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On integral type generalizations of positive linear operators

by

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Abstract. We introduce a sequence of positive linear operators including many integral type generalizations of well known operators. Using the concept of statistical convergence we obtain some Korovkin type approximation theorems for those operators, and compute the rates of statistical convergence. Furthermore, we deal with the local approximation and the rth order generalization of our operators.

1. Introduction. In this paper, we are concerned with Korovkin type theorems for a general sequence of positive linear operators including many integral type generalizations of well known operators in approximation theory via the concept of statistical convergence. The study of Korovkin type approximation is a well established area of research, which deals with the problem of approximating a function f by means of a sequence of positive linear operators $L_n(f)$. Statistical convergence, though introduced over fifty years ago, has only recently become an area of active research. In particular, it has made an appearance in approximation theory [14] (see also [6]–[8]).

The first section of this paper collects some basic ideas related to statistical convergence and introduces a sequence of positive linear operators which generates many Durrmeyer type and Kantorovich type generalizations of well known operators while the second section gives a Korovkin type approximation theorem for these operators on an appropriate weighted space. The third section addresses some problems concerning rates of statistical convergence by means of the modulus of continuity and elements of the Lipschitz class. This section also includes a study of local smoothness of these operators. In the last section, we deal with the approximation properties of the rth order generalization of our operators.

We now introduce some notation and basic definitions used in this paper.

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As usual, the symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of all real numbers, of all natural numbers, and $\mathbb{N} \cup \{0\}$, respectively.

Let $A := (a_{jn})$, $j, n \in \mathbb{N}$, be a non-negative *regular summability matrix*, i.e. $\lim Ax = L$ whenever $\lim x = L$, where $Ax := ((Ax)_j)$ is called the A-*transform* of $x := (x_n)$ and is given by $(Ax)_j := \sum_{n=1}^{\infty} a_{jn} x_n$ provided that the series converges for each $j \in \mathbb{N}$ (see [15]). Then the sequence $x := (x_n)$ is called A-*statistically convergent* to a number L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n : |x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We then write $\mathrm{st}_A\text{-}\lim x = L$ (see [10], and also [12], [20], [23]). If $A = C_1$, the Cesàro matrix of order one, then C_1-statistical convergence is equivalent to statistical convergence [9], [11], [13]. If A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence. Kolk [20] proved that A-statistical convergence is stronger than ordinary convergence if $\lim_j \max_n |a_{jn}| = 0$. The concept of A-statistical convergence may also be defined in normed spaces [19].

Let $I \subset \mathbb{R}$ be an arbitrary interval and let $C(I)$ denote the linear space of all real-valued continuous functions on I. Assume that g is a non-negative increasing function on $[0, \infty)$ with $g(0) = 1$. Consider the function space

$$C_g(I) = \left\{ f \in C(I) : \lim_{|x| \to \infty} \frac{|f(x)|}{(g(|x|))^c} = 0 \text{ for any } c > 0 \right\}.$$

It has been examined in [7]. If $I = [a, b]$ and $g(x) \equiv 1$, then $C_g(I) = C[a, b]$. Now let $\{\mu_{n,k} : n \in \mathbb{N} \text{ and } k \in \mathbb{N}_0\}$ be a collection of measures defined on $(I, \mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel measurable subsets of I. Assume that

$$\int_I d\mu_{n,k}(y) = 1, \tag{1.1}$$

and that

$$\sup_{n \in \mathbb{N},\ k \in \mathbb{N}_0} \int_{I \setminus I_\delta} g(|y|)\, d\mu_{n,k}(y) < \infty \tag{1.2}$$

for any $\delta > 0$, where $I_\delta := [x - \delta, x + \delta] \cap I$.

We now introduce the following operators defined on the space $C_g(I)$:

$$D_n(f; x) = \sum_{k=0}^{\infty} r_{n,k}(x) \int_I f(y)\, d\mu_{n,k}(y) \qquad (f \in C_g(I),\ x \in I,\ n \in \mathbb{N}), \tag{1.3}$$

where $r_{n,k}(x)$ has the following properties:

$$r_{n,k}(x) \geq 0 \qquad x \in I,\ n \in \mathbb{N},\ k \in \mathbb{N}_0, \tag{1.4}$$

$$\sum_{k=0}^{\infty} r_{n,k}(x) = 1,\ x \in I,\ n \in \mathbb{N}. \tag{1.5}$$

The operators D_n are positive and linear. Since condition (1.2) guarantees that f is integrable on I whenever $f \in C_g(I)$, the operators $D_n(f;\cdot)$ in (1.3) are well defined. We also note that conditions (1.1) and (1.5) imply that $D_n(1;x) = 1$.

Applications. **1.** Let $I = [0,1]$. If we choose

$$F_{n,k}(y) = (n+1)\int_0^y \binom{n}{k} t^k (1-t)^{n-k}\, dt, \qquad 0 < y \le 1,$$

then $F_{n,k}$, which is called the *cumulative function*, is increasing with respect to y and continuous on the right. So, there exists a unique Borel measure $\mu_{n,k}$ corresponding to $F_{n,k}$, which satisfies the following Lebesgue–Stieltjes integral equality:

$$\int_I f(y)\, dF_{n,k}(y) = \int_I f(y)\, d\mu_{n,k}(y) \tag{1.6}$$

(see, for instance, [24]).

For $x \in [0,1]$, set $r_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Then conditions (1.4) and (1.5) hold, and also it is clear that

$$dF_{n,k}(y) = (n+1) r_{n,k}(y) dy. \tag{1.7}$$

Using (1.7) in (1.6) we see that our operators turn out to be the Bernstein–Durrmeyer operators (see [3]) which have the following form:

$$M_n(f;x) = (n+1)\sum_{k=0}^{n} r_{n,k}(x) \int_0^1 f(t) p_{n,k}(t)\, dt, \ 0 \le x \le 1.$$

2. Let $I = [0,\infty)$. Now define the cumulative function

$$F_{n,k}(y) = (n-1)\int_0^y \binom{n+k-1}{k} t^k (1+t)^{-n-k}\, dt, \qquad 0 < y < \infty.$$

If $\mu_{n,k}$ is the Borel measure corresponding to $F_{n,k}$, then choosing $r_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$, $x \in [0,\infty)$, and using the above technique we immediately get the Baskakov–Durrmeyer operators (see, for instance, [16]) in $[0,\infty)$:

$$V_n(f;x) = (n-1)\sum_{k=0}^{\infty} \Big(\int_0^{\infty} f(y) r_{n,k}(y)\, dy \Big) r_{n,k}(x).$$

3. Let $I = [0,1]$ and $I_{n,k} := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ for $n \in \mathbb{N}$ and $k = 0,1,\ldots,n$. Now choose the cumulative function

$$F_{n,k}(y) = (n+1)\int_{k/(n+1)}^{y} dt = (n+1)\Big(y - \frac{k}{n+1}\Big), \qquad \frac{k}{n+1} < y \le \frac{k+1}{n+1},$$

and set $r_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$. Let $\mu_{n,k}$ be the Borel measure corresponding to $F_{n,k}$. By the definition of the Lebesgue–Stieltjes integral we have

$$\int\limits_{I_{n,k}} f(y)\, dF_{n,k}(y) = \int\limits_{I_{n,k}} f(y)\, d\mu_{n,k}(y).$$

Then after some simple calculations, we obtain the Bernstein–Kantorovich operators (see [17])

$$U_n(f;x) = (n+1) \sum_{k=0}^{n} r_{n,k}(x) \Big(\int\limits_{I_{n,k}} f(y)\, dy \Big), \qquad x \in [0,1].$$

4. Let $I = [0,\infty)$ and $I_{n,k} := \left[\frac{k}{n}, \frac{k+1}{n}\right]$ for $n \in \mathbb{N}$ and $k = 0,1,\dots,n$. Define

$$F_{n,k}(y) = (n+1) \int\limits_{k/n}^{y} dt = (n+1)\left(y - \frac{k}{n}\right), \qquad \frac{k}{n} < y \le \frac{k+1}{n}.$$

Letting $r_{n,k}(x) = \binom{n}{k} x^k (1+x)^{-n}$, $x \in [0,\infty)$, one can easily get the Balász–Kantorovich operators (see [1])

$$K_n(f;x) = n \sum_{k=0}^{n} r_{n,k}(x) \Bigg(\int\limits_{I_{n,k}} f(y)\, dy \Bigg), \qquad x \in [0,\infty).$$

In a similar manner our operators D_n generate many other Durrmeyer type and Kantorovich type generalizations of operators well known in approximation theory.

2. A-Statistical approximation. In this section we give an A-statistical approximation theorem for the operators D_n defined by (1.3).

THEOREM 2.1. *Let I be an arbitrary interval of $\mathbb{R}$. Let $A = (a_{jn})$ be a non-negative regular summability matrix and fix $x \in I$. Assume that g is a function such that $f_2(y) = y^2$ is in $C_g(I)$ and* (1.2) *holds. Then, for all $f \in C_g(I)$,*

$$\mathrm{st}_A\text{-}\lim_n |D_n(f;x) - f(x)| = 0$$

if and only if

$$\mathrm{st}_A\text{-}\lim_n |D_n(f_v;x) - f_v(x)| = 0 \quad \textit{with} \quad f_v(y) = y^v \ (v = 1,2). \tag{2.1}$$

Proof. Necessity is clear. To prove sufficiency assume that (2.1) holds. Let $f \in C_g(I)$ and fix $x \in I$. By the continuity of f on I, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for $y \in I_\delta$. Then using positivity and linearity of D_n, we find, by (1.1), that

$$\begin{aligned}|D_n(f;x)-f(x)| &\le D_n(|f(y)-f(x)|;x)\\ &= \sum_{k=0}^{\infty} r_{n,k}(x)\int\limits_{I_\delta}|f(y)-f(x)|\,d\mu_{n,k}(y)\\ &\quad+\sum_{k=0}^{\infty} r_{n,k}(x)\int\limits_{I\setminus I_\delta}|f(y)-f(x)|\,d\mu_{n,k}(y)\end{aligned}$$

and, by (1.5),

$$|D_n(f;x)-f(x)|\le \varepsilon+\sum_{k=0}^{\infty} r_{n,k}(x)\int\limits_{I\setminus I_\delta}|f(y)-f(x)|\,d\mu_{n,k}(y). \tag{2.2}$$

It follows from the Hölder inequality that

$$\int\limits_{I\setminus I_\delta}|f(y)-f(x)|\,d\mu_{n,k}(y)\le\Big\{\int\limits_{I\setminus I_\delta}d\mu_{n,k}(y)\Big\}^{1/p}\times\Big\{\int\limits_{I\setminus I_\delta}|f(y)-f(x)|^q\,d\mu_{n,k}(y)\Big\}^{1/q} \tag{2.3}$$

where $1/p+1/q=1$ and $p>1$. By hypothesis and the definition of g we conclude that $f\in C_g(I)$ implies $f^q\in C_g(I)$ and also that there exists a positive number K such that

$$\Big\{\int\limits_{I\setminus I_\delta}|f(y)-f(x)|^q\,d\mu_{n,k}(y)\Big\}^{1/q}<K\quad\text{for } n\in\mathbb{N}\text{ and } k\in\mathbb{N}_0. \tag{2.4}$$

Combining (2.2)–(2.4) we have

$$|D_n(f;x)-f(x)|\le\varepsilon+K\sum_{k=0}^{\infty} r_{n,k}(x)\Big\{\int\limits_{I\setminus I_\delta}d\mu_{n,k}(y)\Big\}^{1/p},$$

and hence

$$|D_n(f;x)-f(x)|\le\varepsilon+K\sum_{k=0}^{\infty} r_{n,k}^{1/q}(x)\Big\{\int\limits_{I\setminus I_\delta}r_{n,k}(x)\,d\mu_{n,x}(y)\Big\}^{1/p}. \tag{2.5}$$

Applying again the Hölder inequality in (2.5) we may write

$$|D_n(f;x)-f(x)|\le\varepsilon+K\Big\{\sum_{k=0}^{\infty} r_{n,k}(x)\Big\}^{1/q}\times\Big\{\sum_{k=0}^{\infty} r_{n,k}(x)\int\limits_{I\setminus I_\delta}d\mu_{n,k}(y)\Big\}^{1/p}. \tag{2.6}$$

If $y \in I \setminus I_\delta$, then $|y-x| \geq \delta$, which implies that $(y-x)^2/\delta^2 \geq 1$. Using this fact and (1.5) we conclude from (2.6) that

$$\begin{aligned} |D_n(f;x) - f(x)| &\leq \varepsilon + \frac{K}{\delta^{2/p}} \Big\{ \sum_{k=0}^{\infty} r_{n,k}(x) \int_{I \setminus I_\delta} (y-x)^2 \, d\mu_{n,k}(y) \Big\}^{1/p} \\ &\leq \varepsilon + \frac{K}{\delta^{2/p}} \Big\{ \sum_{k=0}^{\infty} r_{n,k}(x) \int_I (y-x)^2 \, d\mu_{n,k}(y) \Big\}^{1/p} \\ &= \varepsilon + \frac{K}{\delta^{2/p}} \{ D_n((y-x)^2; x) \}^{1/p} \\ &\leq \varepsilon + \frac{K}{\delta^{2/p}} \{ |D_n(f_2;x) - f_2(x)| \\ &\qquad + 2|x| \, |D_n(f_1;x) - f_1(x)| \}^{1/p}. \end{aligned}$$

Using the inequality $|x+y|^\alpha \leq |x|^\alpha + |y|^\alpha$ for each $\alpha \in (0,1]$, we have

$$\begin{aligned} |D_n(f;x) - f(x)| &\leq \varepsilon + M(x)\{ |D_n(f_2;x) - f_2(x)|^{1/p} \\ &\qquad + |D_n(f_1;x) - f_1(x)|^{1/p} \}, \end{aligned} \tag{2.7}$$

where $M(x) := \max\{K/\delta^{2/p}, K(2|x|/\delta^2)^{1/p}\}$. Observe that condition (2.1) implies

$$\mathrm{st}_A\text{-}\lim_n |D_n(f_v;x) - f_v(x)|^{1/p} = 0, \qquad (v = 1,2). \tag{2.8}$$

Now for a given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$ and define

$$\begin{aligned} U &:= \{ n : |D_n(f;x) - f(x)| \geq r \}, \\ U_1 &:= \Big\{ n : |D_n(f_1;x) - f_1(x)|^{1/p} \geq \frac{r - \varepsilon}{2M(x)} \Big\}, \\ U_2 &:= \Big\{ n : |D_n(f_2;x) - f_2(x)|^{1/p} \geq \frac{r - \varepsilon}{2M(x)} \Big\}. \end{aligned}$$

From (2.7) it is clear that $U \subseteq U_1 \cup U_2$. Thus, for every $j \in \mathbb{N}$, we obtain

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn}. \tag{2.9}$$

Letting $j \to \infty$ in (2.9) and using (2.8) completes the proof. ∎

Since I is an arbitrary interval, note that our A-statistical approximation in Theorem 2.1 works pointwise. But if I is a closed and bounded interval, say $I = [a,b]$, then the above proof leads to the next result immediately.

COROLLARY 2.2. *Let $A = (a_{jn})$. Then, for all $f \in C[a,b]$,*

$$\mathrm{st}_A\text{-}\lim_n \|D_n(f) - f\|_{C[a,b]} = 0$$

if and only if

$$\mathrm{st}_A\text{-}\lim_n \|D_n(f_v) - f_v\|_{C[a,b]} = 0 \quad \textit{with } f_v(y) = y^v \quad (v = 1, 2),$$

where $\|\cdot\|_{C[a,b]}$ *denotes the usual supremum norm in* $C[a,b]$.

Now if we replace the matrix $A = (a_{jn})$ in Corollary 2.2 by the identity matrix, the following classical Korovkin theorem result follows at once.

COROLLARY 2.3. *For all* $f \in C[a,b]$, *the sequence* $\{D_n(f)\}$ *is uniformly convergent to* f *if and only if*, *for each* $v = 1, 2$, $\{D_n(f_v)\}$ *converges uniformly to* f_v.

3. Rates of A-statistical convergence. In this section, we compute the rates of A-statistical convergence in Theorem 2.1 with the help of the modulus of continuity and elements of the Lipschitz class. We also deal with the local approximation properties of the operators D_n given by (1.3).

Let $f \in C(I)$. The *modulus of continuity* of f, denoted by $w(f,\delta)$, is defined to be

$$w(f,\delta) = \sup_{\substack{x,y \in I \\ |x-y| < \delta}} |f(x) - f(y)| \quad (\delta > 0).$$

It is well known that for any constants $c, \delta > 0$,

$$w(f, c\delta) \le (1+c) w(f,\delta) \tag{3.1}$$

(see [2], [21] for details).

Now we have the following

THEOREM 3.1. *Let* I *and* g *be as in Theorem* 2.1. *Assume that condition* (1.1) *holds and fix* $x \in I$. *Then*, *for all* $f \in C_g(I)$,

$$|D_n(f;x) - f(x)| \le 2w(f,\delta_n), \qquad n \in \mathbb{N},$$

where

$$\delta_n := [D_n((y-x)^2; x)]^{1/2}. \tag{3.2}$$

Proof. Let $f \in C_g(I)$ and fix $x \in I$. By (3.1), for any $\delta > 0$, we get

$$\begin{aligned} |D_n(f;x) - f(x)| &\le D_n(|f(y) - f(x)|; x) \le D_n(w(f, |y-x|); x) \\ &\le w(f,\delta) D_n\left(1 + \frac{|y-x|}{\delta}; x\right) \\ &\le w(f,\delta)\left\{1 + \frac{1}{\delta} D_n(|y-x|; x)\right\}. \end{aligned}$$

Then from the Cauchy–Schwarz inequality for positive functionals (see, for instance, [4, p. 31]) and using (1.1), we find

$$|D_n(f;x) - f(x)| \le w(f,\delta)\left\{1 + \frac{1}{\delta} [D_n(|y-x|^2; x)]^{1/2}\right\}.$$

Taking $\delta := \delta_n = [D_n(|y-x|^2; x)]^{1/2}$ we get the result. ∎

Observe that condition (2.1) yields $\mathrm{st}_A\text{-}\lim_n \delta_n = 0$, which implies $\mathrm{st}_A\text{-}\lim_n w(f,\delta_n) = 0$. Hence, Theorem 3.1 gives us the rate of A-statistical convergence in Theorem 2.1 by means of the modulus of continuity.

We will now study the rate of A-statistical convergence of the positive linear operators D_n with the help of elements of the Lipschitz class $\mathrm{Lip}_M(\alpha)$, where $M > 0$ and $0 < \alpha \le 1$.

We recall that a function $f \in C(I)$ belongs to $\mathrm{Lip}_M(\alpha)$ if

$$|f(y) - f(x)| \le M|y-x|^\alpha \qquad (y, x \in I,\ 0 < \alpha \le 1). \tag{3.3}$$

Choose an interval I and a function g such that $C_g(I) \cap \mathrm{Lip}_M(\alpha) \ne \emptyset$. Then we have the following result.

THEOREM 3.2. *Fix $x \in I$. For all $f \in C_g(I) \cap \mathrm{Lip}_M(\alpha)$, $0 < \alpha \le 1$,*

$$|D_n(f;x) - f(x)| \le M\delta_n^\alpha,$$

where δ_n is as in (3.2).

Proof. Let $f \in C_g(I) \cap \mathrm{Lip}_M(\alpha)$ with $0 < \alpha \le 1$, and fix $x \in I$. By linearity and monotonicity of D_n and using (3.3) we have

$$\begin{aligned}|D_n(f;x) - f(x)| &\le D_n(|(f(y) - f(x)|;x) \\ &= \sum_{k=0}^{\infty} r_{n,k}(x) \int_I |f(y) - f(x)|\, d\mu_{n,k}(y) \\ &\le M \sum_{k=0}^{\infty} r_{n,k}(x) \int_I |y-x|^\alpha\, d\mu_{n,k}(y).\end{aligned}$$

Applying the Hölder inequality with $p = 2/\alpha$, $q = 2/(2-\alpha)$ we get

$$\begin{aligned}|D_n(f;x) - f(x)| &\le M \sum_{k=0}^{\infty} r_{n,k}(x) \Big\{ \int_I (y-x)^2\, d\mu_{n,k}(y) \Big\}^{\alpha/2} \\ &\le M \sum_{k=0}^{\infty} (r_{n,k}(x))^{(2-\alpha)/2} \Big\{ \int_I r_{n,k}(x)(y-x)^2\, d\mu_{n,k}(y) \Big\}^{\alpha/2}.\end{aligned}$$

If we use again the Hölder inequality, then we conclude that

$$\begin{aligned}|D_n(f;x) - f(x)| \le M &\Big\{ \sum_{k=0}^{\infty} r_{n,k}(x) \Big\}^{(2-\alpha)/2} \\ &\times \Big\{ \sum_{k=0}^{\infty} r_{n,k}(x) \int_I (y-x)^2\, d\mu_{n,k}(y) \Big\}^{\alpha/2}\end{aligned}$$

and therefore

$$|D_n(f;x) - f(x)| \le M[D_n((y-x)^2;x)]^{\alpha/2}. \tag{3.4}$$

Taking $\delta_n := [D_n((y-x)^2;x)]^{1/2}$ in (3.4) completes the proof. ∎

To close this section we state a local approximation result for the operators D_n given by (1.3). Note that some local smoothness theorems for the Baskakov–Durrmeyer operators may be found in [22].

THEOREM 3.3. *Let* $0 < \alpha \leq 1$ *and* E *be any subset of the interval* I. *Then, if* $f \in C_g(I)$ *is locally* $\mathrm{Lip}(\alpha)$, *i.e.*

$$|f(y) - f(x)| \leq M|y-x|^\alpha, \qquad y \in E \text{ and } x \in I, \tag{3.5}$$

then, for each $x \in I$,

$$|D_n(f;x) - f(x)| \leq M\{\delta_n^\alpha + 2(d(x,E))^\alpha\},$$

where δ_n *is as in* (3.2), M *is a constant depending on* α *and* f, *and* $d(x,E) = \inf\{|y-x| : y \in E\}$.

Proof. Let $\overline{E}$ denote the closure of E in I. Then there exists $x_0 \in \overline{E}$ such that $|x - x_0| = d(x,E)$. Using the triangle inequality

$$|f(y) - f(x)| \leq |f(y) - f(x_0)| + |f(x) - f(x_0)|$$

we get, by (3.5),

$$\begin{aligned}
|D_n(f;x) - f(x)| &\leq D_n(|f(y) - f(x)|;x) \\
&\leq D_n(|f(y) - f(x_0)|;x) + |f(x) - f(x_0)| \\
&\leq M\{D_n(|y-x_0|^\alpha;x) + |x-x_0|^\alpha\} \\
&\leq M\{D_n(|y-x|^\alpha;x) + 2|x-x_0|^\alpha\} \\
&= M\Big\{\sum_{k=0}^{\infty} r_{n,k}(x) \int_I |y-x_0|^\alpha \, d\mu_{n,k}(y) + 2(d(x,E))^\alpha\Big\}.
\end{aligned}$$

As in the proof of Theorem 3.2, using the Hölder inequality with $p = 2/\alpha$, $q = 2/(2-\alpha)$ we find that

$$|D_n(f;x) - f(x)| \leq M\{\delta_n^\alpha + 2(d(x,E))^\alpha\},$$

where δ_n is given by (3.2). ∎

Note that if we set $E = I$ in Theorem 3.3, then the term $d(x,E)$ vanishes, and we get Theorem 3.2 at once.

4. The rth order generalization of the operators D_n. Define

$$C_g^{(r)}(I) = \{f : f^{(r)} \in C_g(I)\} \qquad (r = 0,1,2,\ldots).$$

If $r = 0$, then $C_g^{(0)}(I) = C_g(I)$.

We now consider the rth order generalization of the operators D_n defined as follows:

$$D_n^{[r]}(f;x) = \sum_{k=0}^{\infty} \sum_{i=0}^{r} r_{n,k}(x) \int_I f^{(i)}(y) \, \frac{(x-y)^i}{i!} \, d\mu_{n,k}(y) \tag{4.1}$$

where $f \in C_g^{(r)}(I)$ $(r = 0, 1, 2, \ldots)$, $n \in \mathbb{N}$ and $r_{n,k}(x)$ satisfies conditions (1.4) and (1.5).

This kind of generalization was also considered in [5], [18]. Note that taking $r = 0$ we have

$$D_n^{[0]}(f; x) = D_n(f; x).$$

We now obtain the following approximation theorem for the operators $D_n^{[r]}$ given by (4.1).

THEOREM 4.1. *Let I be an arbitrary interval of the real line. Then for all $f \in C_g^{(r)}(I)$ such that $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, $0 < \alpha \le 1$, and for each $x \in I$, we have*

$$|D_n^{[r]}(f; x) - f(x)| \le C D_n(|x - y|^{\alpha+r}; x)$$

where

$$C = \frac{M\alpha}{\alpha + r}\,\frac{B(\alpha, r)}{(r-1)!},$$

and $B(\alpha, r)$ is the beta function.

Proof. By (4.1) and (1.1) we get

$$\begin{aligned} (4.2)\qquad & f(x) - D_n^{[r]}(f; x) \\ &= \sum_{k=0}^{\infty} r_{n,k}(x) \int_I \left\{ f(x) - \sum_{i=0}^{r} f^{(i)}(y)\,\frac{(x-y)^i}{i!} \right\} d\mu_{n,k}(y). \end{aligned}$$

From the Taylor formula (see [18]),

$$\begin{aligned} (4.3)\qquad & f(x) - \sum_{i=0}^{r} f^{(i)}(y)\frac{(x-y)^i}{i!} \\ &= \frac{(x-y)^r}{(r-1)!} \int_0^1 (1-t)^{r-1} [f^{(r)}(y + t(x-y)) - f^{(r)}(y)]\,dt. \end{aligned}$$

Since $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, we have

$$(4.4)\qquad |f^{(r)}(y + t(x-y)) - f^{(r)}(y)| \le M t^\alpha |x-y|^\alpha.$$

Considering (4.4) in (4.3), and using the beta integral, we conclude that

$$(4.5)\qquad f(x) - \sum_{i=0}^{r} f^{(i)}(y)\,\frac{(x-y)^i}{i!} \le |x-y|^{\alpha+r}\,\frac{M\alpha}{\alpha+r}\,\frac{B(\alpha,r)}{(r-1)!}.$$

By using (4.5) in (4.2), we get

$$\begin{aligned} |f(x) - D_n^{[r]}(f; x)| &\le \frac{M\alpha}{\alpha+r}\,\frac{B(\alpha,r)}{(r-1)!} \sum_{k=0}^{\infty} r_{n,k}(x) \int_I |x-y|^{\alpha+r}\,d\mu_{n,k}(y) \\ &= \frac{M\alpha}{\alpha+r}\,\frac{B(\alpha,r)}{(r-1)!}\,D_n(|x-y|^{\alpha+r}; x), \end{aligned}$$

which gives the desired result. ∎

From Theorem 4.1 one can get the following corollary immediately.

Corollary 4.2. *If condition* (2.1) *holds, then for all* $f \in C_g^{(r)}(I)$ *such that* $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, $0 < \alpha \leq 1$, *we have*

$$\mathrm{st}_A\text{-}\lim_n |D_n^{[r]}(f;x) - f(x)| = 0.$$

Finally, using Theorem 4.1 one can show that Theorems 3.1 and 3.2 contain the following results, respectively.

Corollary 4.3. *Let* δ_n *be as in* (3.2)*. Then, for all* $f \in C_g^{(r)}(I)$ *such that* $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, $0 < \alpha \leq 1$, *we have*

$$|D_n^{[r]}(f;x) - f(x)| \leq M' w(|x-y|^{\alpha+r}, \delta_n)$$

where

$$M' = \frac{2M\alpha}{\alpha + r} \, \frac{B(\alpha, r)}{(r-1)!}.$$

Corollary 4.4. *Under the conditions of Corollary* 4.3, *we have*

$$|D_n^{[r]}(f;x) - f(x)| \leq M'' \, \delta_n,$$

where

$$M'' = \frac{M^2\alpha}{\alpha + r} \, \frac{B(\alpha, r)}{(r-1)!}.$$

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