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Spaces of operators and c_0

by

PAUL LEWIS (Denton, TX)

Abstract. Bessaga and Pełczyński showed that if c_0 embeds in the dual X^* of a Banach space X, then ℓ^1 embeds complementably in X, and ℓ^∞ embeds as a subspace of X^*. In this note the Diestel–Faires theorem and techniques of Kalton are used to show that if X is an infinite-dimensional Banach space, Y is an arbitrary Banach space, and c_0 embeds in $L(X,Y)$, then ℓ^∞ embeds in $L(X,Y)$, and ℓ^1 embeds complementably in $X \otimes_\gamma Y^*$. Applications to embeddings of c_0 in various spaces of operators are given.

All Banach spaces in this note are defined over the real field. If X and Y are Banach spaces, then $L(X,Y)$ is the Banach space of all continuous linear functions (= operators) from X to Y equipped with the usual operator norm, $K(X,Y)$ is the space of compact operators from X to Y, and X^* is the dual of X. We say that X embeds in Y if there is a linear homeomorphism from X into Y, i.e. there is an isomorphic embedding $T : X \to Y$. The canonical unit vector basis of c_0 is denoted by (e_n), and the canonical basis of ℓ^1 is denoted by (e_n^*). If $A \subseteq X$, then $[A]$ denotes the closed linear span of A. The greatest crossnorm tensor product completion of X and Y is denoted by $X \otimes_\gamma Y$. We refer the reader to Lindenstrauss and Tzafriri [LT] or Diestel [D] for undefined notation and terminology.

Numerous authors have noticed that if c_0 embeds in $K(X,Y)$ and either X or Y has a "nice" Schauder decomposition, then ℓ^∞ must embed in $L(X,Y)$ (see e.g. Kalton [K], Feder [F1], [F2], and Emmanuele [E1], [E2]). However, it does not seem to have been observed that the complete analogue of the Bessaga–Pełczyński theorem [BP, Thm. 3] holds in the space $L(X,Y)$ for any infinite-dimensional Banach space X.

THEOREM 1. *If X is infinite-dimensional and c_0 embeds in $L(X,Y)$, then ℓ^∞ embeds in $L(X,Y)$ and ℓ^1 embeds complementably in $X \otimes_\gamma Y^*$. Moreover, $(T(e_n)) \to 0$ in the strong operator topology (of $L(X,Y)$) for each isomorphic embedding $T : c_0 \to L(X,Y)$ if and only if c_0 fails to embed in Y.*

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Proof. We follow the lead of Kalton [K] and consider two cases: c_0 embeds in Y and c_0 does not embed in Y.

Suppose that $T : c_0 \to Y$ is an isomorphic embedding. Use the Josefson–Nissenzweig theorem [D, Chap. XIII], and choose a sequence (x_n^*) in X^* so that $\|x_n^*\| = 1$ for each n and $(x_n^*) \to 0$ in the weak* topology. Define $J : \ell^\infty \to L(X, [T(e_n)])$ by

$$J(b_n)(x) = \sum_{n=1}^{\infty} b_n x_n^*(x) T(e_n)$$

for $x \in X$. It is easy to check that J is continuous, linear, and injective. Further, J^{-1} is continuous since $(T(e_n)) \sim (e_n)$.

Now suppose that c_0 does not embed in Y, and let $B : c_0 \to L(X, Y)$ be an isomorphic embedding. Certainly the weak unconditional convergence of $\sum e_n$ guarantees that

$$\sum_{n=1}^{\infty} |\langle B(e_n)(x), y^*\rangle| < \infty$$

for each $x \in X$ and $y^* \in Y^*$. Thus $\sum B(e_n)x$ is weakly unconditionally convergent in Y. Since c_0 does not embed in Y, $\sum B(e_n)x$ is unconditionally convergent in Y ([BP], [D, p. 45]). Therefore if A is a non-empty subset of $\mathbb{N}$, then $\sum_{n\in A} B(e_n)$ converges unconditionally in the strong operator topology of $L(X, Y)$. Further, an application of the Uniform Boundedness Principle shows that

$$\Big\{ \sum_{n\in A} B(e_n) \text{ (strong operator topology)} : A \subseteq \mathbb{N},\ A \neq \emptyset \Big\}$$

is bounded. Define μ by $\mu(\emptyset) = 0$ and

$$\mu(A) = \sum_{n\in A} B(e_n) \quad \text{(strong operator topology)}$$

for any non-empty subset A of $\mathbb{N}$. It is straightforward to check that μ is bounded and finitely additive on the σ-algebra Σ consisting of all subsets of $\mathbb{N}$. However, $(\mu(n)) \not\to 0$, i.e. μ is not strongly additive. Hence, by the σ-algebra version of the Diestel–Faires theorem ([DU, p. 20], [DF]), $L(E, F)$ contains an isomorphic copy of ℓ^∞.

Next suppose that (x_n) is a bounded sequence in X and (y_n^*) is a bounded sequence in Y^* so that

$$\sum_{n=1}^{\infty} |\langle B(e_n)x_n, y_n^*\rangle - 1| < \infty.$$

(Of course, one can easily arrange to have the preceding infinite series sum to zero.) Note that $L(X, Y^{**})$ is isometrically isomorphic to $(X \otimes_\gamma Y^*)^*$ and that $(x_n \otimes y_n^*)$ is a bounded sequence in $X \otimes_\gamma Y^*$ [DU, Chap. VIII]. An application of the main theorem of Lewis [L] shows that there is a sequence (u_n) consisting of differences of terms of the sequence $(x_n \otimes y_n^*)$ so that $(u_n) \sim (e_n^*)$ and $[u_n]$ is complemented in $X \otimes_\gamma Y^*$.

Now suppose that $T : c_0 \to Y$ is an embedding and $x^* \in X^*$, $\|x^*\| = 1$. Define $J : c_0 \to L(X,Y)$ by $J(u)(x) = x^*(x)T(u)$ for $u \in c_0$ and $x \in X$. It follows that J is an isomorphism and $(J(e_n)) \not\to 0$ in the strong operator topology.

Conversely, if $J : c_0 \to L(X,Y)$ is any operator, $x \in X$, and $(J(e_n)x) \not\to 0$, then $\sum J(e_n)x$ is weakly unconditionally convergent and not unconditionally convergent in Y. Therefore c_0 embeds in Y. ∎

Of course, the converse of the classical Bessaga–Pełczyński theorem is not difficult to verify. That is, if ℓ^1 embeds complementably in X, then certainly c_0 embeds in X^*. However, as we shall see, the converse implication in our setting is false.

It is well known that if $1 < p < q < \infty$, then $L(\ell^q, \ell^p)$ is reflexive and $L(\ell^p, \ell^q)$ is not reflexive (see e.g. [K] or Theorem VIII.4.4 of Diestel and Uhl [DU]). Moreover, Diestel and Uhl [DU, p. 249] pointed out that if $1 < p < \infty$, then $\ell^p \otimes_\gamma \ell^p$ contains a complemented copy of ℓ^1. Consequently, if $X = \ell^p$, $2 < p < \infty$, and $Y = X^*$, then ℓ^1 embeds complementably in $X \otimes_\gamma Y^* = \ell^p \otimes_\gamma \ell^p$, but $L(X,Y) = L(\ell^p, (\ell^p)^*)$ is reflexive and thus does not contain c_0.

Now, again, if $1 < p < q < \infty$, it follows from Theorem 6 of [K] that $L(\ell^p, \ell^q)$ contains a copy of ℓ^∞. Moreover, Kalton remarked in the introduction to [K] that $L(\ell^2, \ell^2)$ contains an isomorphic copy of ℓ^∞. In fact, the techniques of the proof of Theorem 1 allow a more extensive statement. Recall that a sequence $(X_n)_{n=1}^\infty$ of closed linear subspaces of X is called an *unconditional Schauder decomposition* of X [LT, pp. 47–48] if each $x \in X$ has an unconditional and unique expansion of the form $x = \sum x_n$, with $x_n \in X_n$ for each n.

THEOREM 2. *If X has an unconditional Schauder decomposition, then ℓ^∞ embeds in $L(X,X)$ and ℓ^1 embeds complementably in $X \otimes_\gamma X^*$.*

Proof. Suppose that $(X_n)_{n=1}^\infty$ is an unconditional Schauder decomposition of X, and let Q_n be the natural projection of X into X_n. Let $\mathcal{F}$ be the finite-cofinite algebra of subsets of $\mathbb{N}$. Define $\mu(\emptyset)$ to be 0. If $A \in \mathcal{F}$, $A \neq \emptyset$, and A is finite, set $\mu(A) = \sum_{n \in A} Q_n$. If A^{c} is finite, set $\mu(A) = -\mu(A^{\mathrm{c}})$. Then μ is finitely additive and not strongly additive. Further, the unconditionality of the decomposition ensures that μ is bounded. An application of the algebra version of the Diestel–Faires theorem guarantees that c_0 embeds in $L(X,X)$. An application of Theorem 1 finishes the proof. ∎

The following result contrasts sharply with Theorem 2. The reader should compare this theorem with Theorem 3 of Emmanuele [E2].

THEOREM 3. *If neither X nor Y contains a complemented copy of ℓ^1 and each operator from X to Y^* is compact, then $(X \otimes_\gamma Y)^*$ does not contain c_0, and, consequently, $X \otimes_\gamma Y$ does not contain a complemented copy of ℓ^1.*

Proof. Suppose that $(X \otimes_\gamma Y)^*$ does contain c_0. Since $(X \otimes_\gamma Y)^*$ is isometrically isomorphic to $L(X, Y^*)$, we use Theorem 1 and see that ℓ^∞ embeds in $L(X, Y^*)$. Since every operator from X to Y^* is compact, we apply Theorem 4 of [K] and conclude that ℓ^∞ must embed in X^* or in Y^*. However, either case contradicts our hypotheses. ∎

In Theorem 1 of [E1], Emmanuele showed that if there is a non-compact member of $L(X, Y)$, Y is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n), and each operator from X to Z_n is compact for each n, then $K(X, Y)$ must contain a copy of c_0. It is not difficult to see that Emmanuele's hypotheses produce a sequence (T_n) in $K(X, Y)$ so that $\sum_{n=1}^\infty T_n(x)$ converges unconditionally for each $x \in X$ but $(\sum_{n=1}^k T_n)_{k=1}^\infty$ is not Cauchy in $L(X, Y)$. As the next theorem shows, the *compactness* of each T_n and the unconditional *norm* convergence of $\sum T_n(x)$ are not crucial in the determination of the presence of c_0. (Compactness does play a crucial role in other implications in Emmanuele's theorem.)

THEOREM 4. *Let $\mathcal{I}(X, Y)$ be a norm closed operator ideal in $L(X, Y)$. Then c_0 embeds in $\mathcal{I}(X, Y)$ if and only if there is a non-null sequence (T_n) in $\mathcal{I}(X, Y)$ so that $\sum T_n(x)$ is weakly unconditionally convergent in Y for each $x \in X$.*

Proof. Suppose that (T_n) is as in the statement of the theorem, and let $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{N}$. By the Uniform Boundedness Principle, $\{\sum_{n \in A} T_n : A \in \mathcal{F}\}$ is bounded in $L(X, Y)$. Use the finite-cofinite algebra of subsets of $\mathbb{N}$ and the Diestel–Faires theorem as in Theorem 2 to conclude that c_0 embeds in $\mathcal{I}(X, Y)$.

Conversely, suppose that $T : c_0 \to \mathcal{I}(X, Y)$ is an isomorphism, and let $T_n = T(e_n)$, $n \in \mathbb{N}$. Then $\sum T_n(x)$ is weakly unconditionally convergent for each $x \in X$. ∎

REMARK. Theorems 1 and 4 make it clear that ℓ^∞ embeds isomorphically in $L(X, Y)$ if and only if there is a non-null sequence (T_n) in $L(X, Y)$ so that $\sum T_n(x)$ is weakly unconditionally convergent in Y for each $x \in X$. Further, if S is any linear subspace of $L(X, Y)$ which is closed in the strong operator topology and (T_n) is a non-null sequence from S so that $\sum T_n(x)$ converges unconditionally for each $x \in X$, then ℓ^∞ embeds in S. See Feder [F1], [F2] for a discussion of similar conditions.

We conclude by giving a quick application of the preceding results in this note to operators on abstract continuous function spaces and their representing measures. We refer the reader to [BL] or [ABBL] for a complete discussion of this setting. We do note that if $T : C(H,X) \to Y$ is an operator on an abstract continuous function space with representing vector measure m, then T is said to be *strongly bounded* if $(\widetilde{m}(A_n)) \to 0$ on any pairwise disjoint sequence of Borel subsets of the compact Hausdorff space H, where $\widetilde{m}(A)$ denotes the semivariation of m on A.

THEOREM 5. *If* c_0 *does not embed in* $K(X,Y)$, *then every operator* $T : C(H,X) \to Y$ *is strongly bounded. If*, *in addition*, X *is reflexive*, *then every such operator is weakly compact.*

Proof. Suppose that $T : C(H,X) \to Y$ is an operator which is not strongly bounded. By results in Brooks and Lewis [BL] or Dobrakov [Do], T is not unconditionally converging. Therefore T must be an isomorphism on a copy of c_0 ([BP], [D, p. 54]), and Y contains a copy of c_0. Thus c_0 actually embeds in the rank one operators from X to Y, and we have established the contrapositive of the first statement in the theorem.

Now suppose c_0 does not embed in $K(X,Y)$ and that X is reflexive. The preceding paragraph and Theorem 4.1 of [BL] directly show that every operator $T : C(H,X) \to Y$ is weakly compact. ∎

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Department of Mathematics
University of North Texas
Denton, TX 76203, U.S.A.
E-mail: lewis@unt.edu

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