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Subsequences of frames

by

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Abstract. Every frame in Hilbert space contains a subsequence equivalent to an orthogonal basis. If a frame is n-dimensional then this subsequence has length $(1-\varepsilon)n$. On the other hand, there is a frame which does not contain bases with brackets.

1. Introduction. The notion of frame goes back to R. Duffin and A. Schaeffer [D-S] and has been studied extensively since then with relation to nonharmonic Fourier analysis (see [He]). From the geometrical point of view, a frame in a Hilbert space H is the image of an orthonormal basis in a larger Hilbert space under an orthogonal projection onto H, up to equivalence [Ho] (the equivalence constant is called the frame constant). Since frames have nice representation properties (see [D-S], [A]), much attention has been paid to their subsequences that inherit these properties. The most interesting questions arise about subsequences equivalent to an orthogonal basis [Ho], [S], [C1], [C-C1]. P. Casazza [C2] proved that, given an $\varepsilon > 0$, any n-dimensional frame whose norms are well bounded below contains a subsequence of length $(1-\varepsilon)n$ equivalent to an orthogonal basis (the constant of equivalence does not depend on n).

In the present paper this is proved for all frames, without restrictions on norms of the elements. If a frame is n-dimensional then it contains a subsequence of length $(1-\varepsilon)n$ which is C-equivalent to an orthogonal basis. Here C depends only on the frame constant and ε. To put the result in other words, orthogonal projections in Hilbert space preserve orthogonal structure in almost the whole range. Namely, the image of an orthogonal basis under an orthogonal projection P contains a subset of cardinality $(1-\varepsilon)\mathrm{rank}(P)$ which is $C(\varepsilon)$-equivalent to an orthogonal system. This is proved in Section 2.

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An infinite-dimensional version of this result is considered in Section 3. Every infinite-dimensional frame has an infinite subsequence equivalent to an orthogonal basis. However, for some frames this subsequence cannot be complete, as was shown by K. Seip [S] and P. Casazza and O. Christensen [C-C2]. This result is generalized in Section 4 by constructing a frame which does not contain bases with brackets. So our frame (x_j) is "asymptotically indecomposable" in the following sense: if (y_j) is any complete subsequence of (x_j), then the distance from $\operatorname{span}(y_j)_{j\le n}$ to $\operatorname{span}(y_j)_{j>n}$ tends to zero as $n \to \infty$.

In the rest of this section we recall standard definitions and simple known facts about frames. In what follows, H will denote a separable Hilbert space, finite- or infinite-dimensional. Absolute constants will be denoted by $c_1, c_2, \ldots$ A sequence (x_j) in H is called a *frame* if there exist positive numbers A and B such that

$$A\|x\|^2 \le \sum_j |\langle x, x_j\rangle|^2 \le B\|x\|^2 \quad \text{for } x \in H.$$

The number $(B/A)^{1/2}$ is called a *constant* of the frame. We call (x_j) a *tight frame* if $A = B = 1$.

Two sequences (x_j) and (y_j) in possibly different Banach spaces are called *equivalent* if there is an isomorphism $T : [x_j] \to [y_j]$ such that $Tx_j = y_j$ for all j. Here $[x_j]$ denotes the closed linear span of (x_j). Let $c = \|T\| \cdot \|T^{-1}\|$; then the sequences (x_j) and (y_j) are called *c-equivalent*.

The next observation (see [Ho]) allows us to look at frames as at projections of the canonical vector basis (e_j) in l_2.

Proposition 1. *Let $(x_n)_{n=1}^m$ be a frame in H with constant c, where m can be infinity. Then there is an orthogonal projection P in l_2^m such that (x_n) is c-equivalent to (Pe_n). Conversely, if P is an orthogonal projection in l_2^m onto a subspace H, then $(Pe_n)_{n=1}^m$ is a tight frame in H.*

Corollary 2. *Let (x_n) be a frame with constant c. Then (x_n) is c-equivalent to a tight frame.*

Now we present another view on frames. We can regard them as columns of a row-orthogonal matrix (either finite or infinite).

Lemma 3. *Let $n, m \in \mathbb{N} \cup \infty$ and A be an $n \times m$ matrix whose rows are orthonormal. Then the columns of A form a tight frame in l_2^n. Conversely, let $(x_j)_{j=1}^m$ be a frame in H. Then there exists an $n \times m$ matrix A with $n = \dim H$ whose rows are orthonormal and such that the columns form a tight frame equivalent to (x_j).*

Proof. If A is as above then A^* acts as an isometric embedding of l_2^n into l_2^m. Then A acts as a quotient map in a Hilbert space, and we can regard it as an orthogonal projection. On the other hand, the columns of A are equal to Ae_j. Proposition 1 finishes the proof of the first statement. The converse can also be proved by this argument. ∎

LEMMA 4. *Let (x_j) be a tight frame in H. Then $\sum_j \|x_j\|^2 = \dim H$ (which is possibly infinite).*

Proof. By Proposition 1 we may assume that H is a subspace of l_2 and $x_j = Pe_j$, where P is the orthogonal projection in l_2 onto H. Then the Hilbert–Schmidt norm $\|P\|_{\mathrm{HS}}$ is $(\sum_j \|x_j\|^2)^{1/2}$. On the other hand, $\|P\|_{\mathrm{HS}} = (\dim H)^{1/2}$. ∎

2. Finite-dimensional frames.

In this section we prove

THEOREM 5. *There is a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following holds. Suppose (x_j) is an n-dimensional frame with constant c. Then for every $\varepsilon > 0$ there is a set σ of indices with $|\sigma| > (1-\varepsilon)n$ such that the system $(x_j)_{j\in\sigma}$ is C-equivalent to an orthogonal basis, where $C = h(\varepsilon)c$.*

We will need a result of A. Lunin on norms of restrictions of operators to coordinate subspaces ([L]; for improvements see [K-Tz]).

THEOREM 6 (A. Lunin). *Let $T : l_2^m \to l_2^n$ be a linear operator. Then there is a set $\sigma \subset \{1,\ldots,m\}$ with $|\sigma| = n$ such that*

$$\|T|_{\mathbb{R}^\sigma}\| \le c_1\sqrt{n/m}\,\|T\|.$$

Given an $h > 0$, a system of vectors (x_j) in a Hilbert space is called *h-Hilbertian* if

$$\Big\|\sum_j a_j x_j\Big\| \le h\Big(\sum_j |a_j|^2\Big)^{1/2}$$

for all sequences (a_j) of scalars. Then Theorem 6 can be reformulated as follows. Suppose $(x_j)_{1\le j\le m}$ is a 1-Hilbertian system in l_2^n. Then there is a set $\sigma \subset \{1,\ldots,m\}$ with $|\sigma| = n$ such that $(\sqrt{m/n}\,x_j)_{j\in\sigma}$ is c_1-Hilbertian.

Next, we will use a result of J. Bourgain and L. Tzafriri on invertibility of large submatrices ([B-Tz], Theorem 1.2):

THEOREM 7 (J. Bourgain, L. Tzafriri). *Let $T : l_2^n \to l_2^n$ be a linear operator such that $\|Te_j\| = 1$ for all j. Then there is a set $\sigma \subset \{1,\ldots,n\}$ with $|\sigma| \ge c_2 n/\|T\|^2$ such that*

$$\|Tx\| \ge c_2\|x\| \quad \text{for every } x \in \mathbb{R}^\sigma.$$

Given a $b > 0$, a system of vectors (x_j) in a Hilbert space is called *b-Besselian* if

$$b\Big\|\sum_j a_j x_j\Big\| \ge \Big(\sum_j |a_j|^2\Big)^{1/2}$$

for all sequences (a_j) of scalars. Then Theorem 7 can be reformulated as follows. Suppose $(x_j)_{1\le j\le n}$ is an h-Hilbertian system in l_2^n and $\|x_j\| \ge \alpha$ for all $1 \le j \le n$. Then there is a set $\sigma \subset \{1,\dots,n\}$ with $|\sigma| \ge c_2(\alpha/h)^2 n$ such that the system $(\alpha^{-1}x_j)_{j\in\sigma}$ is c_3-Besselian.

Clearly, every tight frame is 1-Hilbertian.

LEMMA 8. *Let $(y_j)_{1\le j\le m}$ be a tight frame in l_2^n with $\|y_j\| = \sqrt{n/m}$ for all j. Let P be a k-dimensional orthogonal projection in l_2^n. Then for $\delta > 0$,*

$$|\{j : \|(I-P)y_j\| \ge \delta\sqrt{n/m}\}| \ge (1-\delta^2-k/n)m.$$

Proof. Let $\tau = \{j : \|(I-P)y_j\| \ge \delta\sqrt{n/m}\}$. Since $((I-P)y_j)_{1\le j\le m}$ is a tight frame in the $(n-k)$-dimensional space $(I-P)l_2^n$, Lemma 4 yields

$$\begin{aligned} n-k = \sum_{j=1}^m \|(I-P)y_j\|^2 &\le \sum_{j\in\tau}\|y_j\|^2 + \sum_{j\in\tau^c}\|(I-P)y_j\|^2 \\ &\le |\tau|\cdot(n/m) + m\cdot\delta^2(n/m) = (|\tau|/m+\delta^2)n. \end{aligned}$$

The required estimate follows. ∎

Now we proceed to the proof of Theorem 5. As in P. Casazza's proof [C2], the set σ will be constructed by an iteration procedure. Our proof consists of several parts.

I. *Splitting.* By Corollary 2, we may assume that the frame $(x_j) \subset l_2^n$ is tight and all of its terms are nonzero. First, we split (x_j) to get almost equal norms of the terms. Note that if we substitute any member x_j of the frame by k elements $x_j/\sqrt{k},\dots,x_j/\sqrt{k}$, we still get a tight frame. Fix $\nu > 0$. Splitting each x_j as above, we obtain a new tight frame $(y_j)_{1\le j\le m}$ such that

(i) elements of (y_j) are multiples of the ones from (x_j);
(ii) there is a $\lambda > 0$ such that $\lambda \le \|y_j\| \le (1+\nu)\lambda$ for all $j = 1,\dots,m$.

The constant λ can be evaluated using Lemma 4:

$$(1+\nu)^{-1}\sqrt{n/m} \le \|y_j\| \le (1+\nu)\sqrt{n/m} \quad \text{for } j = 1,\dots,m.$$

Clearly, it is enough to prove the theorem for (y_j) instead of (x_j). We can choose the parameter $\nu = \nu(\varepsilon) > 0$ arbitrarily small. To make the proof more readable, we simply assume that $\nu = 0$. The reader will easily adjust the argument to the general case. So we have

$$\|y_j\| = \sqrt{n/m}, \quad j = 1,\dots,m.$$

We can also assume that $(\varepsilon/2)m \ge n$.

II. *Iterative construction.* Let $\delta = \sqrt{\varepsilon/2}$.

STEP 1. Set $\tau_0 = \{1, \ldots, m\}$. The system $(y_j)_{j\in\tau_0}$ is 1-Hilbertian. Lunin's theorem yields the existence of a set $\sigma_1' \subset \tau_0$ with $|\sigma_1'| = n$ such that

$$\text{the system } (\sqrt{m/n}\, y_j)_{j\in\sigma_1'} \text{ is } c_1\text{-Hilbertian.}$$

Note that $\|\sqrt{m/n}\, y_j\| = 1$ for $j \in \sigma_1'$. Then Bourgain–Tzafriri's theorem gives us a set $\sigma_1 \subset \sigma_1'$ with $|\sigma_1| \geq (c_2/c_1^2)n$ such that

$$\text{the system } (\sqrt{m/n}\, y_j)_{j\in\sigma_1} \text{ is } c_3\text{-Besselian.}$$

So we have already found a subsequence $(y_j)_{j\in\sigma_1}$ of length proportional to n which is well equivalent to an orthogonal basis. If $|\sigma_1| \geq (1-\varepsilon)n$, then we are done. Otherwise we proceed to the next step.

STEP 2. Let P_1 be the orthogonal projection in l_2^n onto $[y_j]_{j\in\sigma_1}$. Let

$$\tau_1 = \{j : \|(I - P_1)y_j\| \geq \delta\sqrt{n/m}\}.$$

Clearly, $\tau_1 \subset \sigma_1^{\mathrm{c}}$. By Lemma 8,

$$|\tau_1| \geq (1 - \delta^2 - |\sigma_1|/n)m.$$

As $|\sigma_1| < (1-\varepsilon)n$, we obtain

$$|\tau_1| > (1 - \delta^2 - (1-\varepsilon))m = (\varepsilon/2)m.$$

The system $(y_j)_{j\in\tau_1}$ is 1-Hilbertian and $|\tau_1| \geq n$ by the choice of m. Lunin's theorem yields the existence of a set $\sigma_2' \subset \tau_1$ with $|\sigma_2'| = n$ such that

$$\text{the system } (\sqrt{|\tau_1|/n}\, y_j)_{j\in\sigma_2'} \text{ is } c_1\text{-Hilbertian.}$$

Then the system $(\sqrt{|\tau_1|/n}\,(I - P_1)y_j)_{j\in\sigma_2'}$ is also c_1-Hilbertian. By the definition of τ_1, it has not too small norms:

$$\left\|\sqrt{|\tau_1|/n}\,(I - P_1)y_j\right\| \geq \delta\sqrt{|\tau_1|/m}, \quad j \in \sigma_2'.$$

Then Bourgain–Tzafriri's theorem gives us a set $\sigma_2 \subset \sigma_2'$ with

$$|\sigma_2| \geq c_2(\delta^2|\tau_1|/(mc_1^2))n \geq (c_2/c_1^2)\delta^2((1-\delta^2)n - |\sigma_1|)$$

such that

$$\text{the system } (\sqrt{m/n}\,(I - P_1)y_j)_{j\in\sigma_2} \text{ is } (c_3\delta^{-1})\text{-Besselian.}$$

If $|\sigma_1| + |\sigma_2| \geq (1-\varepsilon)n$, then we stop. Otherwise we proceed to the next step.

STEP $k+1$. We assume that the sets $\sigma_1, \ldots, \sigma_k$ are already constructed and

$$\sum_{i=1}^{k} |\sigma_i| < (1-\varepsilon)n. \tag{1}$$

Let P_k be the orthogonal projection in l_2^n onto $[y_j]_{j\in\sigma_1\cup\ldots\cup\sigma_k}$. Let

$$\tau_k = \{j : \|(I - P_k)y_j\| \geq \delta\sqrt{n/m}\}.$$

Clearly, $\tau_k \subset (\sigma_1 \cup \ldots \cup \sigma_k)^c$. By Lemma 8,

$$|\tau_k| \geq \Big(1 - \delta^2 - \sum_{i=1}^{k} |\sigma_i|/n\Big)m.$$

By (1),

$$|\tau_k| > (1 - \delta^2 - (1 - \varepsilon))m = (\varepsilon/2)m.$$

The system $(y_j)_{j\in\tau_k}$ is 1-Hilbertian and $|\tau_k| \geq n$ by the choice of m. Lunin's theorem yields the existence of a set $\sigma'_{k+1} \subset \tau_k$ with $|\sigma'_{k+1}| = n$ such that

$$\text{the system } (\sqrt{|\tau_k|/n}\, y_j)_{j\in\sigma'_{k+1}} \text{ is } c_1\text{-Hilbertian.}$$

Then the system $(\sqrt{|\tau_k|/n}\,(I - P_k)y_j)_{j\in\sigma'_{k+1}}$ is also c_1-Hilbertian. By the definition of τ_k, it has not too small norms:

$$\|\sqrt{|\tau_k|/n}\,(I - P_k)y_j\| \geq \delta\sqrt{|\tau_k|/m}, \qquad j \in \sigma'_{k+1}.$$

Then Bourgain–Tzafriri's theorem gives us a set $\sigma_{k+1} \subset \sigma'_{k+1}$ with

$$(2) \qquad |\sigma_{k+1}| \geq c_2(\delta^2|\tau_k|/(mc_1^2))n \geq (c_2/c_1^2)\delta^2\Big((1 - \delta^2)n - \sum_{i=1}^{k} |\sigma_i|\Big)$$

such that

$$\text{the system } (\sqrt{m/n}\,(I - P_k)y_j)_{j\in\sigma_{k+1}} \text{ is } (c_3\delta^{-1})\text{-Besselian.}$$

If $\sum_{i=1}^{k+1} |\sigma_i| \geq (1 - \varepsilon)n$, then we stop. Otherwise we proceed to the next step.

III. *When we stop.* Let k_0 be the number of the last step, that is, the smallest integer such that

$$\sum_{i=1}^{k_0} |\sigma_i| \geq (1 - \varepsilon)n.$$

We claim that k_0 exists and there is a function $K(\varepsilon)$ such that $k_0 \leq K(\varepsilon)$. Indeed, let $K(\varepsilon) = [4c_1^2c_2^{-1}\varepsilon^{-2}] + 2$. If the claim were not true, then

$$\sum_{i=1}^{k} |\sigma_i| < (1 - \varepsilon)n \qquad \text{for } k = 1, \ldots, K(\varepsilon).$$

Then by (2) for all $k = 2, \ldots, K(\varepsilon)$,

$$|\sigma_k| \geq (c_2/c_1^2)\delta^2((1 - \delta^2) - (1 - \varepsilon))n = (c_2/c_1^2)(\varepsilon^2/4)n.$$

Thus

$$\sum_{i=1}^{K(\varepsilon)} |\sigma_i| \geq (K(\varepsilon)-1)\cdot(c_2/c_1^2)(\varepsilon^2/4)n \geq n.$$

This contradiction proves the claim.

Now set $\sigma = \sigma_1 \cup \ldots \cup \sigma_{k_0}$; then $|\sigma| > (1-\varepsilon)n$. To complete the proof of the theorem, it remains to check that the system $(\sqrt{m/n}\, y_j)_{j\in\sigma}$ is well equivalent to an orthonormal basis.

IV. *Equivalence to the orthogonal basis within blocks σ_k.* Recall that for every $k < k_0$ the size of τ_k is comparable to m, namely $|\tau_k| \geq (\varepsilon/2)m$. Then we conclude from the construction the existence of functions $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that for every $k = 1, \ldots, k_0$,

(3) the system $(\sqrt{m/n}\, y_j)_{j\in\sigma_k}$ is $c_1(\varepsilon)$-Hilbertian,

(4) the system $(\sqrt{m/n}\,(I-P_{k-1})y_j)_{j\in\sigma_k}$ is $c_2(\varepsilon)$-Besselian.

V. *The system $(\sqrt{m/n}\, y_j)_{j\in\sigma}$ is h-Hilbertian for some function $h = h(\varepsilon)$.* Indeed, fix scalars $(a_j)_{j\in\sigma}$ such that $\sum_{j\in\sigma}|a_j|^2 = 1$. Then

$$\begin{aligned}
\Big\|\sum_{j\in\sigma} a_j(\sqrt{m/n}\, y_j)\Big\| &\leq \sum_{k=1}^{k_0}\Big\|\sum_{j\in\sigma_k} a_j(\sqrt{m/n}\, y_j)\Big\| \\
&\leq \sqrt{k_0}\Big(\sum_{k=1}^{k_0}\Big\|\sum_{j\in\sigma_k} a_j(\sqrt{m/n}\, y_j)\Big\|^2\Big)^{1/2} \\
&\leq \sqrt{k_0}\, c_1(\varepsilon)\Big(\sum_{k=1}^{k_0}\sum_{j\in\sigma_k}|a_j|^2\Big)^{1/2} \qquad \text{by (3)} \\
&= \sqrt{K(\varepsilon)}\, c_1(\varepsilon).
\end{aligned}$$

VI. *The system $(\sqrt{m/n}\, y_j)_{j\in\sigma}$ is b-Besselian for some function $b = b(\varepsilon)$.* We follow P. Casazza [C2]. Choose $r = r(\varepsilon) > 2$ large enough (to be specified later). Let $a = a(\varepsilon) > 0$ be such that $r^{k_0+1}a < 1$. Fix scalars $(a_j)_{j\in\sigma}$ such that $\sum_{j\in\sigma}|a_j|^2 = 1$. Suppose

(5) $1 \leq k' \leq k_0$ is the largest integer such that

$$\Big(\sum_{j\in\sigma_{k'}}|a_j|^2\Big)^{1/2} \geq r^{k_0-k'}a.$$

The k' must exist, since otherwise

$$\Big(\sum_{j\in\sigma}|a_j|^2\Big)^{1/2} \leq \sum_{k=1}^{k_0}\Big(\sum_{j\in\sigma_k}|a_j|^2\Big)^{1/2} \leq \sum_{k=1}^{k_0} r^k a \leq r^{k_0+1}a < 1,$$

contradicting the choice of a. We have

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma} a_j(\sqrt{m/n}\,y_j)\Big\| \\
&\qquad \geq \Big\|\sum_{k=1}^{k'}\sum_{j\in\sigma_k} a_j(\sqrt{m/n}\,y_j)\Big\| - \sum_{k=k'+1}^{k_0}\Big\|\sum_{j\in\sigma_k} a_j(\sqrt{m/n}\,y_j)\Big\| \\
&\qquad \geq \Big\|(I-P_{k'-1})\sum_{k=1}^{k'}\sum_{j\in\sigma_k} a_j(\sqrt{m/n}\,y_j)\Big\| \\
&\qquad\quad - c_1(\varepsilon)\sum_{k=k'+1}^{k_0}\Big(\sum_{j\in\sigma_k}|a_j|^2\Big)^{1/2} \quad \text{by (3)} \\
&\qquad \geq \Big\|\sum_{j\in\sigma_{k'}} a_j(\sqrt{m/n}\,(I-P_{k'-1})y_j)\Big\| - c_1(\varepsilon)\sum_{k=k'+1}^{k_0} r^{k_0-k}a \quad \text{by (5)} \\
&\qquad \geq c_2(\varepsilon)^{-1}\Big(\sum_{j\in\sigma_{k'}}|a_j|^2\Big)^{1/2} - c_1(\varepsilon)\frac{r^{k_0-k'}}{r-1}a \quad \text{by (4)} \\
&\qquad \geq (c_2(\varepsilon)^{-1} - c_1(\varepsilon)(r-1)^{-1})r^{k_0-k'}a \quad \text{by (5)} \\
&\qquad \geq (c_2(\varepsilon)^{-1} - c_1(\varepsilon)(r-1)^{-1})a.
\end{aligned}$$

If r was chosen so that $c_2(\varepsilon)^{-1} - c_1(\varepsilon)(r-1)^{-1} > c_2(\varepsilon)^{-1}/2$, we are done. The proof is complete. ∎

REMARK 1. C tends to 1 as $\varepsilon \to 1$. This is a consequence of a restriction theorem [K-Tz] which we use in the following special case (see also [B-Tz], Theorem 1.6).

THEOREM 9 (B. Kashin, L. Tzafriri). *Let T be a linear operator in l_2^n with 0's on the diagonal and $\|T\| = 1$. Let $1/n \le \delta < 1$. Then there exists a set $\sigma \subset \{1,\dots,n\}$ with $|\sigma| \ge \delta n/4$ for which*

$$\|R_\sigma T R_\sigma\| \le c_5\delta^{1/2}.$$

First, Theorem 5 gives us a set σ_1 of indices with $|\sigma_1| \ge n/2$ such that the system $(x_j/\|x_j\|)_{j\in\sigma_1}$ is c_6c-equivalent to the canonical vector basis of $l_2^{\sigma_1}$. Let $\delta = 1-\varepsilon$ and $z_j = x_j/\|x_j\|$ for $j \in \sigma_1$. Consider the linear operator T in $l_2^{\sigma_1}$ which sends e_j to z_j for $j\in\sigma_1$. Then the operator $T^*T - I$ has 0's on the diagonal and is of norm at most $2c_6^2c^2$. Applying Theorem 9 we get a set $\sigma \subset \sigma_1$ with $\sigma \ge \delta|\sigma_1|/4$ such that for any sequence (a_j) of scalars with $\sum_{j\in\sigma}|a_j|^2 = 1$,

$$\Big\|\Big\langle (T^*T - I)\sum_{j\in\sigma} a_je_j, \sum_{j\in\sigma} a_je_j\Big\rangle\Big\| \le (2c_6^2c^2)c_5\delta^{1/2} = c_7c^2\delta^{1/2}.$$

Thus

$$\Big|\Big\langle \sum_{j\in\sigma} a_j z_j, \sum_{j\in\sigma} a_j z_j \Big\rangle - \sum_{j\in\sigma} |a_j|^2 \Big| \le c_7 c^2 \delta^{1/2}.$$

Therefore the sequence $(z_j)_{j\in\sigma}$ is $g(\delta)$-equivalent to $(e_j)_{j\in\sigma}$ for a function $g(\delta)$ which tends to 1 as $\delta \to 0$. This proves Remark 1.

REMARK 2. $h(\varepsilon)$ tends to infinity as $\varepsilon \to 0$. This is verified for the following tight frame $(x_j)_{1\le j\le n+1}$, $n \ge 2$, considered by P. Casazza and O. Christensen in [C-C2]:

$$x_j = e_j - n^{-1} \sum_{j=1}^{n} e_j \quad \text{for } j = 1, \dots, n;$$

$$x_{n+1} = n^{-1/2} \sum_{j=1}^{n} e_j.$$

Indeed, let $\sigma \subset \{1, \dots, n\}$ be such that $|\sigma| > (1-\varepsilon)n$ and the system $(x_j)_{j\in\sigma}$ is M-equivalent to an orthogonal basis. By a change of coordinates, the system $(x_j)_{1\le j\le |\sigma|-1}$ must be M-equivalent to an orthogonal basis as well. However,

$$\Big\| \sum_{j=1}^{|\sigma|-1} x_j \Big\|^2 \le 2(\varepsilon n + 1)$$

while $\|x_j\| \ge 1/2$ for all j. Therefore M cannot be bounded independently of n as $\varepsilon \to 0$. This proves Remark 2.

3. Almost orthogonal subsequences of frames. In this section we prove an infinite-dimensional version of Theorem 5.

THEOREM 10. *Given an $\varepsilon > 0$, every infinite-dimensional frame has a subsequence $(1-\varepsilon)$-equivalent to an orthogonal basis of l_2.*

Given two sets A and B in H, we put by definition

$$\theta(A, B) = \sup_{a\in A} \operatorname{dist}(a, B) = \sup_{a\in A} \inf\{\|a-b\| : b \in B\}.$$

LEMMA 11. *Let (x_j) be a frame in an infinite-dimensional H. Let $A = \{x_j/\|x_j\|\}$. Then for any finite-dimensional subspace $E \subset H$,*

$$\theta(A, E) = 1.$$

Proof. Let $z_j = x_j/\|x_j\|$ for all j. Assume that, on the contrary, there is a $\delta < 1$ such that

$$\operatorname{dist}(z_j, E) < \delta \quad \text{for all } j.$$

Let P be the orthogonal projection in H onto E. Then

$$\|Pz_j\| > \sqrt{1-\delta^2} \quad \text{for all } j,$$

so that

$$\|Px_j\| \geq \sqrt{1-\delta^2} \cdot \|x_j\| \quad \text{for all } j. \tag{6}$$

Since P is finite-dimensional, Lemma 4 yields that the sequence $\|Px_j\|$ is square summable. Then, by (6), $\|x_j\|$ must also be square summable. Thus, from Lemma 4, (x_j) is finite-dimensional. This contradiction completes the proof. ∎

LEMMA 12. *Let ε_j be a sequence of fast decreasing positive numbers* (2^{-j-1} *will do*). *Let (z_j) be a normalized sequence in H such that*

$$\langle z_i, z_j \rangle < \varepsilon_j \quad \textit{whenever } i < j.$$

Then (z_j) is equivalent to an orthonormal basis.

The proof is simple.

Proof of Theorem 10. First note that, given an $\varepsilon > 0$, every subsequence equivalent to the canonical vector basis of l_2 is weakly null, therefore has a subsequence which is $(1-\varepsilon)$-equivalent to the canonical vector basis of l_2. Hence by Corollary 2 we may assume that our given frame (x_j) is tight. Let $z_j = x_j/\|x_j\|$ for all j. We will find a subsequence (z_{j_k}) equivalent to an orthogonal basis by induction. Put $j_1 = 1$. Let $j_1, \ldots, j_{k-1}$ be defined and let $E = \operatorname{span}(z_{j_1}, \ldots, z_{j_{k-1}})$. Choose j_k from Lemma 11 so that

$$\operatorname{dist}(z_{j_k}, E) > 1 - 2^{-2k}.$$

Then it is easy to check that the constructed subsequence (z_{j_k}) satisfies the assumption of Lemma 12. This finishes the proof. ∎

4. A frame not containing bases with brackets

DEFINITION 13. A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is called a *basis with brackets* if there are numbers $1 < n_1 < n_2 < \ldots$ such that every vector $x \in X$ admits a unique representation of the form

$$x = \lim_j \sum_{n=1}^{n_j} a_n x_n, \qquad a_n \in \mathbb{R}.$$

Clearly, every basis is a basis with brackets. The difference between bases and bases with brackets is that the latter require the convergence only of *some* partial sums in the representation.

The following lemma is known [L-T].

LEMMA 14. *Let $(x_n)_{n=1}^{\infty}$ be a basis with brackets*, *and numbers* $1 < n_1 < n_2 < \ldots$ *be as in Definition* 13. *Consider the projection P_j onto* $[x_n : n \leq n_j]$ *parallel to* $[x_n : n > n_j]$. *Then* $\sup_j \|P_j\| < \infty$.

Clearly, the converse also holds: if $\sup_j \|P_j\| < \infty$ for some sequence $1 < n_1 < n_2 < \ldots$, then (x_n) is a basis with brackets.

In this section we prove

THEOREM 15. *There exists a frame in l_2 which does not contain bases with brackets.*

Moreover, this frame is tight and has norms bounded from below.

LEMMA 16. *There is an orthonormal basis (z_j) in l_2^n such that, given any set $J \subset \{1, \dots, n\}$ with $|J| \ge n-2$, one has*

$$\operatorname{dist}(e_1, [z_j : j \in J,\ j \ge j_0]) \le 4/\sqrt{n} \quad \text{for } 1 \le j_0 < n/2,$$
$$\operatorname{dist}(e_n, [z_j : j \in J,\ j < j_0]) \le 4/\sqrt{n} \quad \text{for } n/2 \le j_0 \le n.$$

Proof. By rotation, it is enough to find normalized vectors v_1, v_2 in l_2^n such that $\langle v_1, v_2 \rangle = 0$ and, given a set J as in the hypothesis,

$$\operatorname{dist}(v_1, [e_j : j \in J,\ j \ge j_0]) \le 4/\sqrt{n} \quad \text{for } 1 \le j_0 < n/2,$$
$$\operatorname{dist}(v_2, [e_j : j \in J,\ j < j_0]) \le 4/\sqrt{n} \quad \text{for } n/2 \le j_0 \le n.$$

Clearly, one may take

$$v_1 = \lceil n/2 \rceil^{-1/2} \cdot (\underbrace{1, \dots, 1}_{\lceil n/2 \rceil}, 0, \dots, 0),$$
$$v_2 = \lceil n/2 \rceil^{-1/2} \cdot (0, \dots, 0, \underbrace{1, \dots, 1}_{\lceil n/2 \rceil}). \ \blacksquare$$

We will construct our frame (x_j) by defining blocks $(x_j : j \in J(n))$, where

$$J(1) = \{1\}, \quad J(2) = \{2, 3\}, \quad J(3) = \{4, 5, 6\}, \quad J(4) = \{7, 8, 9, 10\}, \ \dots$$

The supports of x_j's from block $J(n)$ will lie in an interval $I(n)$, where

$$I(1) = \{1\}, \quad I(2) = \{1, 2\}, \quad I(3) = \{2, 3, 4\}, \quad I(4) = \{4, 5, 6, 7\}, \ \dots$$

Let $i(n)$ be the first element of $I(n)$.

$$\begin{array}{ccccccccccc}
* & * & * & & & & & 0 & & & \\
 & * & * & * & * & * & & & & & \\
 & & & * & * & * & & & & & \\
 & & & * & * & * & * & * & * & * & \\
 & & & & & & * & * & * & * & \\
 & & 0 & & & & * & * & * & * & \\
 & & & & & & * & * & * & * & \cdots
\end{array}$$

The columns of this infinite matrix form the frame elements x_j, the asterisks marking their supports. Consider the shift operator $T_n : l_2^n \to l_2$ which sends $(e_i)_{i=1}^n$ to $(e_i : i \in I(n))$. Choose an orthonormal basis $(z_j : j \in J(n))$ in l_2^n satisfying the conclusion of Lemma 16, and define

$$x_j = T_n z_j \quad \text{for } j \in J(n).$$

LEMMA 17. *(x_j) is a frame in l_2.*

Proof. Indeed, look at the rows in the picture, that is, the vectors $y_i = (x_1(i), x_2(i), \ldots)$. Since the vectors x_j, $j \in J(n)$, are orthonormal for fixed n, the vectors y_i are orthogonal. Moreover, their norms are either 2 (if $i = i(n)$ for some n) or 1 (otherwise). Now we pass again from the rows y_i to the columns x_j. Lemma 3 yields that (x_j) is a frame. ∎

Let J be a set of positive integers such that the sequence $(x_j)_{j\in J}$ is complete in l_2. We shall prove that it is not a basis with brackets.

LEMMA 18. *$|J(n) \cap J| \geq n - 2$ for every n.*

Proof. Let P be the orthogonal projection onto those $n-2$ coordinates in $I(n)$ which do not belong to the other blocks $I(n_1)$, i.e. onto $[e_i : i \in I(n) \setminus \{i(n), i(n+1)\}]$. Thus P sends to zero all x_j with $j \notin J(n)$. Hence $\mathrm{Im}(P) = P([x_j : j \in J(n) \cap J])$. Since $\mathrm{Im}(P)$ is an $(n-2)$-dimensional space, the lemma follows. ∎

In what follows we consider large blocks $J(n)$, i.e. with $n \to \infty$. Given a vector v and a subspace L in l_2 (both possibly depending on n), we say that v is *close* to L if $\mathrm{dist}(x, L) \leq c/\sqrt{n}$. Here c is some absolute constant, whose value may be different in different occurrences.

LEMMA 19. (1) *$e_{i(n)}$ is close to $[x_j : j \in J(n-1) \cap J]$.*

(2) *$e_{i(n+1)}$ is close to $[x_j : j \in J(n+1) \cap J]$.*

(3) *For each $j_0 \in J(n)$, either $e_{i(n)}$ is close to $[x_j : j \in J(n) \cap J,\ j \geq j_0]$, or $e_{i(n+1)}$ is close to $[x_j : j \in J(n) \cap J,\ j < j_0]$.*

Proof. Note that T_n sends e_1 to $e_{i(n)}$ and e_n to $e_{i(n+1)}$. Then all three statements of the lemma follow from Lemma 16. ∎

The next (and last) lemma, in tandem with Lemma 14, completes the proof of Theorem 15.

LEMMA 20. *For every $j_0 \in J(n)$ there is a normalized vector x in l_2 which is close to both subspaces $E = [x_j : j \in J,\ j \geq j_0]$ and $F = [x_j : j \in J,\ j < j_0]$.*

Proof. We make use of Lemma 19. By (3), we take either $x = e_{i(n)}$ to have x close to E, or $x = e_{i(n+1)}$ to have x close to F. In the first case x is also close to F by (2), and in the second case x is close to E by (1). The proof is complete. ∎

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