

Hilbert space factorization and Fourier type of operators

by

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Abstract. It is an open question whether every Fourier type 2 operator factors through a Hilbert space. We show that at least the natural gradations of Fourier type 2 norms and Hilbert space factorization norms are not uniformly equivalent. A corresponding result is also obtained for a number of other sequences of ideal norms instead of the Fourier type 2 gradation including the Walsh function analogue of Fourier type. Our main tools are ideal norms and random matrices.

1. Introduction. Let \mathcal{F} denote the Fourier transform on the real line given by

$$\mathcal{F}f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-ist} dt.$$

X and Y are always complex Banach spaces. $\mathcal{L}(X, Y)$ denotes the Banach space of bounded linear operators from X to Y . An operator $T \in \mathcal{L}(X, Y)$ is said to be of *Fourier type* p for some $p \in (1, 2]$ if the operator $\mathcal{F} \otimes T$, which is, by the classical Hausdorff–Young inequality, a well defined operator between the algebraic tensor products $L_p(\mathbb{R}) \otimes X$ and $L_q(\mathbb{R}) \otimes Y$, has a bounded linear extension acting between the vector-valued spaces $L_p^X(\mathbb{R})$ and $L_q^Y(\mathbb{R})$. Here q is the conjugate number of p given by $1/p + 1/q = 1$. We say that X has Fourier type p if the identity operator I_X does.

The notion of Fourier type was introduced by Peetre [Pee] in the case of Banach spaces. He studied interpolation properties of such spaces. Kwapien [Kwa] proved that X has Fourier type 2 if and only if X is isomorphic to a Hilbert space. The deepest result in this direction is due to Bourgain [Bou1], [Bou2] who showed that X has some Fourier type $p > 1$ if and only if it is B-convex, i.e. has some Rademacher type $p > 1$.

Bourgain's theorem fails for operators, i.e. there exist operators of Rademacher type 2 failing to have any Fourier type $p > 1$ (see [PW]). The

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operator case of Kwapień’s result is still an open question. It is not known whether every operator of Fourier type 2 factors through a Hilbert space (see [DF], [PW]). It is the main purpose of this paper to shed some light onto this problem.

Both notions of Fourier type and factorization through a Hilbert space are local notions in the sense that they depend only on the finite-dimensional pieces of the operator under consideration. This is made precise in the following two theorems. Proofs of both of them can be found in [PW]. Both theorems are mainly due to Kwapień [Kwa] (see also [Koe] and [Pis]).

THEOREM 1.1. *For any $T \in \mathcal{L}(X, Y)$ the following assertions are equivalent.*

- (i) T has Fourier type 2.
- (ii) There exists $c > 0$ such that for all sequences x_1, x_2, \dots in X ,

$$\int_0^1 \left\| \sum_{k=1}^{\infty} \exp(2\pi ikt)Tx_k \right\|^2 dt \leq c^2 \sum_{k=1}^{\infty} \|x_k\|^2.$$

- (iii) There exists $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$,

$$(1) \quad \sum_{h=1}^n \left\| \sum_{k=1}^n \frac{1}{\sqrt{n}} \exp(2\pi i hk/n)Tx_k \right\|^2 \leq c^2 \sum_{k=1}^n \|x_k\|^2.$$

THEOREM 1.2. *For any $T \in \mathcal{L}(X, Y)$ the following assertions are equivalent.*

(i) T factors through a Hilbert space, i.e. there exists a Hilbert space H and operators $B \in \mathcal{L}(X, H)$ and $A \in \mathcal{L}(H, Y)$ such that $T = AB$.

(ii) There exists $c > 0$ such that for all orthonormal systems (a_1, a_2, \dots) in some Hilbert space $L_2(M, \mu)$ and all sequences x_1, x_2, \dots in X ,

$$\int_M \left\| \sum_{k=1}^{\infty} a_k(t)Tx_k \right\|^2 d\mu(t) \leq c^2 \sum_{k=1}^{\infty} \|x_k\|^2.$$

(iii) There exists $c > 0$ such that for all $n \in \mathbb{N}$, $n \times n$ -matrices $A_n = (\alpha_{hk})$ satisfying $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ and $x_1, \dots, x_n \in X$,

$$(2) \quad \sum_{h=1}^n \left\| \sum_{k=1}^n \alpha_{hk}Tx_k \right\|^2 \leq c^2 \sum_{k=1}^n \|x_k\|^2.$$

REMARK. A characterization similar to Theorem 1 is possible for Fourier type p operators (see [PW]). We restricted ourselves to the case $p = 2$ to make the similarity between both theorems more striking.

For a fixed $n \times n$ -matrix A_n and $T \in \mathcal{L}(X, Y)$, define $\kappa(T|A_n)$ as the smallest constant c such that (2) holds for all $x_1, \dots, x_n \in X$. The functional

$\kappa(A_n) : T \rightarrow \kappa(T|A_n)$ is the *Kwapień ideal norm* associated with A_n . The *universal Kwapień ideal norm* of T is defined as

$$\kappa_n(T) = \sup\{\kappa(T|A_n) : \|A_n : l_2^n \rightarrow l_2^n\| \leq 1\}.$$

For the general theory of ideal norms associated with matrices and orthonormal systems and, in particular, for properties of the ideal norms just defined we refer the reader to [PW]. Observe that Theorem 1.2 says that the sequence (κ_n) of ideal norms is a natural gradation in the sense of [Tom] of the Hilbert space factorization norm generally defined as $\inf \|A\| \cdot \|B\|$ where the infimum is taken over all factorizations $T = AB$ as in (i). Similarly, if we let $E_n = (\exp(2\pi i h k/n)/\sqrt{n})$, then Theorem 1.1 states that $(\kappa(E_n))$ is a natural gradation of the Fourier type 2 norm, which may be defined as the operator norm of $\mathcal{F} \otimes T : L_2^X(\mathbb{R}) \rightarrow L_2^Y(\mathbb{R})$.

Considering the problem whether every Fourier type 2 operator factors through a Hilbert space it is natural to ask whether the sequences of ideal norms κ_n and $\kappa(E_n)$ are uniformly equivalent, i.e. whether there exists a constant c such that $\kappa_n(T) \leq c\kappa(T|E_n)$ for all operators T . The following theorem shows that this is not true.

THEOREM 1.3. *For any sequence (A_n) of $n \times n$ -matrices which satisfies $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ there exists a sequence (T_n) of operators such that*

$$\kappa_n(T_n) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} n^\alpha \kappa(T_n|A_n) = 0 \quad \text{for } \alpha < 1/10.$$

This theorem also answers Problem 2.3.16 in [PW] about the existence of universal matrices for Kwapień ideal norms. We even show that sets of matrices needed to approximate $\kappa_n(T)$ for all T up to some fixed constant factor (independent of n) must have big size. See Theorem 5.2 for a detailed statement.

The proof of Theorem 1.3 is nonconstructive. We use operators $T_n : l_1^n \rightarrow l_\infty^n$ whose representing matrices have random entries of modulus 1. In Section 2, we clarify the conditions under which such an operator can have large value of $\kappa(T_n|A_n)$ for a fixed $n \times n$ -matrix A_n . Section 3 contains an estimate for the probability that this happens, which will be shown to be negligible. In Section 4, we derive from the results of [BGN] about the norm of such random matrices that the probability of having $\kappa_n(T_n)$ large is almost 1. This is used to prove Theorems 1.3 and 5.2. Finally, applications to the behavior of ideal norms associated with the trigonometric system and the Walsh system are contained in Section 6.

2. Operators from l_1^n to l_∞^n . The purpose of this section is to provide necessary conditions for an operator in $\mathcal{L}(l_1^n, l_\infty^n)$ to have large $\kappa(T|A_n)$ for a fixed matrix A_n . By a *c-net* \mathcal{N} in a metric space M we mean a subset of M such that the balls of radius c with centers in \mathcal{N} cover M .

PROPOSITION 2.1. *Let $T \in \mathcal{L}(l_1^n, l_\infty^n)$ be an operator induced by a matrix (τ_{ij}) . Let $A_n = (\alpha_{hk})$ be an $n \times n$ -matrix with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$. For some $c > 0$, let \mathcal{N} be a $c/8$ -net in $B_{l_2^n} \cap \frac{2}{c\sqrt{n}}B_{l_\infty^n}$ in l_2^n -distance. If $\kappa(T|A_n) \geq c\sqrt{n} \|T\|$, then there exists a partition $\{1, \dots, n\} = \bigcup_{i \in F} M_i$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{N}$ such that*

- (i) $F \subset \{1, \dots, n\}$, $1 \leq |M_i| \leq 16/c^2$ for $i \in F$ and
- (ii) $\sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \tau_{ji} \right|^2 \geq \frac{c^2 n \|T\|^2}{64}$.

Proof. Because of homogeneity we can assume that T is a norm one operator, that is, $|\tau_{ij}| \leq 1$ for $i, j = 1, \dots, n$. Let e_1, \dots, e_n denote the unit vectors in l_1^n . An extreme point argument (see [PW], p. 22) shows that

$$(3) \quad \kappa(T|A_n)^2 = \sum_{h=1}^n \left\| \sum_{k=1}^n \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2$$

for some map $\varphi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and numbers μ_1, \dots, μ_n with $\sum_{k=1}^n |\mu_k|^2 = 1$. Let

$$A = \{k \in \{1, \dots, n\} : |\mu_k| \leq 2/(c\sqrt{n})\} \quad \text{and} \quad A^c = \{1, \dots, n\} \setminus A.$$

Then $1 = \sum_{k=1}^n |\mu_k|^2 \geq (4/(c^2 n))|A^c|$ implies $|A^c| \leq c^2 n/4$. It follows from (3) and the assumption $\kappa(T|A_n) \geq c\sqrt{n}$ that

$$\begin{aligned} c^2 n &\leq \sum_{h=1}^n \left\| \sum_{k=1}^n \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2 \\ &\leq 2 \sum_{h=1}^n \left\| \sum_{k \in A} \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2 + 2 \sum_{h=1}^n \left\| \sum_{k \in A^c} \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2. \end{aligned}$$

Now the bound on the cardinality of A^c , $\|T\| \leq 1$, and $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ imply that

$$\begin{aligned} \sum_{h=1}^n \left\| \sum_{k \in A^c} \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2 &\leq \sum_{h=1}^n \left(\sum_{k \in A^c} |\mu_k| \cdot |\alpha_{hk}| \right)^2 \\ &\leq \sum_{h=1}^n \left(\sum_{k \in A^c} |\mu_k|^2 \right) \left(\sum_{k \in A^c} |\alpha_{hk}|^2 \right) \leq \sum_{k \in A^c} \sum_{h=1}^n |\alpha_{hk}|^2 \leq |A^c| \leq c^2 n/4. \end{aligned}$$

Hence

$$c^2 n/4 \leq \sum_{h=1}^n \left\| \sum_{k \in A} \mu_k T e_{\varphi(k)} \alpha_{hk} \right\|^2.$$

We define the level sets $M_i = \{k \in A : \varphi(k) = i\}$ for $i = 1, \dots, n$. Then

$$c^2n/4 \leq \sum_{h=1}^n \left\| \sum_{i=1}^n \left(\sum_{k \in M_i} \mu_k \alpha_{hk} \right) T e_i \right\|^2 = \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \sum_{k \in M_i} \mu_k \alpha_{hk} \tau_{ji} \right|^2.$$

Let

$$B = \{i \in \{1, \dots, n\} : |M_i| \leq 16/c^2\} \quad \text{and} \quad B^c = \{1, \dots, n\} \setminus B.$$

We conclude from the preceding inequality that

$$\begin{aligned} c^2n/4 &\leq 2 \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in B} \sum_{k \in M_i} \mu_k \alpha_{hk} \tau_{ji} \right|^2 \\ &\quad + 2 \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in B^c} \sum_{k \in M_i} \mu_k \alpha_{hk} \tau_{ji} \right|^2. \end{aligned}$$

Since the sets M_i form a partition of A , we find that

$$n \geq |A| = \sum_{i=1}^n |M_i| \geq \frac{16}{c^2} |B^c|,$$

which in turn yields $|B^c| \leq c^2n/16$. Using again the assumptions $|\tau_{ji}| \leq 1$ and $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$, we derive that

$$\begin{aligned} \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in B^c} \sum_{k \in M_i} \mu_k \alpha_{hk} \tau_{ji} \right|^2 &\leq \sum_{h=1}^n \left(\sum_{i \in B^c} \left| \sum_{k \in M_i} \mu_k \alpha_{hk} \right| \right)^2 \\ &\leq |B^c| \sum_{h=1}^n \sum_{i \in B^c} \left| \sum_{k \in M_i} \mu_k \alpha_{hk} \right|^2 \leq \frac{c^2n}{16} \sum_{i \in B^c} \sum_{k \in M_i} |\mu_k|^2 \leq \frac{c^2n}{16}. \end{aligned}$$

Hence we arrive at

$$(4) \quad c^2n/16 \leq \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in B} \sum_{k \in M_i} \mu_k \alpha_{hk} \tau_{ji} \right|^2,$$

which is nearly what we want. In the last step of the proof we replace (μ_k) by $(\lambda_k) \in \mathcal{N}$. To this end we define

$$\lambda_k^o = \begin{cases} \mu_k & \text{if } k \in M_i \text{ for some } i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

We extend B to a set F and define M_i for $i \in F \setminus B$ such that (M_i) forms a partition of $\{1, \dots, n\}$ with $|M_i| \leq 16/c^2$ for all $i \in F$. Removing the empty M_i 's we may and do assume that $|F| \leq n$ and $F \subset \{1, \dots, n\}$. Then we can rewrite inequality (4) as

$$(5) \quad c^2n/16 \leq \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in F} \sum_{k \in M_i} \lambda_k^o \alpha_{hk} \tau_{ji} \right|^2.$$

Our construction yields $(\lambda_1^o, \dots, \lambda_n^o) \in B_{l_2^n} \cap \frac{2}{c\sqrt{n}}B_{l_\infty^n}$. Thus we can find $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{N}$ such that

$$\sum_{k=1}^n |\lambda_k^o - \lambda_k|^2 \leq c^2/64.$$

Now it follows from (5) that

$$\begin{aligned} c^2n/16 &\leq 2 \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in F} \sum_{k \in M_i} \lambda_k \alpha_{hk} \tau_{ji} \right|^2 \\ &\quad + 2 \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in F} \sum_{k \in M_i} (\lambda_k^o - \lambda_k) \alpha_{hk} \tau_{ji} \right|^2. \end{aligned}$$

Finally, we use once more the assumptions $|\tau_{ji}| \leq 1$ and $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ to conclude that

$$\begin{aligned} \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{i \in F} \sum_{k \in M_i} (\lambda_k^o - \lambda_k) \alpha_{hk} \tau_{ji} \right|^2 &\leq \sum_{h=1}^n \left(\sum_{i \in F} \left| \sum_{k \in M_i} (\lambda_k^o - \lambda_k) \alpha_{hk} \right| \right)^2 \\ &\leq |F| \sum_{i \in F} \sum_{h=1}^n \left| \sum_{k \in M_i} (\lambda_k^o - \lambda_k) \alpha_{hk} \right|^2 \leq n \sum_{i \in F} \sum_{k \in M_i} |\lambda_k^o - \lambda_k|^2 \leq c^2n/64, \end{aligned}$$

which in turn implies the claimed property (ii). ■

REMARK. The proposition is sharp in the following sense. If there exist a partition $\{1, \dots, n\} = \bigcup_{i \in F} M_i$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in B_{l_2^n}$ such that (ii) holds then $\kappa(T|A_n) \geq c\sqrt{n} \|T\|/8$. This can be seen directly from the definition of $\kappa(T|A_n)$ by choosing $x_k = \lambda_k e_i$ for $k \in M_i$.

3. Random matrices. Let \mathcal{M}_n be the set of all $n \times n$ -matrices with entries ± 1 , viewed as operators in $\mathcal{L}(l_1^n, l_\infty^n)$, equipped with the normalized counting measure P , that is, $P(T) = 2^{-n^2}$ for all $T \in \mathcal{M}_n$. The next theorem which will be the main tool in the following sections shows that the probability of $\kappa(T : l_1^n \rightarrow l_\infty^n | A_n)$ being large is very small for a fixed $n \times n$ -matrix A_n . In contrast, the probability of T factoring nicely through a Hilbert space is also very small. As we will see, this follows quite easily from well known norm estimates for random matrices. Consequently, for any given $n \times n$ -matrix A_n with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$, there must be $n \times n$ -matrices T with ± 1 entries which have, as operators in $\mathcal{L}(l_1^n, l_\infty^n)$, large $\kappa_n(T)$ but considerably smaller $\kappa(T|A_n)$. We now exploit this approach in detail.

THEOREM 3.1. *For $n = 1, 2, \dots$, every $n \times n$ -matrix A_n satisfying $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ and $1/\sqrt{n} \leq c \leq 1$, we have*

$$\mathbb{P}(\kappa(T|A_n) \geq c\sqrt{n}) \leq 2^{-2^{-33} c^{10} n^2 + 4n(4 + \log_2 n)}.$$

Proof. We fix a $c/8$ -net \mathcal{N} in $B_{l_2^n} \cap \frac{2}{c\sqrt{n}}B_{l_\infty^n}$ in l_2^n -distance. Let \mathcal{M}_n^o be the set of all $T \in \mathcal{M}_n$ satisfying the conclusion of Proposition 2.1. That means that for each $T \in \mathcal{M}_n^o$ there exist a partition $(M_i)_{i \in F}$ of $\{1, \dots, n\}$ with nonempty M_i and $\lambda \in \mathcal{N}$ with properties (i) and (ii). Proposition 2.1 tells us that $\mathbb{P}(\kappa(T|A_n) \geq c\sqrt{n}) \leq \mathbb{P}(\mathcal{M}_n^o)$. Let \mathcal{P} be the collection of all partitions $(M_i)_{i \in F}$ of $\{1, \dots, n\}$ satisfying (i). For fixed $P \in \mathcal{P}$ and $\lambda \in \mathcal{N}$, we define $\mathcal{M}(P, \lambda) = \{T \in \mathcal{M}_n : \text{(ii) is also satisfied}\}$. Then it follows from

$$\mathcal{M}_n^o = \bigcup_{P \in \mathcal{P}} \bigcup_{\lambda \in \mathcal{N}} \mathcal{M}(P, \lambda)$$

that

$$\mathbb{P}(\mathcal{M}_n^o) \leq \sum_{P \in \mathcal{P}} \sum_{\lambda \in \mathcal{N}} \mathbb{P}(\mathcal{M}(P, \lambda)).$$

The key lemma to be proved below is as follows.

LEMMA 3.2. *For all $P \in \mathcal{P}$ and $\lambda \in \mathcal{N}$, we have*

$$\mathbb{P}(\mathcal{M}(P, \lambda)) \leq 2^{-2^{-33}c^{10}n^2 + 2n(1 + \log_2 n)}.$$

Together with the above considerations, this lemma now implies

$$(6) \quad \mathbb{P}(\kappa(T|A_n) \geq c\sqrt{n}) \leq |\mathcal{P}| \cdot |\mathcal{N}| 2^{-2^{-33}c^{10}n^2 + 2n(1 + \log_2 n)}.$$

It remains to estimate the cardinalities of \mathcal{P} and \mathcal{N} . A trivial bound on the cardinality of \mathcal{P} is

$$(7) \quad |\mathcal{P}| \leq n^n \leq 2^{n(1 + \log_2 n)}.$$

Moreover, it follows from well known entropy estimates that \mathcal{N} can be chosen such that

$$|\mathcal{N}| \leq (17/c)^{2n} \leq 2^{2n \log_2(17/c)}.$$

Since we assumed $c \geq 1/\sqrt{n}$, this implies

$$(8) \quad |\mathcal{N}| \leq 2^{2n \log_2(17\sqrt{n})} \leq 2^{n(10 + \log_2 n)}.$$

Now (6)–(8) provide the claimed estimate in the theorem. ■

Proof of Lemma 3.2. For $h = 1, \dots, n$, we define

$$N_h = \left\{ \varepsilon \in \{+1, -1\}^F : \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \varepsilon_i \right| \geq c/16 \right\}.$$

First, we observe that, for any $\varepsilon \in \{+1, -1\}^F$, the set $A(\varepsilon) = \{h \in \{1, \dots, n\} : \varepsilon \in N_h\}$ satisfies the cardinality estimate

$$(9) \quad |A(\varepsilon)| \leq 256/c^2.$$

Indeed, this follows from $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ since

$$|A(\varepsilon)| \frac{c^2}{256} \leq \sum_{h=1}^n \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \varepsilon_i \right|^2 \leq \sum_{i \in F} \sum_{k \in M_i} |\lambda_k|^2 \leq 1.$$

Let now $T \in \mathcal{M}(P, \lambda)$ and define

$$B = \left\{ h \in \{1, \dots, n\} : \max_{1 \leq j \leq n} \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \tau_{ji} \right| \geq c/16 \right\}$$

and $B^c = \{1, \dots, n\} \setminus B$. Then

$$\sum_{h \in B^c} \max_{1 \leq j \leq n} \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \tau_{ji} \right|^2 \leq |B^c| \frac{c^2}{256} = \frac{c^2}{256} (n - |B|)$$

and

$$\begin{aligned} \sum_{h \in B} \max_{1 \leq j \leq n} \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \tau_{ji} \right|^2 &\leq \sum_{h \in B} \left(\sum_{k=1}^n |\lambda_k| \cdot |\alpha_{hk}| \right)^2 \\ &\leq \sum_{h \in B} \left(\sum_{k=1}^n |\lambda_k|^2 \right) \left(\sum_{k=1}^n |\alpha_{hk}|^2 \right) \leq |B|. \end{aligned}$$

Now it follows from the assumed property (ii) that

$$\frac{c^2 n}{64} \leq |B| + (n - |B|) \frac{c^2}{256} \leq \frac{c^2 n}{256} + |B|,$$

which implies $|B| \geq c^2 n / 128$. Let

$$B_j = \left\{ h \in \{1, \dots, n\} : \left| \sum_{i \in F} \left(\sum_{k \in M_i} \lambda_k \alpha_{hk} \right) \tau_{ji} \right| \geq c/16 \right\} = A((\tau_{ji})_i).$$

Then $B = \bigcup_{j=1}^n B_j$. Furthermore, let $N = \bigcup_{h=1}^n N_h$ and

$$C = \{j \in \{1, \dots, n\} : (\tau_{ji})_{i=1}^n \in N\} = \{j \in \{1, \dots, n\} : B_j \neq \emptyset\}.$$

Then it follows from (9) that

$$\frac{c^2 n}{128} \leq |B| \leq \sum_{j=1}^n |B_j| = \sum_{j \in C} |B_j| \leq \frac{256}{c^2} |C|,$$

which gives $|C| \geq c^4 n / 2^{15}$.

Let \mathcal{C} denote the collection of all subsets $C \subset \{1, \dots, n\}$ satisfying this cardinality estimate. For $C \in \mathcal{C}$, let

$$\mathcal{T}(C) = \{T \in \mathcal{M}(P, \lambda) : (\tau_{ji})_{i=1}^n \in N \text{ for } j \in C\}.$$

The above considerations tell us that $\bigcup_{C \in \mathcal{C}} \mathcal{T}(C) = \mathcal{M}(P, \lambda)$, which implies that

$$(10) \quad \mathbb{P}(\mathcal{M}(P, \lambda)) \leq \sum_{C \in \mathcal{C}} \mathbb{P}(\mathcal{T}(C)).$$

To estimate $\mathbb{P}(\mathcal{T}(C))$, we have to estimate the size of N . It follows from the well known tail behavior estimate of Rademacher series with real coefficients that if $\varepsilon_1, \dots, \varepsilon_m$ are independent random variables identically distributed with $\mathbb{P}(\varepsilon_k = +1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$ and $\zeta_1, \dots, \zeta_m \in C$, then

$$\mathbb{P}\left(\left|\sum \varepsilon_k \zeta_k\right| > \delta\right) \leq 4 \exp\left(-\frac{\delta^2}{4 \sum |\zeta_k|^2}\right) \quad \text{for } \delta > 0.$$

Using this, we see from the definition of N_h that

$$|N_h| \leq 2^{|F|+2} \exp\left(-\frac{2^{-10} c^2}{\sum_{i \in F} \left|\sum_{k \in M_i} \lambda_k \alpha_{hk}\right|^2}\right).$$

Moreover, we conclude from $|M_i| \leq 16/c^2$ for $i \in F$ and $|\lambda_k| \leq 2/c\sqrt{n}$ that

$$\begin{aligned} (11) \quad \sum_{i \in F} \left|\sum_{k \in M_i} \lambda_k \alpha_{hk}\right|^2 &\leq \sum_{i \in F} \left(\sum_{k \in M_i} |\lambda_k| \cdot |\alpha_{hk}|\right)^2 \leq \frac{4}{c^2 n} \sum_{i \in F} \left(\sum_{k \in M_i} |\alpha_{hk}|\right)^2 \\ &\leq \frac{4}{c^2 n} \sum_{i \in F} |M_i| \sum_{k \in M_i} |\alpha_{hk}|^2 \leq \frac{64}{c^4 n} \sum_{k=1}^n |\alpha_{hk}|^2 \leq \frac{64}{c^4 n}. \end{aligned}$$

Together with $|F| \leq n$ this implies

$$|N_h| \leq 2^{n+2} \exp(-2^{-18} c^6 n) = 2^{n(1-2^{-18} c^6 \log_2 e)+2}$$

and

$$|N| \leq \sum_{h=1}^n |N_h| \leq 2^{n(1-2^{-18} c^6)+2+\log_2 n}.$$

Now we are able to estimate the size of $\mathcal{T}(C)$ as

$$\begin{aligned} |\mathcal{T}(C)| &\leq |N|^{|C|} (2^n)^{n-|C|} \leq 2^{n(1-2^{-18} c^6)|C|+(2+\log_2 n)|C|+n^2-n|C|} \\ &\leq 2^{n^2-2^{-18} c^6 n|C|+n(2+\log_2 n)} \leq 2^{n^2(1-2^{-33} c^{10})+n(2+\log_2 n)}. \end{aligned}$$

Hence

$$\mathbb{P}(\mathcal{T}(C)) \leq 2^{-2^{-33} c^{10} n^2+n(2+\log_2 n)}.$$

Finally, observing that $|C| \leq n^n = 2^{n \log_2 n}$, an appeal to (10) finishes the proof. ■

REMARK. If A_n is a unimodular orthogonal $n \times n$ -matrix, which means that $|\alpha_{hk}| = 1/\sqrt{n}$ for $h, k = 1, \dots, n$, the estimate (11) can be improved to

$$\sum_{i \in F} \left|\sum_{k \in M_i} \lambda_k \alpha_{hk}\right|^2 \leq \frac{1}{n} \sum_{i \in F} \left(\sum_{k \in M_i} |\lambda_k|\right)^2 \leq \frac{1}{n} \sum_{i \in F} |M_i| \sum_{k \in M_i} |\lambda_k|^2 \leq \frac{16}{c^2 n}.$$

This yields a better estimate in Theorem 3.1 for such matrices, in fact the fraction $2^{-33} c^{10}$ can be replaced by $2^{-31} c^8$.

The next corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Let $c > 0$ and let $(A_n)_{n \geq 1}$ be a sequence of $n \times n$ -matrices with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$. Then $\mathbb{P}(\kappa(T|A_n) \geq c\sqrt{n}) \rightarrow 0$ as $n \rightarrow \infty$.*

This corollary can be made more quantitative by keeping track of the constants.

COROLLARY 3.4. *Let $(A_n)_{n \geq 1}$ be a sequence of $n \times n$ -matrices with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$. Then $\mathbb{P}(\kappa(T|A_n) \geq n^\alpha) \rightarrow 0$ as $n \rightarrow \infty$ for any $\alpha > 2/5$.*

Proof. Setting $c_n = n^{\alpha-1/2}$ and applying Theorem 3.1 gives

$$\mathbb{P}(\kappa(T|A_n) \geq n^\alpha) = \mathbb{P}(\kappa(T|A_n) \geq c_n \sqrt{n}) \leq 2^{-2^{-33} n^{10\alpha-3} + 4n(4+\log_2 n)},$$

which tends to 0 as long as $\alpha > 2/5$. ■

By use of the remark following the proof of Lemma 3.2, the preceding corollary can be strengthened for unimodular orthogonal matrices to

COROLLARY 3.5. *Let $(A_n)_{n \geq 1}$ be a sequence of unimodular orthogonal $n \times n$ -matrices with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$. Then $\mathbb{P}(\kappa(T|A_n) \geq n^\alpha) \rightarrow 0$ as $n \rightarrow \infty$ for any $\alpha > 3/8$.*

4. Hilbert space factorization. We start this section with the following easy observation.

PROPOSITION 4.1. *Let $T \in \mathcal{L}(l_1^n, l_\infty^n)$ be an operator induced by a matrix (τ_{ij}) with $\tau_{ij} \in \pm 1$. Then*

$$\kappa_n(T : l_1^n \rightarrow l_\infty^n) \geq \|T : l_2^n \rightarrow l_2^n\|^{-1} n.$$

Proof. Let e_1, \dots, e_n be the unit vectors in l_1^n . Then

$$\sum_{k=1}^n \|e_k\|^2 = n \quad \text{and} \quad \sum_{h=1}^n \left\| \sum_{k=1}^n \tau_{hk} T e_k \right\|^2 = \sum_{h=1}^n \max_{1 \leq j \leq n} \left| \sum_{k=1}^n \tau_{hk} \tau_{jk} \right|^2 = n^3$$

imply that $\kappa(T|T) \geq n$. Now a glance at the definition of $\kappa_n(T)$ proves the claim. ■

Estimates on norms of random matrices are well known. We need a special case of Lemma 4 from [BGN]:

LEMMA 4.2. *There exist constants c_1 and c_2 such that, for all $t > 0$,*

$$\mathbb{P}(\|T : l_2^n \rightarrow l_2^n\| \geq c_1 \sqrt{(1+t)n}) \leq \exp(-(t-c_2)n).$$

COROLLARY 4.3. *There exists a constant $\chi > 0$ such that*

$$\mathbb{P}(\kappa_n(T) \leq \chi \sqrt{n}) \leq e^{-3n}.$$

Proof. Set $t = 3 + 4c_2$ in the preceding lemma and define $\chi = 1/(2c_1\sqrt{1 + c_2})$. Then the lemma tells us that

$$\mathbb{P}(\|T : l_2^n \rightarrow l_2^n\| \geq \sqrt{n}/\chi) \leq e^{-3(1+c_2)n} \leq e^{-3n}.$$

Now the assertion follows from the above proposition. ■

It can be immediately seen from Corollaries 3.3 and 4.3 that there cannot exist a sequence (A_n) of $n \times n$ -matrices with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$ which is universal for the sequence of ideal norms κ_n . We prefer here to give a more quantitative statement which also implies Theorem 1.3.

THEOREM 4.4. *Let $(A_n)_{n \geq 1}$ be a sequence of $n \times n$ -matrices with $\|A_n : l_2^n \rightarrow l_2^n\| \leq 1$. Then $\mathbb{P}(\kappa_n(T) \geq n^\beta \kappa(T|A_n)) \rightarrow 1$ as $n \rightarrow \infty$ for any $\beta < 1/10$. If the A_n are unimodular orthogonal matrices then this even holds for $\beta < 1/8$.*

Proof. Since

$$\mathbb{P}(\kappa_n(T) \geq n^\beta \kappa(T|A_n)) \geq 1 - \mathbb{P}(\kappa_n(T) \leq \chi\sqrt{n}) - \mathbb{P}(\kappa(T|A_n) \geq \chi n^{1/2-\beta}),$$

this follows from Corollaries 3.4, 3.5 and 4.3. ■

5. The size of universal sets of matrices. For $n \geq 1$, let \mathcal{M}_n^2 be the set of all $n \times n$ -matrices with operator norm on l_2^n less than or equal to 1. We say that a sequence (M_n) of subsets $M_n \subset \mathcal{M}_n^2$ is universal for κ_n if there exists $c > 0$ such that

$$\kappa_n(T) \leq c \sup_{A_n \in M_n} \kappa(T|A_n) \quad \text{for } n = 1, 2, \dots \text{ and all operators } T.$$

In this section, we want to clarify what size the members M_n of a universal sequence for κ_n must have. We first show an upper bound using entropy estimates. With the help of the results from the preceding sections we then derive a lower bound which is essentially the same.

THEOREM 5.1. *There exists a universal sequence (M_n) for κ_n for which $|M_n| \leq 2^{n^2(5+\log_2 n)}$ for $n = 1, 2, \dots$*

Proof. M_n is going to be a $1/2$ -net in \mathcal{M}_n^2 . Let us first see that such a sequence is indeed universal for κ_n .

Given $B_n = B_n^{(0)} \in \mathcal{M}_n^2$, we inductively choose $A_n^{(k)} \in M_n$ such that $B_n^{(k+1)} = B_n^{(k)} - 2^{-k} A_n^{(k)}$ satisfies $\|B_n^{(k+1)} : l_2^n \rightarrow l_2^n\| \leq 2^{-k-1}$. Then

$$B_n = \sum_{k=0}^{\infty} 2^{-k} A_n^{(k)},$$

and it easily follows from the definition of $\kappa(T|A_n)$ that

$$\kappa(T|B_n) \leq \sum_{k=0}^{\infty} 2^{-k} \kappa(T|A_n^{(k)}) \leq 2 \max_{A_n \in M_n} \kappa(T|A_n),$$

showing the required inequality.

It remains to prove that there is a $1/2$ -net M_n in \mathcal{M}_n^2 with $|M_n| \leq 2^{2n^2(5+\log_2 n)}$. To this end, we identify \mathcal{M}_n^2 with a subset of \mathbb{C}^{n^2} in the obvious way. Using the well known estimate

$$\|A_n : l_2^n \rightarrow l_2^n\| \leq \left(\sum_{h=1}^n \sum_{k=1}^n |\alpha_{hk}|^2 \right)^{1/2}$$

for an $n \times n$ -matrix $A_n = (\alpha_{hk})$, we see that it is enough to find a $1/2$ -net in euclidian norm with the required cardinality.

Pick now a point x_1 in \mathcal{M}_n^2 . Assume that we have found points x_1, \dots, x_k in \mathcal{M}_n^2 such that $\|x_i - x_j\| \geq 1/2$ for $i \neq j$, where $\|\cdot\|$ now denotes euclidian norm. If possible, we choose a further point $x_{k+1} \in \mathcal{M}_n^2$ with $\|x_{k+1} - x_i\| \geq 1/2$ for $i = 1, \dots, k$. This process eventually stops leaving us with points $x_1, \dots, x_K \in \mathcal{M}_n^2$ such that $\|x_i - x_j\| \geq 1/2$ for $i \neq j$ and such that the union of the balls with centers x_i and radius $1/2$ covers \mathcal{M}_n^2 . Hence the set $M_n = (x_1, \dots, x_K)$ is indeed a $1/2$ -net in euclidian norm.

Moreover, since also

$$\left(\sum_{h=1}^n \sum_{k=1}^n |\alpha_{hk}|^2 \right)^{1/2} \leq \sqrt{n} \|A_n : l_2^n \rightarrow l_2^n\|,$$

we find that $\mathcal{M}_n^2 \subset \sqrt{n}B$ where B is the euclidian unit ball in \mathbb{C}^{n^2} . Hence the balls with centers x_i and radius $1/4$ are contained in $(\sqrt{n} + 1/4)B$. They are also mutually disjoint, so volume comparison yields

$$K \left(\frac{1}{4} \right)^{2n^2} \leq \left(\sqrt{n} + \frac{1}{4} \right)^{2n^2},$$

which finally implies

$$|M_n| = K \leq (4\sqrt{n} + 1)^{2n^2} \leq 2^{n^2(5+\log_2 n)}. \blacksquare$$

THEOREM 5.2. *If the sequence (M_n) is universal for κ_n then there exists $\alpha > 0$ such that $|M_n| \geq 2^{\alpha n^2}$ for $n = 1, 2, \dots$*

Proof. Assume that (M_n) is universal for κ_n with constant $c > 0$. We may also assume that each M_n is finite. In particular,

$$\kappa_n(T) \leq c \max_{A_n \in M_n} \kappa(T|A_n) \quad \text{for all } T \in \mathcal{M}_n.$$

It then follows from Corollary 4.3 and Theorem 3.1 that

$$\begin{aligned}
 1 - e^{-3n} &\leq \mathbb{P}(\kappa_n(T) > \chi\sqrt{n}) \leq \mathbb{P}(\max_{A_n \in M_n} \kappa(T|A_n) > (\chi/c)\sqrt{n}) \\
 &\leq |M_n|2^{-2^{-33}\chi^{10}n^2/c^{10}+4n(4+\log_2 n)}.
 \end{aligned}$$

This easily gives the lower bound on $|M_n|$ as claimed in the theorem. ■

6. Application to some other ideal norms. In this section, we make use of the notation of [PW]. The reader not familiar with this notation is advised to consult this reference for further information. We briefly review some things we need here.

$\mathcal{E}_n = (e_1, \dots, e_n)$ stands for the first n trigonometric characters $e_k(t) = \exp(2\pi ikt)$ in $L_2(0, 1)$. $\mathcal{W}_n = (w_1, \dots, w_n)$ stands for the first n Walsh functions in $L_2(0, 1)$ in some natural ordering. W_{2^n} is the $n+1$ -st Hadamard–Walsh matrix inductively defined by

$$W_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \quad \text{and} \quad W_{2^{n+1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} +W_{2^n} & +W_{2^n} \\ +W_{2^n} & -W_{2^n} \end{pmatrix}.$$

The matrices E_n were already defined in the introduction.

The n th trigonometric and Walsh type norms of an operator T are denoted by $\varrho(T|\mathcal{E}_n, \mathcal{I}_n)$ and $\varrho(T|\mathcal{W}_n, \mathcal{I}_n)$, respectively. Given T , we let

$$\begin{aligned}
 \varrho(T|\mathcal{E}_n^{\max}, \mathcal{I}_n) &= \max\{\varrho(T|\mathcal{E}(F), \mathcal{I}(F)) : F \subset N \text{ and } |F| \leq n\}, \\
 \varrho(T|\mathcal{W}_n^{\max}, \mathcal{I}_n) &= \max\{\varrho(T|\mathcal{W}(F), \mathcal{I}(F)) : F \subset N \text{ and } |F| \leq n\}.
 \end{aligned}$$

We now state the main theorem of this section.

THEOREM 6.1. *None of the following sequences of ideal norms is uniformly equivalent to the sequence κ_n :*

$$\kappa(E_n), \quad \varrho(\mathcal{E}_n, \mathcal{I}_n), \quad \varrho(\mathcal{E}_n^{\max}, \mathcal{I}_n), \quad \varrho(\mathcal{W}_n, \mathcal{I}_n), \quad \varrho(\mathcal{W}_n^{\max}, \mathcal{I}_n).$$

Moreover the sequence $\kappa(W_{2^n})$ is not uniformly equivalent to the sequence κ_{2^n} .

Proof. It is shown in [Bou2] that $\varrho(\mathcal{W}_n^{\max}, \mathcal{I}_n) \leq c\varrho(\mathcal{W}_n, \mathcal{I}_n)$ with c independent of n (see also [PW], p. 326). This means that the sequences $\varrho(\mathcal{W}_n^{\max}, \mathcal{I}_n)$ and $\varrho(\mathcal{W}_n, \mathcal{I}_n)$ are uniformly equivalent. We conclude from Theorem 1.3 that the sequence $\kappa(W_{2^n})$ is not uniformly equivalent to the sequence κ_{2^n} . Since $\varrho(\mathcal{W}_{2^n}, \mathcal{I}_{2^n}) = \kappa(W_{2^n})$, we conclude that neither $\varrho(\mathcal{W}_n, \mathcal{I}_n)$ nor $\varrho(\mathcal{W}_n^{\max}, \mathcal{I}_n)$ can be uniformly equivalent to κ_n .

As already observed in [Hin], the methods of [Bou2] imply that there exists $c > 0$ such that $\varrho(\mathcal{E}_n^{\max}, \mathcal{I}_n) \leq c\sqrt{1 + \log n} \varrho(\mathcal{E}_n, \mathcal{I}_n)$. Since the sequence $\varrho(\mathcal{E}_n, \mathcal{I}_n)$ is uniformly equivalent to the sequence $\kappa(E_n)$ (see [PW], p. 285), we infer from Theorem 1.3 that none of the sequences $\kappa(E_n)$, $\varrho(\mathcal{E}_n, \mathcal{I}_n)$, and $\varrho(\mathcal{E}_n^{\max}, \mathcal{I}_n)$ is uniformly equivalent to κ_n . ■

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