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Uniform factorization for compact sets of weakly compact operators

by

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Abstract. We prove uniform factorization results that describe the factorization of compact sets of compact and weakly compact operators via Hölder continuous homeomorphisms having Lipschitz continuous inverses. This yields, in particular, quantitative strengthenings of results of Graves and Ruess on the factorization through ℓ_p-spaces and of Aron, Lindström, Ruess, and Ryan on the factorization through universal spaces of Figiel and Johnson. Our method is based on the isometric version of the Davis–Figiel–Johnson–Pełczyński factorization construction due to Lima, Nygaard, and Oja.

1. Introduction. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y, and by $\mathcal{F}(X,Y)$, $\overline{\mathcal{F}}(X,Y)$, $\mathcal{K}(X,Y)$, and $\mathcal{W}(X,Y)$ its subspaces of finite rank, approximable, compact, and weakly compact operators. If $\mathcal{A}$ is $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{K}$, $\mathcal{W}$, or $\mathcal{L}$, then $\mathcal{A}_{w^*}(X^*,Y)$ denotes the subspace of $\mathcal{A}(X^*,Y)$ consisting of those operators which are weak*-weak continuous.

In 1987, Graves and Ruess (see [GR2, Theorem 2.1]) proved the following factorization result for compact operators between special spaces.

Theorem 1 (Graves and Ruess). *Let X be an $\mathcal{L}_1$-space (respectively, an $\mathcal{L}_\infty$-space) and let Y be a Banach space. Let $\mathcal{C}$ be a relatively compact subset of $\mathcal{K}(X,Y)$. Then there exist an operator $u \in \mathcal{K}(X,\ell_1)$ (respectively, $u \in \mathcal{K}(X,c_0)$) and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{K}(\ell_1,Y)$ (respectively, of $\mathcal{K}(c_0,Y)$) such that $S = A_S \circ u$ for all $S \in \mathcal{C}$.*

The uniform factorization of compact operators in a general setting was studied by Aron, Lindström, Ruess, and Ryan. In 1999, the following result was obtained (see [ALRR, Theorem 1]) where Z_{FJ} denotes a universal factor-

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ization space of Figiel [F] and Johnson [J] (for instance, $Z_{\mathrm{FJ}} = (\sum_{W\subset C_p} W)_p$ where W runs through the closed subspaces of C_p for any fixed p).

THEOREM 2 (Aron *et al.*). *Let X and Y be Banach spaces and let $\mathcal{C}$ be a relatively compact subset of $\mathcal{K}_{w^*}(X^*,Y)$. Then there exist operators $u \in \mathcal{K}_{w^*}(X^*, Z_{\mathrm{FJ}})$ and $v \in \mathcal{K}(Z_{\mathrm{FJ}}, Y)$, and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{K}(Z_{\mathrm{FJ}}, Z_{\mathrm{FJ}})$ such that $S = v \circ A_S \circ u$ for all $S \in \mathcal{C}$.*

Since, in the setting of Theorem 1, every single compact operator factors compactly through ℓ_1 (see [R, Corollary 7]) or, respectively, through c_0 (see [T, p. 252] or [Da, Proposition 5.12]), Theorem 2 easily implies Theorem 1 (see [ALRR, Corollary 4]).

Theorems 1 and 2, together with their proofs in [GR2] and [ALRR], do not give much information about mapping properties of the correspondence $S \mapsto A_S$, $S \in \mathcal{C}$. For instance, one does not even have any estimate for $\operatorname{diam}\{A_S : S \in \mathcal{C}\}$.

A purpose of this article is to get quantitative strengthenings of Theorems 1 and 2 (see Theorems 14 and 10). Actually, we apply a general unified approach, different from [GR2] and [ALRR], and, in our opinion, much easier, to obtain uniform factorization results for compact subsets of compact operators as well as of weakly compact operators. The idea (see Lemmas 4 and 5) consists in constructing a mapping $S \mapsto A_S$ from a compact subset $\mathcal{C}$ of weakly compact operators that preserves compact operators, as well as finite rank operators. This mapping is Hölder continuous, being also bijective and having a 1-Lipschitz continuous inverse, and $\operatorname{diam}\{A_S : S \in \mathcal{C}\} = \operatorname{diam}\mathcal{C}$ whenever $0 \in \mathcal{C}$.

Our construction will be based on the isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization lemma [DFJP] due to Lima, Nygaard, and Oja [LNO]. For comparison, let us remark that the technical proof in [GR2] relies on Ruess's characterization [Ru] of relatively compact sets in $\mathcal{K}_{w^*}(X^*,Y)$, and uses Saphar's tensor products machinery [S]. The paper [ALRR] presents two different methods of proof: one being essentially based on Grothendieck's characterization [G] of relatively compact sets in the projective tensor product of Banach spaces, the other—on the Banach–Dieudonné theorem.

Our notation is rather standard. A Banach space X will always be regarded as a subspace of its bidual X^{**} under the canonical embedding. The closed unit ball of X is denoted by B_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The linear span of A is denoted by $\operatorname{span} A$ and the closed convex hull by $\overline{\operatorname{conv}} A$. Let us recall that $T \in \mathcal{L}(X^*,Y)$ is weak*-weak continuous if and only if $\operatorname{ran} T^* \subset X$. Recall also that $\mathcal{L}_{w^*}(X^*,Y) = \mathcal{W}_{w^*}(X^*,Y)$ (if $T \in \mathcal{L}(X^*,Y)$ is weak*-weak continuous, then $T(B_{X^*})$ is weakly compact because B_{X^*} is compact in the weak* topology).

For the definition and basic properties of $\mathcal{L}_{p,\lambda}$-spaces and $\mathcal{L}_p$-spaces, $1 \le p \le \infty$, $1 \le \lambda < \infty$, the reader is referred to [LP] and [LR], or [JL, pp. 57–60]. For the universal spaces C_p, $1 \le p \le \infty$, see [J] or, e.g., [Si, pp. 422–426]. We use the symbol ℓ_∞ for the Banach space of null sequences, usually denoted by c_0.

2. Main factorization lemmas for compact subsets of weakly compact operators. Our main Lemmas 4 and 5 below rely on Lemma 3 which is an isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization lemma [DFJP] due to Lima, Nygaard, and Oja [LNO]. Let us recall the relevant construction.

Define $f : (1,\infty) \to (0,\infty)$ by

$$f(a) = \Big(\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2}\Big)^{1/2}.$$

The function f is continuous, strictly decreasing, $\lim_{a\to 1+} f(a) = \infty$, and $\lim_{a\to\infty} f(a) = 0$. Hence, there exists a unique $a \in (1,\infty)$ such that $f(a) = 1$. Let us fix this a.

Let Y be a Banach space and let K be a closed absolutely convex subset of B_Y. For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_Y$. The gauge of B_n gives an equivalent norm $\|\cdot\|_n$ on Y. Set

$$\|y\|_K = \Big(\sum_{n=1}^{\infty} \|y\|_n^2\Big)^{1/2},$$

define $Y_K = \{y \in Y : \|y\|_K < \infty\}$, and let $J_K : Y_K \to Y$ denote the identity embedding.

Lemma 3 (see [DFJP] and [LNO]). *With notation as above*, *the following holds*:

- (i) $Y_K = (Y_K, \|\cdot\|_K)$ *is a Banach space and* $\|J_K\| \le 1$.
- (ii) $K \subset B_{Y_K} \subset B_Y$.
- (iii) *If* $y \in K$, *then* $\|y\|_K^2 \le (1/4 + 1/\ln a)\|y\|$.
- (iv) *The* Y-*norm and* Y_K-*norm topologies coincide on* K.
- (v) J_K^{**} *is injective.*
- (vi) J_K *is compact if and only if* K *is compact*; *in this case* Y_K *is separable.*
- (vii) Y_K *is reflexive if and only if* K *is weakly compact.*

Remark. By [LNO] a “good” estimate of a is $\exp(4/9)$. This is an estimate from below. Hence

$$1/4 + 1/\ln a < 5/2.$$

LEMMA 4. *Let X and Y be Banach spaces. Let $\mathcal{C}$ be a compact subset of $\mathcal{L}_{w^*}(X^*,Y)$. Then there exist a weakly compact absolutely convex subset K of B_Y, which is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*,Y)$, and a linear mapping $\Phi:\operatorname{span}\mathcal{C}\to\mathcal{L}_{w^*}(X^*,Y_K)$ such that $S=J_K\circ\Phi(S)$ for all $S\in\operatorname{span}\mathcal{C}$ and $\|J_K\|=1$. Moreover, if $S\in\operatorname{span}\mathcal{C}$, then*

(i) *S has finite rank if and only if $\Phi(S)$ has finite rank,*
(ii) *S is compact if and only if $\Phi(S)$ is compact.*

The mapping Φ restricted to $\mathcal{C}\cup\{0\}$ is a homeomorphism satisfying

$$\begin{aligned}\|S-T\| &\le \|\Phi(S)-\Phi(T)\|\\ &\le \min\{\mathrm{d},\mathrm{d}^{1/2}(1/4+1/\ln a)^{1/2}\|S-T\|^{1/2}\},\qquad S,T\in\mathcal{C}\cup\{0\},\end{aligned}$$

where

$$\mathrm{d}=\operatorname{diam}\mathcal{C}\cup\{0\}.$$

In particular, if $-S\in\mathcal{C}$ for some $S\in\mathcal{C}$, then

$$\|\Phi(S)\|\le\min\{\mathrm{d}/2,(\mathrm{d}/2)^{1/2}(1/4+1/\ln a)^{1/2}\|S\|^{1/2}\}.$$

Proof. Let

$$K=\overline{\operatorname{conv}}\{\mathrm{d}^{-1}(S-T)x^*: S,T\in\mathcal{C}\cup\{0\},\ x^*\in B_{X^*}\}.$$

Then K is contained in B_Y, K is closed and absolutely convex, hence weakly closed.

To prove that K is weakly compact, fix an arbitrary $\varepsilon>0$. We shall find a weakly compact subset K_ε of Y such that $K\subset K_\varepsilon+\varepsilon B_Y$. Then the weak compactness of K will be immediate from Grothendieck's lemma (see, e.g., [D, p. 227]). Let $\{U_1,\dots,U_n\}$ be an ε-net in the compact subset

$$\{\mathrm{d}^{-1}(S-T): S,T\in\mathcal{C}\cup\{0\}\}$$

of $\mathcal{L}_{w^*}(X^*,Y)$. Denoting by K_ε the closed convex hull of the weakly compact set $\overline{U_1(B_{X^*})}\cup\dots\cup\overline{U_n(B_{X^*})}$, which is weakly compact by a classical result of Krein and Šmulian, it is straightforward to verify that $K\subset K_\varepsilon+\varepsilon B_Y$ as desired.

If $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*,Y)$, then K_ε is compact (by a theorem of Mazur), implying that also K is compact.

Let the Banach space Y_K and the identity embedding $J_K:Y_K\to Y$ with $\|J_K\|\le 1$ be as in Lemma 3. Since $K\subset B_{Y_K}$,

$$\|J_K\|=\sup_{z\in B_{Y_K}}\|z\|\ge\sup_{z\in K}\|z\|\ge\mathrm{d}^{-1}\sup_{S,T\in\mathcal{C}\cup\{0\}}\|S-T\|=1.$$

Hence $\|J_K\|=1$.

Let $S \in \operatorname{span} \mathcal{C}$. Then

$$\begin{aligned}\operatorname{ran} S &\subset \operatorname{span}\{Sx^* : S \in \mathcal{C},\, x^* \in X^*\} \\ &\subset \operatorname{span}\{(S-T)x^* : S,T \in \mathcal{C} \cup \{0\},\, x^* \in B_{X^*}\} \\ &\subset \operatorname{span} K \subset \operatorname{span} B_{Y_K} = Y_K.\end{aligned}$$

This permits us to define $\Phi(S) : X^* \to Y_K$ by

$$\Phi(S)x^* = Sx^*, \qquad x^* \in X^*.$$

Since $\Phi(S)$ is algebraically the same operator as S, we see that $\Phi(S)$ is linear, and $S = J_K \circ \Phi(S)$.

Let $S,T \in \mathcal{C} \cup \{0\}$. Then $\mathrm{d}^{-1}(S-T)x^* \in K \subset B_{Y_K}$ for all $x^* \in B_{X^*}$. Hence

$$\|\Phi(S-T)\| = \sup_{x^* \in B_{X^*}} \|(S-T)x^*\|_K \le \mathrm{d}, \qquad S,T \in \mathcal{C} \cup \{0\}. \tag{1}$$

This implies, in particular, that $\|\Phi(S)\| < \infty$ for all $S \in \operatorname{span} \mathcal{C}$. Every $\Phi(S)$, $S \in \operatorname{span} \mathcal{C}$, is also weak*-weak continuous because, $J_K^*(Y^*)$ being norm dense in Y_K^* (since J_K^{**} is injective by Lemma 3), we have

$$\begin{aligned}(\Phi(S))^*(Y_K^*) &= (\Phi(S))^*(\overline{J_K^*(Y^*)}) \subset \overline{((\Phi(S))^* \circ J_K^*)(Y^*)} \\ &= \overline{S^*(Y^*)} \subset \overline{X} = X.\end{aligned}$$

Consequently, Φ is a linear mapping from $\operatorname{span} \mathcal{C}$ to $\mathcal{L}_{w^*}(X^*, Y_K)$.

Since $S \in \operatorname{span} \mathcal{C}$ and $\Phi(S)$ are algebraically the same operators, clearly (i) holds. Condition (ii) holds by Lemma 3(iv) (and by the linearity of Φ) because $\mathrm{d}^{-1}S(B_{X^*}) \subset K$ for all $S \in \mathcal{C}$.

Finally, let $S,T \in \mathcal{C} \cup \{0\}$. Then, by (1),

$$\|S-T\| \le \|J_K\|\,\|\Phi(S-T)\| = \|\Phi(S) - \Phi(T)\| \le \mathrm{d}.$$

Since $\mathrm{d}^{-1}(S-T)x^* \in K$ for all $x^* \in B_{X^*}$, using Lemma 3(iii), we also have

$$\begin{aligned}\|\Phi(S) - \Phi(T)\| &= \sup_{x^* \in B_{X^*}} \|(S-T)x^*\|_K \\ &\le \mathrm{d}^{1/2}(1/4 + 1/\ln a)^{1/2} \sup_{x^* \in B_{X^*}} \|(S-T)x^*\|^{1/2} \\ &= \mathrm{d}^{1/2}(1/4 + 1/\ln a)^{1/2}\|S-T\|^{1/2}.\end{aligned}$$

If, in particular, $S, -S \in \mathcal{C}$, then the desired estimate for the norm of $\Phi(S) = (\Phi(S) - \Phi(-S))/2$ immediately follows from the above. ∎

LEMMA 5. *Let X and Y be Banach spaces. Let $\mathcal{C}$ be a compact subset of $\mathcal{L}_{w^*}(X^*, Y)$. Then there exist a reflexive Banach space Z, a norm one operator $J \in \mathcal{L}_{w^*}(X^*, Z)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}(Z, Y)$ satisfying conditions* (i) *and* (ii) *of Lemma 4 such that $S = \Phi(S) \circ J$ for all $S \in \operatorname{span} \mathcal{C}$. Moreover, $Z = X_K^*$ and $J = J_K^*$ for some weakly compact absolutely convex subset K of B_X, and if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then*

Z is separable and $J \in \mathcal{K}_{w^*}(X^*, Z)$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 4.

Proof. Applying Lemma 4 to the compact subset $\mathcal{C}^* = \{S^* : S \in \mathcal{C}\}$ of $\mathcal{L}_{w^*}(Y^*, X)$, we can find a weakly compact absolutely convex subset K of B_X, which is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (since S^* is compact if and only if S is). We can also find a linear mapping

$$\Psi : \operatorname{span} \mathcal{C}^* \to \mathcal{L}_{w^*}(Y^*, X_K)$$

satisfying the conclusions of Lemma 4 such that $S^* = J_K \circ \Psi(S^*)$ for all $S \in \operatorname{span} \mathcal{C}$, and we know that $\|J_K\| = 1$.

Let $Z = X_K^*$ and $J = J_K^*$. Then Z is reflexive by Lemma 3(vii), $\|J\| = 1$, and $J \in \mathcal{L}_{w^*}(X^*, Z)$ since $\operatorname{ran} J_K^{**} \subset X$ because Z is reflexive. The reflexive space Z is separable and the operator J is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 3(vi)).

Define $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}(Z, Y)$ by

$$\Phi(S) = (\Psi(S^*))^*, \qquad S \in \operatorname{span} \mathcal{C}.$$

The properties of Ψ clearly imply that Φ is a linear mapping satisfying conditions (i) and (ii) of Lemma 4. If $S \in \operatorname{span} \mathcal{C}$, then $S^{**} = S$ (because $S^* \in \mathcal{L}_{w^*}(Y^*, X)$) and therefore $S = (J_K \circ \Psi(S^*))^* = \Phi(S) \circ J$. Since $\|S - T\| = \|S^* - T^*\|$ and $\|\Phi(S) - \Phi(T)\| = \|\Psi(S^*) - \Psi(T^*)\|$ for $S, T \in \operatorname{span} \mathcal{C}$, the mapping Φ restricted to $\mathcal{C} \cup \{0\}$ obviously satisfies the conclusions of Lemma 4. ∎

REMARK. Observe that $\operatorname{diam} \Phi(\mathcal{C} \cup \{0\}) = \operatorname{diam} \mathcal{C} \cup \{0\}$ in Lemmas 4 and 5.

3. Quantitative versions of the uniform factorization for compact sets of operators. For Banach spaces X and Y, let us consider the following infinite direct sum in the sense of ℓ_2:

$$Z_{(X,Y)} = \Big(\sum_K X_K^*\Big)_2 \oplus_2 \Big(\sum_L Y_L\Big)_2$$

where K and L run through the weakly compact absolutely convex subsets of B_X and B_Y, respectively. The space $Z_{(X,Y)}$ is reflexive (see Lemma 3(vii)). In Theorems 6–8 below, $Z_{(X,Y)}$ will serve as a universal factorization space for all compact sets of the space $\mathcal{L}_{w^*}(X^*, Y)$.

THEOREM 6. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}_{w^*}(X^*, Z_{(X,Y)})$ which preserves finite rank and compact operators and a norm one operator $v \in \mathcal{L}(Z_{(X,Y)}, Y)$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 4. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then $v \in \mathcal{K}(Z_{(X,Y)}, Y)$.*

Proof. Let L and φ be, respectively, the weakly compact absolutely convex subset of B_Y and the linear mapping from $\operatorname{span}\mathcal{C}$ to $\mathcal{L}_{w^*}(X^*, Y_L)$ given by Lemma 4. Let $I_L : Y_L \to Z_{(X,Y)}$ denote the natural norm one embedding and $P_L : Z_{(X,Y)} \to Y_L$ the natural norm one projection. It is straightforward to verify that the mappings Φ, defined by $\Phi(S) = I_L \circ \varphi(S)$, $S \in \operatorname{span}\mathcal{C}$, and $v = J_L \circ P_L$ have the desired properties (use Lemma 3(v) and the fact that $1 = \|J_L\| = \|v \circ I_L\| \le \|v\|$; for the "moreover" part, use Lemma 3(vi)). ∎

THEOREM 7. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist a norm one operator $u \in \mathcal{L}_{w^*}(X^*, Z_{(X,Y)})$ and a linear mapping $\Phi : \operatorname{span}\mathcal{C} \to \mathcal{L}(Z_{(X,Y)}, Y)$ which preserves finite rank and compact operators such that $S = \Phi(S) \circ u$ for all $S \in \operatorname{span}\mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 4. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then $u \in \mathcal{K}_{w^*}(X^*, Z_{(X,Y)})$.*

Proof. The proof relies on Lemma 5 and it is similar to that of Theorem 6. ∎

THEOREM 8. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist norm one operators $u \in \mathcal{L}_{w^*}(X^*, Z_{(X,Y)})$ and $v \in \mathcal{L}(Z_{(X,Y)}, Y)$, and a linear mapping $\Phi : \operatorname{span}\mathcal{C} \to \mathcal{L}(Z_{(X,Y)}, Z_{(X,Y)})$ which preserves finite rank and compact operators such that $S = v \circ \Phi(S) \circ u$ for all $S \in \operatorname{span}\mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying*

$$\begin{aligned}\|S - T\| &\le \|\Phi(S) - \Phi(T)\| \\ &\le \min\{\mathrm{d}, \mathrm{d}^{3/4}(1/4 + 1/\ln a)^{3/4}\|S - T\|^{1/4}\}, \qquad S, T \in \mathcal{C} \cup \{0\},\end{aligned}$$

where $\mathrm{d} = \operatorname{diam}\mathcal{C} \cup \{0\}$. *In particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then*

$$\|\Phi(S)\| \le \min\{\mathrm{d}/2, (\mathrm{d}/2)^{3/4}(1/4 + 1/\ln a)^{3/4}\|S\|^{1/4}\}.$$

Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then $u \in \mathcal{K}_{w^*}(X^*, Z_{(X,Y)})$ and $v \in \mathcal{K}(Z_{(X,Y)}, Y)$.

Proof. Let $K \subset B_X$, $J = J_K^* \in \mathcal{L}_{w^*}(X^*, X_K^*)$, and $\varphi : \operatorname{span}\mathcal{C} \to \mathcal{L}(X_K^*, Y)$ be, respectively, the weakly compact absolutely convex subset, the norm one operator, and the linear mapping given by Lemma 5.

Since $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{L}(X_K^*, Y) = \mathcal{L}_{w^*}(X_K^*, Y)$ (recall that X_K^* is reflexive), we can apply Lemma 4. Let $L \subset B_Y$ and $\psi : \operatorname{span}\varphi(\mathcal{C}) \to \mathcal{L}_{w^*}(X_K^*, Y_L)$ be, respectively, the weakly compact subset and the linear mapping given by Lemma 4.

Let $I_K : X_K^* \to Z_{(X,Y)}$ and $I_L : Y_L \to Z_{(X,Y)}$ denote the natural norm one embeddings, and let $P_K : Z_{(X,Y)} \to X_K^*$ and $P_L : Z_{(X,Y)} \to Y_L$ denote the natural norm one projections. It is straightforward to verify (observing that $\operatorname{diam}\varphi(\mathcal{C}\cup\{0\}) = \mathrm{d})$ that the mappings $u = I_K \circ J$, Φ defined by $\Phi(S) = I_L \circ \psi(\varphi(S)) \circ P_K$ for $S \in \operatorname{span}\mathcal{C}$, and $v = J_L \circ P_L$ have the desired properties. In particular, for all $S \in \operatorname{span}\mathcal{C}$,

$$\begin{aligned} S &= \varphi(S) \circ J = J_L \circ \psi(\varphi(S)) \circ J = J_L \circ P_L \circ I_L \circ \psi(\varphi(S)) \circ P_K \circ I_K \circ J \\ &= v \circ \Phi(S) \circ u. \end{aligned}$$

The "moreover" part uses the fact that $J \in \mathcal{K}_{w^*}(X^*, X_K^*)$ whenever $\mathcal{C} \subset \mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 5) and that, in this case, $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{K}_{w^*}(X_K^*, Y)$, implying (by Lemmas 4 and 3(vi)) the compactness of the operator J_L. ∎

Remark. Observe that $\operatorname{diam}\Phi(\mathcal{C}\cup\{0\}) = \operatorname{diam}\mathcal{C}\cup\{0\}$ in Theorems 6–8.

Remark. Theorem 8 represents a quantitative strengthening of the following result by Aron *et al.* (see [ALRR, Proposition 2]): *for Banach spaces X and Y, there exists a reflexive Banach space $Z = Z(X,Y)$ such that, for every relatively compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*,Y)$, there exist operators $u \in \mathcal{L}_{w^*}(X^*,Z)$ and $v \in \mathcal{L}(Z,Y)$, and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{L}(Z,Z)$ such that $S = v \circ A_S \circ u$ for all $S \in \mathcal{C}$.* Note that our definition of $Z_{(X,Y)}$ is much simpler than that of $Z(X,Y)$, but similar.

Since $\mathcal{W}(X,Y)$ and $\mathcal{L}_{w^*}(X^{**},Y) = \mathcal{W}_{w^*}(X^{**},Y)$, and also $\mathcal{K}(X,Y)$ and $\mathcal{K}_{w^*}(X^{**},Y)$ are canonically isometrically isomorphic under the mapping $S \mapsto S^{**}$, Theorems 6–8 yield immediate applications to factoring compact subsets of $\mathcal{W}(X,Y)$ and $\mathcal{K}(X,Y)$. We only state the corresponding application of Theorem 8, the others being similar.

Corollary 9. *Let X and Y be Banach spaces, and let $Z = Z_{(X^*,Y)}$. For every compact subset $\mathcal{C}$ of $\mathcal{W}(X,Y)$, there exist norm one operators $u \in \mathcal{W}(X,Z)$ and $v \in \mathcal{W}(Z,Y)$, and a linear mapping $\Phi : \operatorname{span}\mathcal{C} \to \mathcal{W}(Z,Z)$ which preserves finite rank and compact operators such that $S = v \circ \Phi(S) \circ u$ for all $S \in \operatorname{span}\mathcal{C}$. The mapping Φ restricted to $\mathcal{C}\cup\{0\}$ is a homeomorphism satisfying the conclusions of Theorem* 8. *Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X,Y)$, then $u \in \mathcal{K}(X,Z)$ and $v \in \mathcal{K}(Z,Y)$.*

4. Quantitative versions of the uniform factorization for compact sets of compact operators. Theorem 8, combined with the well known factorization methods by Johnson [J] and Figiel [F], yield the following quantitative strengthening of Theorem 2 of Aron *et al.* (see the Introduction).

Theorem 10. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*,Y)$. Then, for every $\varepsilon > 0$, there exist operators $u \in$*

$\mathcal{K}_{w^*}(X^*, Z_{\mathrm{FJ}})$ *and* $v \in \mathcal{K}(Z_{\mathrm{FJ}}, Y)$ *with* $1 \le \|u\|, \|v\| \le 1 + \varepsilon$, *and a linear mapping* $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{\mathrm{FJ}}, Z_{\mathrm{FJ}})$ *such that* $S = v \circ \Phi(S) \circ u$ *for all* $S \in \operatorname{span} \mathcal{C}$. *The mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Theorem* 8.

Proof. Let $A \in \mathcal{K}_{w^*}(X^*, Z_{(X,Y)})$, $B \in \mathcal{K}(Z_{(X,Y)}, Y)$, and $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{(X,Y)}, Z_{(X,Y)})$ be the norm one operators and the linear mapping given by Theorem 8. It follows from the proofs of [J, Theorem 1] and [F, Proposition 3.1] that there exist $V \in \mathcal{K}(Z_{(X,Y)}, Z_{\mathrm{FJ}})$ and $v \in \mathcal{K}(Z_{\mathrm{FJ}}, Y)$ such that $\|V\| = 1$, $1 \le \|v\| \le 1 + \varepsilon$, and $B = v \circ V$. Similarly to [J, Proposition 1], we can show that the subspace of $\overline{\mathcal{F}}_{w^*}(X^*, Z)$ (where Z is any Banach space) consisting of operators T which admit a factorization $T = \beta \circ \alpha$ for some operators $\alpha \in \overline{\mathcal{F}}_{w^*}(X^*, C_p)$ and $\beta \in \overline{\mathcal{F}}(C_p, Z)$, is a Banach space under the norm

$$\|T\|_{C_p} = \inf\{\|\beta\|\,\|\alpha\| : T = \beta \circ \alpha,\ \alpha \in \overline{\mathcal{F}}_{w^*}(X^*, C_p),\ \beta \in \overline{\mathcal{F}}(C_p, Z)\}.$$

This enables us to obtain, arguing similarly to the proofs of [J, Theorem 1] and [F, Proposition 3.1], for $A \in \mathcal{K}_{w^*}(X^*, Z_{(X,Y)})$ with $\|A\| = 1$, two operators $u \in \mathcal{K}_{w^*}(X^*, Z_{\mathrm{FJ}})$ and $U \in \mathcal{K}(Z_{\mathrm{FJ}}, Z_{(X,Y)})$ such that $1 \le \|u\| \le 1 + \varepsilon$, $\|U\| = 1$, and $A = U \circ u$. Since, for all $S \in \operatorname{span} \mathcal{C}$,

$$S = B \circ \varphi(S) \circ A = v \circ V \circ \varphi(S) \circ U \circ u,$$

the mapping Φ defined by $\Phi(S) = V \circ \varphi(S) \circ U$, $S \in \operatorname{span} \mathcal{C}$, has the desired properties. ∎

In the same vein like the space $Z_{(X,Y)}$ was "replaced" by Z_{FJ} in Theorem 8 to obtain Theorem 10, one can "replace" $Z_{(X,Y)}$ by Z_{FJ} also in Theorems 6 and 7 (or, equivalently, one may base on Lemmas 4 and 5 instead of Theorems 6 and 7). We shall not state the corresponding results.

Instead, we would like to point out an important case when $Z_{(X,Y)}$ may be "replaced" by any C_p, $1 \le p \le \infty$. Here we only present the result that relies on Theorem 6. Similar results based on Theorems 7 and 8 can then be easily stated and proved.

THEOREM 11. *Let X and Y be Banach spaces such that Y has the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist a linear mapping Φ : $\operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}(X^*, C_p)$ and an operator $v \in \mathcal{K}(C_p, Y)$ with $1 \le \|v\| \le 1 + \varepsilon$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma* 4.

Proof. Let $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}(X^*, Z_{(X,Y)})$ and $B \in \mathcal{K}(Z_{(X,Y)}, Y)$ be the linear mapping and the norm one operator given by Theorem 6. Since Y has the approximation property, it is well known that $\mathcal{K}(Z_{(X,Y)}, Y) = \overline{\mathcal{F}}(Z_{(X,Y)}, Y)$. Therefore, from the proof of [J, Theorem 1], it follows that

B admits a factorization $B = v \circ V$ with $V \in \mathcal{K}(Z_{(X,Y)}, C_p)$, $\|V\| = 1$, and $v \in \mathcal{K}(C_p, Y)$, $1 \le \|v\| \le 1 + \varepsilon$. The mapping Φ defined by $\Phi(S) = V \circ \varphi(S)$, $S \in \operatorname{span} \mathcal{C}$, has the needed properties. ∎

If we use, in the above proof, the factorization argument [J, Theorem 2] by Johnson instead of [J, Theorem 1], then we get the following quantitative strengthening of the symmetric version of Theorem 1 of Graves and Ruess for all $\mathcal{L}_p$-spaces, $1 \le p \le \infty$. Recall that ℓ_∞ denotes the space c_0.

THEOREM 12. *Let X be a Banach space and let Y be an $\mathcal{L}_{p,\lambda}$-space* ($1 \le p \le \infty$, $1 \le \lambda < \infty$). *If $\mathcal{C}$ is a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$, then, for every $\varepsilon > 0$, there exist a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}(X^*, \ell_p)$ and an operator $v \in \mathcal{K}(\ell_p, Y)$ with $1 \le \|v\| \le \lambda + \varepsilon$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} \mathcal{C}$, and the mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma* 4.

Recall that $\mathcal{K}(X, Y) = \mathcal{K}_{w^*}(X^{**}, Y)$. Therefore Theorems 10 and 11 together with their analogs and Theorem 12 immediately yield seven corresponding factorization results for compact subsets of $\mathcal{K}(X, Y)$.

Let us now point out an application of Theorems 11 and 12 to representing compact subsets of the injective tensor product $X \check{\otimes} Y$ of Banach spaces X and Y.

COROLLARY 13. *Let $1 \le p \le \infty$ and let X and Y be Banach spaces such that X has the approximation property* (*respectively*, *is an $\mathcal{L}_{p,\lambda}$-space*). *Let $\mathcal{C}$ be a compact subset of $X \check{\otimes} Y$. Then*, *for every $\varepsilon > 0$*, *there exist a linear mapping Φ from* $\operatorname{span} \mathcal{C}$ *to $C_p \check{\otimes} Y$* (*respectively*, *to $\ell_p \check{\otimes} Y$*) *and an operator $A \in \mathcal{K}(C_p, X)$ with $1 \le \|A\| \le 1 + \varepsilon$* (*respectively*, *$A \in \mathcal{K}(\ell_p, X)$ with $1 \le \|A\| \le \lambda + \varepsilon$*) *such that $u = (A \otimes \mathrm{Id})(\Phi u)$ for all $u \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma* 4.

Proof. Recall that $X \check{\otimes} Y = \mathcal{K}_{w^*}(Y^*, X)$ whenever X or Y has the approximation property (see [G, Ch. I, p. 165]). Recall also that $\mathcal{L}_p$-spaces, C_p, and ℓ_p have the approximation property, and apply Theorems 11 and 12. To verify the equality $u = (A \otimes \mathrm{Id})(\Phi u)$, rely on the (easy) fact that if $v \in C_p \check{\otimes} Y$ (respectively, $v \in l_p \check{\otimes} Y$) is canonically identified with $\widehat{v} \in \mathcal{K}_{w^*}(Y^*, C_p)$ (respectively, $\widehat{v} \in \mathcal{K}_{w^*}(Y^*, \ell_p)$), then $(A \otimes \mathrm{Id})v \in X \check{\otimes} Y$ is canonically identified with the operator $A \circ \widehat{v} \in \mathcal{K}_{w^*}(Y^*, X)$. ∎

Let us consider the particular case of Corollary 13 when $Y = C(K)$, the Banach space of continuous functions on a compact Hausdorff space K. Then $X \check{\otimes} Y = C(K; X)$, the Banach space of continuous X-valued functions on K. And Corollary 13 yields the representation of $\mathcal{C} \subset C(K; X)$ through the subset $\Phi(\mathcal{C})$ of $C(K; C_p)$ (respectively, of $C(K; \ell_p)$) so that $f = A \circ (\Phi f)$

for all $f \in \operatorname{span} \mathcal{C}$. Corollary 13 may also be applied to identifications of $X \check{\otimes} Y$ as spaces of X-valued measures (e.g., when $Y = L_1(\mu)$ or $Y = \mathrm{ba}(\mathcal{B})$, $\mathcal{B}$ being a Boolean algebra; see [DU, pp. 223–224] and [GR1]).

We conclude with a quantitative strengthening of Theorem 1 of Graves and Ruess (see the Introduction) for all $\mathcal{L}_p$-spaces, $1 \le p \le \infty$, which is a symmetric version of Theorem 12. Let us recall that X is an $\mathcal{L}_p$-space if and only if X^* is an $\mathcal{L}_q$-space where $1/q + 1/p = 1$ with $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.

THEOREM 14. *Let $1 \le p \le \infty$ and $1 \le \lambda < \infty$. Let X be an $\mathcal{L}_p$-space such that X^* is an $\mathcal{L}_{q,\lambda}$-space (where $1/q + 1/p = 1$ with $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$) and let Y be a Banach space. If $\mathcal{C}$ is a compact subset of $\mathcal{K}(X, Y)$, then, for every $\varepsilon > 0$, there exist a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(\ell_p, Y)$ and an operator $u \in \mathcal{K}(X, \ell_p)$ with $1 \le \|u\| \le \lambda + \varepsilon$ such that $S = \Phi(S) \circ u$ for all $S \in \operatorname{span} \mathcal{C}$, and the mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma* 4.

Proof. Suppose first that $1 \le p < \infty$. Observe that $\mathcal{C}^* = \{S^* : S \in \mathcal{C}\}$ is a compact subset of $\mathcal{K}_{w^*}(Y^*, X^*)$ (recall that $\operatorname{ran} S^{**} \subset Y$ whenever $S \in \mathcal{K}(X, Y)$) and apply Theorem 12. Let $\varepsilon > 0$ and let $\varphi : \operatorname{span} \mathcal{C}^* \to \mathcal{K}_{w^*}(Y^*, \ell_q)$ and $v \in \mathcal{K}(\ell_q, X^*)$ be given by Theorem 12. Then $\ell_q^* = \ell_p$ and $(\varphi(S^*))^* \in \mathcal{K}(\ell_p, Y)$ if $S \in \operatorname{span} \mathcal{C}$. Define $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(\ell_p, Y)$ by $\Phi(S) = (\varphi(S^*))^*$, $S \in \operatorname{span} \mathcal{C}$, and $u \in \mathcal{K}(X, \ell_p)$ by $u = v^*|_X$. These mappings have the desired properties. In particular, if $S \in \operatorname{span} \mathcal{C}$, then

$$\Phi(S) \circ u = ((\varphi(S^*))^* \circ v^*)|_X = (v \circ \varphi(S^*))^*|_X = S^{**}|_X = S.$$

Suppose now that $p = \infty$. Let Z be the reflexive space, $U \in \mathcal{K}(X, Z)$ the norm one operator, and $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z, Y)$ the linear mapping given by Theorem 7. Observe that, in fact, $U \in \overline{\mathcal{F}}(X, Z)$ because X^*, being an $\mathcal{L}_1$-space, has the approximation property.

Let $\varepsilon > 0$. Consider any $T \in \mathcal{F}(X, Z)$. Then $T^* \in \mathcal{F}(Z^*, X^*)$ and, similarly to the proof of [J, Theorem 2], we can choose a finite-dimensional subspace E of X^* with $\operatorname{ran} T^* \subset E$, a positive integer n, and an isomorphism L from E onto ℓ_1^n such that, e.g., $\|L\| = 1$ and $1 \le \|L^{-1}\| < \lambda + \varepsilon/2$. Denoting by $V : Z^* \to E$ the astriction of T^* and by $j : E \to X^*$ the identity embedding, we have $V^* \in \mathcal{F}(E^*, Z)$ and $T = V^* \circ j^*|_X$. Hence, T admits a factorization $T = \beta \circ \alpha$ for some operators $\alpha \in \mathcal{F}(X, \ell_\infty^n)$ and $\beta \in \mathcal{F}(\ell_\infty^n, Z)$ with $\|\alpha\| \le \lambda + \varepsilon/2$ and $\|\beta\| \le \|T\|$. Since c_0 is isometrically isomorphic to the infinite direct sum $(\sum_n \ell_\infty^n)_\infty$ in the sense of c_0, we have, for the norm $\|\cdot\|_{c_0}$ introduced in [J, Proposition 1],

$$\begin{aligned}\|T\|_{c_0} &= \inf\{\|\beta\|\,\|\alpha\| : T = \beta \circ \alpha,\ \alpha \in \overline{\mathcal{F}}(X, c_0),\ \beta \in \overline{\mathcal{F}}(c_0, Z)\} \\ &\le (\lambda + \varepsilon/2)\|T\|.\end{aligned}$$

Consequently, $\|\cdot\|_{c_0}$ is equivalent to the operator norm on $\mathcal{F}(X,Z)$ and, since $\mathcal{F}(X,Z)$ is dense in $\overline{\mathcal{F}}(X,Z)$, it follows from [J, Proposition 1] that, in particular, U admits a factorization $U = v \circ u$ with $u \in \overline{\mathcal{F}}(X,c_0)$, $v \in \overline{\mathcal{F}}(c_0,Z)$, and $\|U\| = 1 \le \|v\|\,\|u\| \le (\lambda+\varepsilon)\|U\| = \lambda+\varepsilon$. We may clearly assume that $1 \le \|u\| \le \lambda+\varepsilon$ and $\|v\| = 1$. Since, for all $S \in \operatorname{span}\mathcal{C}$,

$$S = \varphi(S)\circ U = \varphi(S)\circ v\circ u,$$

the mapping Φ defined by $\Phi(S) = \varphi(S)\circ v$, $S \in \operatorname{span}\mathcal{C}$, has the desired properties. ■

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