|\,|(\nabla\eta)(x)|^2 + |\eta(x)|^2) \leq |B_{2r}|(4br^{\gamma(A)-2} + 1).$$

Since $\gamma(A) \leq 2$ it follows that there is a $c > 0$, whose value is independent of the coefficients of H_Ω, such that $\|\eta\|^2_{D(h_\Omega)} \leq (b+1)cr^{d+\gamma(A)-2}$ for all $r \leq 1$. Moreover, the estimate is uniform in y. Now let $B_{r_i} = B(y_i; r_i)$ be a countable family of balls with $r_i \leq \delta \leq 1$ such that $B_{r_i} \cap B \neq \emptyset$ and $B \subset \bigcup_i B_{r_i}$. Then

$$\operatorname{cap}_\Omega(B) \leq \sum_i \operatorname{cap}_\Omega(B_{r_i} \cap \Omega) \leq (b+1)c \sum_i r_i^{d+\gamma(A)-2}$$

by the foregoing estimate. Therefore

$$\operatorname{cap}_\Omega(B) \leq (b+1)c\mathcal{H}^{d+\gamma(A)-2}(B) \leq (b+1)c\mathcal{H}^{d+\gamma(A)-2}(A).$$

But if $d + \gamma(A) - 2 > d_{\rm H}(A)$ then $\mathcal{H}^{d+\gamma(A)-2}(A) = 0$ and it follows that $\operatorname{cap}_\Omega(B) = 0$.

Finally consider the third statement of the proposition. The proof follows from the reasoning used for the Laplacian in [EvG92] and [MZ97]. It is based on the estimates $\|\nabla\eta\|_2^2 \leq cr^{d-2}$ for the function $\eta \in C_c^\infty(B_r)$ introduced in the foregoing paragraph. These estimates are now replaced by the bounds $h_\Omega(\eta) \leq (b+1)cr^{d+\gamma(A)-2}$. Care has to be taken, however, as the capacity in [EvG92] and [MZ97] is defined by the seminorm $\varphi \mapsto \|\nabla\varphi\|_2$ and not the graph norm $\varphi \mapsto (\|\nabla\varphi\|_2^2 + \|\varphi\|_2^2)^{1/2}$.

Again it suffices to prove that $\mathrm{cap}_\Omega(B) = 0$ for each bounded measurable subset of A. As a preliminary, note that $\mathcal{H}^{d+\gamma(A)-2}(B) \le \mathcal{H}^{d+\gamma(A)-2}(A) < \infty$ by assumption. Now we follow the reasoning of [EvG92, Section 4.7.2].

Let V_1 be an open set with $B \subset V_1 \subset B_\delta$ where B_δ is the δ-neighbourhood of B, i.e. $B_\delta = \{x \in \mathbb{R}^d : d(x;B) < \delta\}$, and $\delta \in \langle 0,1]$. Then following [EvG92] one constructs a sequence of open sets V_n with $B \subset V_n$ and $\overline{V}_{n+1} \subset V_n$ together with a sequence of C^∞-functions η_n with $0 \le \eta_n \le 1$, $\mathrm{supp}\,\eta_n \subset V_n$, $\eta_n = 1$ on V_{n+1} and

$$h_\Omega(\eta_n) \le a(\mathcal{H}^{d+\gamma(A)-2}(B) + 1).$$

The value of the positive constant a depends on d, B and the constant b occurring in the bounds on $\|C(x)\|$. Next one sets $T_N = \sum_{n=1}^N n^{-1}$ and $\psi_N = T_N^{-1} \sum_{n=1}^N n^{-1}\varphi_n$. Note that $0 \le \psi_N \le 1$, $\psi_N = 1$ on $V_{N+1} \supset B$ and $\mathrm{supp}\,\psi_N \subset V_1 \subset B_\delta$. Now it follows that

$$\begin{aligned}\mathrm{cap}_\Omega(B) \le \|\psi_N\|^2_{D(h_\Omega)} &\le T_N^{-2} \sum_{n=1}^N n^{-2} h_\Omega(\varphi_n) + \|\psi_N\|_2^2 \\ &\le a(\mathcal{H}^{d+\gamma(A)-2}(B) + 1) T_N^{-2} \sum_{n=1}^N n^{-2} + |B_\delta|.\end{aligned}$$

where we have used $\mathrm{supp}\,|\nabla\varphi_n| \subset V_n \setminus \overline{V}_{n+1}$. But $T_N \to \infty$ as $N \to \infty$ and $|B_\delta| \to |B| = 0$ as $\delta \to 0$. Thus taking the limit $N \to \infty$ followed by the limit $\delta \to 0$ one concludes that $\mathrm{cap}_\Omega(B) = 0$. ∎

Optimal application of Proposition 4.4 depends on verification that the Hausdorff measure is finite at the critical value of discontinuity. This is not always the case. If, however, B is bounded and is the graph of a Lipschitz function then $d_{\mathrm{H}}(B) = d - 1$ and $\mathcal{H}^s(B)$, at the critical value $s = d_{\mathrm{H}}(B)$, is equal to the (finite) surface area of B (see [EvG92, Sections 2.4.2 and 3.3.4]). Therefore one has the following corollary of Proposition 4.4 and Theorem 1.1.

Corollary 4.5. *Assume $H_\Omega \in \mathcal{E}_\Omega$ and $\partial\Omega$ is locally Lipschitz. Further assume the coefficients of H_Ω satisfy the growth condition* (1.4) *and, in addition, $\|C(x)\| \to 0$ as $x \to \partial\Omega$. Then H_Ω is L_1-unique.*

Proof. Let $B \subset \partial\Omega$ be a bounded measurable subset which is the graph of a Lipschitz function. Then $\gamma(B) \ge 1$ because the $c_{ij} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ tend to zero on $\partial\Omega$. Hence one can apply the third statement of Proposition 4.4 with $d_{\mathrm{H}}(B) = d - 1$ and $\gamma(B) = 1$. It follows that $\mathrm{cap}_\Omega(B) = 0$. Since this conclusion holds for all possible choices of B one deduces from the additivity properties of the capacity that $\mathrm{cap}_\Omega(\partial\Omega) = 0$. Then L_1-uniqueness follows from Theorem 1.1. ∎

Finally we note that if $\gamma(A) \geq 2$ then one even has a simple criterion for L_2-uniqueness.

LEMMA 4.6. *Assume $H_\Omega \in \mathcal{E}_\Omega$ and $|\partial\Omega| = 0$. If $\gamma(A) \geq 2$ for all bounded measurable $A \subset \partial\Omega$ and if $\mu(s) \to \infty$ as $s \to \infty$ then H is L_2-unique.*

Proof. Let ρ_n denote the functions introduced in the proof of IV⇒III in Theorem 1.1. Since $\mu(s) \to \infty$ as $s \to \infty$ it follows that ρ_n converges pointwise to $\mathbb{1}_\Omega$ as $n \to \infty$. Moreover, $\|\Gamma(\rho_n)\|_\infty \leq b^2 n^{-2}$. Next define χ_n on $[0, \infty\rangle$ by $\chi_n(s) = 1$ if $s \in [0, n^{-1}\rangle$, $\chi_n(s) = -\log s / \log n$ if $s \in [n^{-1}, 1]$ and $\chi_n(s) = 0$ if $s \geq 1$. Then define ξ_n on Ω by $\xi_n(x) = \chi_n(d(x; \partial\Omega))$. Finally define η_n by $\eta_n = \rho_n(\mathbb{1}_\Omega - \xi_n)$. It follows that η_n converges pointwise to $\mathbb{1}_\Omega$ as $n \to \infty$. In addition

$$\|\Gamma(\eta_n)\|_\infty \leq 2\|\Gamma(\rho_n)\|_\infty + 2a\|\Gamma(\xi_n)\|_\infty \leq 2b^2 n^{-2} + 2a(\log n)^{-2} \to 0$$

as $n \to \infty$, where we have used $\|\Gamma(\xi_n)\|_\infty \leq a(\log n)^{-2}$. The latter estimate follows from the degeneracy assumption. Hence H_Ω is L_2-unique by Proposition 6.1 of [RS11], with $p = 2$, and a regularization argument. ∎

Note that in Lemma 4.6 there is no restraint on the dimension of the boundary $\partial\Omega$. Therefore the conclusion is valid for sets Ω with arbitrarily rough boundaries, in particular for fractal boundaries. The result is, however, not optimal if the boundary is smooth. Indeed if $d = 1$ and $\Omega = \langle 0, \infty\rangle$ then a degeneracy of order 1 at the origin is necessary and sufficient for L_1-uniqueness and a degeneracy of order $3/2$ is necessary and sufficient for L_2-uniqueness (see [RS10, Theorem 2.7]).

4.3. Negligible sets. The foregoing discussion indicates that sets of Hausdorff dimension lower than $d - 2$ are insignificant for L_1-uniqueness. In this subsection we establish a precise statement of this nature for non-degenerate operators on $\mathbb{R}^d$ and also derive an analogous statement for L_2-uniqueness.

PROPOSITION 4.7. *Let $H \in \mathcal{E}_{\mathbb{R}^d}$ with $d \geq 2$ and assume the coefficients of H satisfy the growth condition* (1.4). *Further let Γ be a submanifold of $\mathbb{R}^d$ with $d_{\mathrm{H}}(\Gamma) = m$. The following conditions are equivalent:*

I. $m \leq d - 2$.
II. $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ *is an L_1-core of H.*

Proof. The proof is based on the observation that both conditions of the proposition are equivalent to the capacity of the set Γ being zero. But there are three different capacities involved in the argument.

Let $A \subset \Gamma$ be a measurable subset. First define $\operatorname{cap}(A)$ as the capacity of the set measured with respect to H. Explicitly

$$\text{(4.2)}\qquad \operatorname{cap}(A) = \inf\{\|\psi\|^2_{D(h)} : \psi \in D(h) \text{ and there exists an open set } U \subset \mathbb{R}^d \text{ such that } U \supseteq A \text{ and } \psi \geq 1 \text{ a.e. on } U\}$$

with h the closed quadratic form associated with H. Secondly, let $\Omega = \mathbb{R}^d \backslash \Gamma$, set $\mathcal{D} = C_c^\infty(\mathbb{R}^d \backslash \Gamma)$ and introduce $H_\Omega = H|_{\mathcal{D}}$. Then $H_\Omega \in \mathcal{E}_\Omega$ and we define $\operatorname{cap}_\Omega(A)$ as the capacity measured with respect to H_Ω. Thus $\operatorname{cap}_\Omega(A)$ is given by (4.2) with h replaced by $h_{\Omega,N}$. Thirdly, define $\operatorname{cap}_{1,2}(A)$ as the capacity given by (4.2) but with $D(h)$ replaced by $W^{1,2}(\mathbb{R}^d)$. It can be understood as the capacity associated with the form of the Laplacian.

In fact only two of these capacities are distinct in general. Since $h_{\Omega,N} \supseteq h$ it follows that $\operatorname{cap}_\Omega(A) \leq \operatorname{cap}(A)$. But both of these capacities can be calculated with functions which are equal to one in an open neighbourhood of A and on such functions the two forms coincide. Therefore $\operatorname{cap}_\Omega(A) = \operatorname{cap}(A)$.

After these preliminary definitions the proof of the proposition consists of establishing three equivalences.

I $\Leftrightarrow$ $\operatorname{cap}_{1,2}(\Gamma) = 0$. This is the key observation of the proof. It is a standard property of the Laplacian (see, for example, Corollary 5.1.15 of [AH96]).

$\operatorname{cap}_{1,2}(\Gamma) = 0 \Leftrightarrow \operatorname{cap}(\Gamma) = 0$. Let $A \subset \Gamma$ be a bounded measurable set. Assume $\operatorname{cap}(\Gamma) = 0$. Hence $\operatorname{cap}(A) = 0$. Then since $C_c^\infty(\mathbb{R}^d)$ is a core of h there exist a sequence $\chi_n \in C_c^\infty(\mathbb{R}^d)$ and a decreasing sequence of bounded open subsets $U_n \supset A$ such that $0 \leq \chi_n \leq 1$, $\chi_n = 1$ on U_n and $h(\chi_n) + \|\chi_n\|_2^2 \leq n^{-1}$. Now fix an $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on U_1 and hence on each U_n. Set $\varphi_n = \chi_n \eta$. It follows that $\varphi_n \in D(h)$, $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on U_n and $h(\varphi_n) + \|\varphi_n\|_2^2 \leq a n^{-1}$ with $a = 2(\|\nabla\eta\|_\infty^2 + 1)$. Moreover, if $K = \operatorname{supp}\eta$ then $\operatorname{supp}\varphi_n \subseteq K$ for all n. But it follows from local boundedness and strict positivity of the matrix of coefficients C that there exist $\mu_K, \nu_K > 0$ such that

$$\mu_K \|\varphi\|^2_{W^{1,2}(\mathbb{R}^d)} \leq \|\varphi\|^2_{D(h)} \leq \nu_K \|\varphi\|^2_{W^{1,2}(\mathbb{R}^d)}$$

for all $\varphi \in W^{1,2}(K)$. Therefore if $\operatorname{cap}(A) = 0$ then $\operatorname{cap}_{1,2}(A) = 0$ by the lower bound. The converse implication is similar, with the capacities interchanged and using the upper bound. Since A was an arbitrary bounded subset of Γ the desired conclusion follows immediately.

$\operatorname{cap}(\Gamma) = 0 \Leftrightarrow$ II. First, observe that H is L_1-unique by the second statement of Proposition 4.1. But H is L_1-unique if and only if $\overline{H}^1$ is the generator of an L_1-continuous semigroup. Now suppose condition II is valid. Then

$\overline{H}^1 = \overline{H_\Omega}^1$. Therefore $\overline{H_\Omega}^1$ is the generator of an L_1-continuous semigroup. Consequently, H_Ω is L_1-unique. But L_1-uniqueness of H_Ω is equivalent to $\mathrm{cap}_\Omega(\Gamma) = 0$ by Theorem 1.1, which in turn is equivalent to $\mathrm{cap}(\Gamma) = 0$.

Conversely, if H_Ω is L_1-unique then $\overline{H_\Omega}^1$ is the generator of an L_1-continuous semigroup. But $\overline{H}^1$ is also a generator and $\overline{H}^1 \supseteq \overline{H_\Omega}^1$ by definition. Since a semigroup generator cannot have a proper generator extension one must have $\overline{H}^1 = \overline{H_\Omega}^1 = \overline{H|_{\mathcal{D}}}^1$. Thus $\mathcal{D}$ is an L_1-core of H. ■

It follows from the assumptions of Proposition 4.7 and the first statement of Proposition 4.1 that H is L_2-unique, i.e. H is essentially self-adjoint. Then the closed form h associated with H is the form of the L_2-closure $\overline{H}$ of H. Therefore $h(\varphi) = \|\overline{H}^{1/2}\varphi\|_2^2$ for all $\varphi \in D(h) = D(\overline{H}^{1/2})$. This observation provides a relation between L_1- and L_2-cores.

Corollary 4.8. *Under the assumptions of Proposition* 4.7, *the following conditions are equivalent:*

I. $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ *is an* L_1*-core of* H.
II. $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ *is an* L_2*-core of* $\overline{H}^{1/2}$.

Proof. Again set $\Omega = \mathbb{R}^d \setminus \Gamma$, $\mathcal{D} = C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ and $H_\Omega = H|_{\mathcal{D}}$. Then $H_\Omega \in \mathcal{E}_\Omega$. Moreover, it follows from the proof of Proposition 4.7 that condition I is equivalent to the condition $\mathrm{cap}_\Omega(\Gamma) = 0$. But condition II is also equivalent to this capacity condition by the proof of Theorem 1.1. ■

There is also an L_2-version of Proposition 4.7.

Proposition 4.9. *Assume* $H \in \mathcal{E}_{\mathbb{R}^d}$ *with* $d \geq 4$ *and that* $\mu(s) \to \infty$ *as* $s \to \infty$. *Further let* Γ *be a submanifold of* $\mathbb{R}^d$ *with* $d_{\mathrm{H}}(\Gamma) = m$. *The following conditions are equivalent:*

I. $m \leq d - 4$.
II. $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ *is an* L_2*-core of* H.

Proof. It follows from the first statement of Proposition 4.1 that H is essentially self-adjoint. Then we define the capacity $\mathrm{Cap}(A)$ of the measurable subset $A \subset \mathbb{R}^d$ associated with the self-adjoint L_2-closure $\overline{H}$ by

$$\mathrm{Cap}(A) = \inf\{\|\psi\|_{D(\overline{H})}^2 : \psi \in D(\overline{H}) \text{ and there exists an open set } U \subset \mathbb{R}^d \text{ such that } U \supseteq A \text{ and } \psi \geq 1 \text{ a.e. on } U\}. \tag{4.3}$$

Analogously we define $\mathrm{cap}_{2,2}(A)$ as the capacity measured with respect to the $W^{2,2}(\mathbb{R}^d)$-norm, i.e. $\mathrm{cap}_{2,2}(A)$ is given by (4.3) but with $D(\overline{H})$ replaced by $W^{2,2}(\mathbb{R}^d)$. We will argue that both conditions of the proposition are equivalent to $\mathrm{Cap}(\Gamma) = 0$ or, equivalently, $\mathrm{Cap}_{2,2}(\Gamma) = 0$. The reasoning is similar to the proof of Proposition 4.7.

I $\Leftrightarrow$ $\mathrm{cap}_{2,2}(\Gamma) = 0$. This equivalence is again a consequence of Corollary 5.1.15 of [AH96].

$\mathrm{cap}_{2,2}(\Gamma) = 0 \Leftrightarrow \mathrm{Cap}(\Gamma) = 0$. If A is a bounded measurable subset of Γ with $\mathrm{Cap}(A) = 0$ then there exist a sequence $\chi_n \in C_c^\infty(\mathbb{R}^d)$ and a decreasing sequence of bounded open subsets $U_n \supset A$ such that $0 \le \chi_n \le 1$, $\chi_n = 1$ on U_n and $\|\chi_n\|^2_{D(\overline{H})} \le n^{-1}$. But the sequence χ_n can be modified by a variation of the argument used in the proof of Proposition 4.7 to yield a sequence with similar properties but with each element of the sequence supported by a fixed compact set K. Explicitly, fix $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on U_1 and hence on each U_n. Let $K = \mathrm{supp}\,\eta$ and set $\varphi_n = \chi_n \eta$. It follows that $0 \le \varphi_n \le 1$, $\mathrm{supp}\,\varphi_n \subseteq K$, $\varphi_n = 1$ on U_n and $\|\varphi_n\|_2 \le \|\chi_n\|_2$. But

$$\overline{H}\varphi_n = (H\chi_n)\eta + \chi_n(H\eta) + 2\Gamma(\chi_n;\eta)$$

where $\Gamma(\cdot\,;\cdot)$ is the *carré du champ* associated with H. Therefore

$$\|\overline{H}\varphi_n\|_2^2 \le 3(\|H\eta\|_\infty^2 + \|\eta\|_\infty^2)\|\chi_n\|^2_{D(\overline{H})} + 12\int \Gamma(\chi_n;\eta)^2 .$$

But

$$\int \Gamma(\chi_n;\eta)^2 \le \int \Gamma(\chi_n)\Gamma(\eta) \le |\Gamma(\eta)|_\infty h(\chi_n) \le 2^{-1}|\Gamma(\eta)|_\infty \|\chi_n\|^2_{D(\overline{H})} .$$

Combining these estimates one deduces that there is an $a > 0$ such that $\|\varphi_n\|^2_{D(\overline{H})} \le a\|\chi_n\|^2_{D(\overline{H})} \le a n^{-1}$ for all n.

Next since the coefficients $c_{ij} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $C = (c_{ij}) > 0$ it follows that there exist $\mu_K, \nu_K > 0$ such that

$$\nu_K \|\varphi\|^2_{W^{2,2}(\mathbb{R}^d)} \ge \|\varphi\|^2_{D(\overline{H})} \ge \mu_K \|\varphi\|^2_{W^{2,2}(\mathbb{R}^d)}$$

for all $\varphi \in C_c^\infty(K)$ (see, for example, the appendix of [RS11]). Therefore replacing φ by φ_n and taking the limit $n \to \infty$ one deduces from the lower bound in this estimate that $\mathrm{cap}_{2,2}(A) = 0$. Since this is valid for all bounded measurable subsets of Γ it follows that $\mathrm{Cap}(\Gamma) = 0$ implies $\mathrm{cap}_{2,2}(\Gamma) = 0$. The proof of the converse is similar.

$\mathrm{Cap}(\Gamma) = 0 \Leftrightarrow$ II. First suppose condition II is valid. Secondly, fix $\psi \in C_c^\infty(\mathbb{R}^d) \subset D(H)$ with $\psi = 1$ on an open neighbourhood of the bounded subset $A \subset \Gamma$. Then, by condition II, there is a sequence $\psi_n \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ such that $\|\psi - \psi_n\|_{D(\overline{H})} \to 0$ as $n \to \infty$. Set $\varphi_n = \psi - \psi_n$. It follows that $\varphi_n \in D(H)$, $\varphi_n = 1$ on an open neighbourhood U_n of A and $\|\varphi_n\|_{D(\overline{H})} \to 0$ as $n \to \infty$. Therefore $\mathrm{Cap}(A) = 0$. Since this holds for an arbitrary bounded subset A of Γ it follows that $\mathrm{Cap}(\Gamma) = 0$.

Conversely, suppose $\mathrm{Cap}(\Gamma) = 0$. Therefore $\mathrm{Cap}(A) = 0$ for each bounded measurable subset A of Γ. Then since $C_c^\infty(\mathbb{R}^d)$ is a core of H, by definition, there exist a sequence $\chi_n \in C_c^\infty(\mathbb{R}^d)$ and a sequence of open subsets U_n of A

such that $0 \leq \chi_n \leq 1$, $\chi_n = 1$ on U_n and $\|\chi_n\|_{D(\overline{H})} \to 0$ as $n \to \infty$. Now fix $\psi \in C_c^\infty(\mathbb{R}^d)$ and set $\psi_n = (\mathbb{1} - \chi_n)\psi$. It follows that $\psi_n \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)$. Moreover, $\psi - \psi_n = \chi_n \psi$. But

$$H(\psi - \psi_n) = (H\chi_n)\psi + \chi_n(H\psi) + 2\Gamma(\chi_n; \psi)$$

and consequently

$$\|H(\psi - \psi_n)\|_2^2 \leq 3(\|H\psi\|_\infty^2 + \|\psi\|_\infty^2 + 2\|\Gamma(\psi)\|_\infty))\|\chi_n\|_{D(\overline{H})}^2.$$

Since $\|\psi - \psi_n\|_2 \leq \|\psi\|_\infty \|\chi_n\|_2$ it follows that $\|(\psi - \psi_n)\|_{D(\overline{H})} \to 0$ as $n \to \infty$. Therefore each $\psi \in C_c^\infty(\mathbb{R}^d)$ can be approximated by a sequence $\psi_n \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ in the $D(\overline{H})$-graph norm. But as $C_c^\infty(\mathbb{R}^d)$ is a core of H one concludes that $C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ is also a core. ∎

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