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Local integrated C-semigroups

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Abstract. We introduce the notion of a local n-times integrated C-semigroup, which unifies the classes of local C-semigroups, local integrated semigroups and local C-cosine functions. We then study its relations to the C-wellposedness of the $(n+1)$-times integrated Cauchy problem and second order abstract Cauchy problem. Finally, a generation theorem for local n-times integrated C-semigroups is given.

1. Introduction. The first systematic local theory for illposed abstract Cauchy problems appeared in 1990. Tanaka and Okazawa [TO] defined local C-semigroups and local integrated semigroups, and a real characterization was obtained under the assumption that $D(A)$ and $R(C)$ are dense. A generation theorem for local C-semigroups with nondensely defined generator was given by Zou [Zo]. Sun [Su] proved some properties of local C-semigroups and (once) integrated semigroups. In [LZ] and [ZL], Liu and Zhao discussed some properties of local integrated C-semigroups and their applications to the abstract Cauchy problem. Local C-cosine functions were introduced and investigated by F. Huang and T. Huang [HH] in the case when $R(C)$ is dense. Furthermore, a generation theorem giving a sufficient condition for a densely defined operator A to be the generator of a local C-semigroup or C-cosine function appeared in [Ga].

On the other hand, W. Arendt *et al.* [AEK] proceeded in a different way. They defined the wellposedness of the $(n+1)$-times integrated Cauchy problem (see $C_{n+1}(\tau)$ below with Cx replaced by x), and then characterized it by the resolvent of A ([AEK, Theorems 2.1, 2.2]). The operator-valued function which governs the problem there was called the n-times integrated semigroup generates by A. Moreover, an interesting extension property of the solution was given.

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In this paper, we define local n-times integrated C-semigroups which unify the classes of local C-semigroups, local integrated semigroups and local C-cosine functions. We then study the relations between local n-times integrated C-semigroups and C-wellposedness of the $(n+1)$-times integrated Cauchy problem

$$C_{n+1}(\tau) \qquad \begin{cases} v \in C([0,\tau); D(A)) \cap C^1([0,\tau); X), \\ v'(t) = Av(t) + \dfrac{t^n}{n!}Cx, \quad t \in [0,\tau), \\ v(0) = 0. \end{cases}$$

(See Section 2 for definitions.) A generation theorem for local integrated C-semigroups is also given.

Section 2 clarifies the relations between the local n-times integrated C-semigroups and the C-wellposedness of $C_{n+1}(\tau)$. We show in Theorem 2.5 that the C-wellposedness of $C_{n+1}(\tau)$ implies that A generates a local n-times integrated C-semigroup. Moreover, if $C_{n+1}(\tau)$ is C-wellposed, then

$$C_0(\tau) \qquad \begin{cases} u \in C([0,\tau); D(A)) \cap C^1([0,\tau); X), \\ u'(t) = Au(t), \quad t \in [0,\tau), \\ u(0) = Cx, \end{cases}$$

has a unique solution for each $x \in D(A^{n+1})$.

In Section 3 we consider second order Cauchy problems. Proposition 3.1 gives some properties of local C-cosine functions and their generators. It was shown in [WW] that a second order Cauchy problem is C-wellposed if and only if A generates a local C-cosine function. In terms of local integrated C-semigroups, we show in Theorem 3.3 that the second order problem is C-wellposed if and only if the reduced first order Cauchy problem is $\mathcal{C} := \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$-wellposed. So the example in [HH] can be modified to show that the generator of a local integrated C-semigroup can have empty C-resolvent. This is different from local integrated semigroups since it was proved in [AEK] that the generator of a local integrated semigroup always has nonempty resolvent.

In [AEK, Theorem 4.1] it is proved that if $C_{n+1}(\tau)$ is wellposed, then $C_{2n+1}(2\tau)$ is wellposed as well. That is, the solution can be extended if one is ready to give up regularity. Wang and Gao [WG] have generalized it to local regularized semigroups and local regularized cosine functions. For local integrated C-semigroups, we also have analogous extensions (Theorem 4.1).

Section 5 is devoted to the generation of local integrated C-semigroups. First we prove that if $C_{k+1}(\tau)$ is C-wellposed then A has an asymptotic C-resolvent. Then, by using the Arendt–Widder theorem on the Laplace transforms of vector-valued functions, we show that if A has an asymptotic C-resolvent, then $C_{k+2}(\tau)$ is C-wellposed (Theorem 5.2); when A is densely defined, we get in fact the C-wellposedness of $C_{k+1}(\tau)$ (Corollary 5.3). Our proof simplifies those for local C-semigroups and local C-cosine functions (see [TO], [HH], [Zo], [Ga]).

Throughout this paper, C is an injective operator on a Banach space X. For an operator A, we denote by $D(A)$, $R(A)$ its domain and range, respectively.

2. Local n-times integrated C-semigroups and the C-wellposedness of $C_{n+1}(\tau)$. First we give the definition of local n-times integrated C-semigroups. For details on n-times integrated C-semigroups defined on $[0,\infty)$, see [LS].

DEFINITION 2.1. Let $\tau \in (0,\infty]$ and $n \in \mathbb{N}$. A strongly continuous family $\{T(t) : 0 \le t < \tau\} \subset B(X)$ is called a *local n-times integrated C-semigroup* on X if it satisfies:

(i) $T(0) = 0$ and $T(t)C = CT(t)$ for $t \in [0,\tau)$.

(ii) $T(t)T(s)x = \frac{1}{(n-1)!}\Big(\int_0^{t+s} - \int_0^t - \int_0^s\Big)(s+t-r)^{n-1}T(r)Cx\,dr$

for $x \in X$ and $0 \le s,t,s+t < \tau$.

$T(\cdot)$ is said to be *nondegenerate* if $T(t)x = 0$ for all $t \in [0,\tau)$ implies $x = 0$.

The *generator*, A, of a nondegenerate local n-times integrated C-semigroup $T(\cdot)$ is defined by

$$x \in D(A) \text{ with } Ax = y \;\Leftrightarrow\; T(t)x - \frac{t^n}{n!}Cx = \int_0^t T(s)y\,ds,\ \forall t \in [0,\tau).$$

The *C-resolvent set* of A, $\varrho_C(A)$, is the set of all complex numbers λ such that $\lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)$.

If $C = I$, a local integrated C-semigroup is in fact a local integrated semigroup. We also call a local C-semigroup a *local 0-times integrated C-semigroup*.

DEFINITION 2.2. Let $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\tau > 0$. The Cauchy problem $C_{n+1}(\tau)$ is *C-wellposed* if for every $x \in X$ there exists a unique solution of $C_{n+1}(\tau)$.

Now we demonstrate the relations between local n-times integrated C-semigroups and the C-wellposedness of $C_{n+1}(\tau)$. To this end, we give a result analogous to [AEK, Proposition 2.3]. The proof is also similar, so it is omitted.

PROPOSITION 2.3. *Let $n \in \mathbb{N}_0$ and $0 < \tau \in \mathbb{R}$. Assume that $C_{n+1}(\tau)$ is C-wellposed. Then there exists a unique nondegenerate strongly continuous* *function* $S:[0,\tau)\to B(X)$ *such that for every* $x\in X$, $\int_0^t S(s)x\,ds\in D(A)$ *and*

$$A\int_0^t S(s)x\,ds = S(t)x-\frac{t^n}{n!}Cx,\qquad t\in[0,\tau). \tag{1}$$

PROPOSITION 2.4. *Let* $\{S(t): t\in[0,\tau)\}$ *be a nondegenerate strongly continuous family of bounded operators such that* $S(t)A\subseteq AS(t)$ *and* $CS(t) = S(t)C$. *Suppose that for every* $x\in X$, $\int_0^t S(s)x\,ds\in D(A)$ *and satisfies* (1). *Then*:

(i) $S(t)$ *is a local* n*-times integrated* C*-semigroup whose generator is an extension of* A.

(ii) $C_{k+1}(\tau)$ *is* C*-wellposed.*

Proof. (i) Fix $t\in(0,\tau)$. For $0<r<t$, since A commutes with $S(\cdot)$, we have

$$\begin{aligned}
&\frac{d}{dr}S(t-r)\int_0^r S(\sigma)x\,d\sigma\\
&\quad= -S(t-r)A\int_0^r S(\sigma)x\,d\sigma-\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(\sigma)Cx\,d\sigma+S(t-r)S(r)x\\
&\quad= -S(t-r)S(r)x+S(t-r)\frac{r^n}{n!}Cx-\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(\sigma)Cx\,d\sigma\\
&\qquad\ +S(t-r)S(r)x\\
&\quad= \frac{r^n}{n!}S(t-r)Cx-\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(\sigma)Cx\,d\sigma.
\end{aligned}$$

Integrating with respect to r from 0 to s, where $0<s<t$, gives

$$S(t-s)\int_0^s S(\sigma)x\,d\sigma=\int_0^s\frac{r^n}{n!}S(t-r)Cx\,dr-\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(\sigma)Cx\,d\sigma\,dr.$$

Thus,

$$\begin{aligned}
S(t-s)S(s)x&=S(t-s)A\int_0^s S(\sigma)x\,d\sigma+S(t-s)\frac{s^n}{n!}Cx\\
&=AS(t-s)\int_0^s S(\sigma)x\,d\sigma+S(t-s)\frac{s^n}{n!}Cx\\
&=A\int_0^s\frac{r^n}{n!}S(t-r)Cx\,dr-A\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}\int_0^r S(\sigma)Cx\,d\sigma\,dr
\end{aligned}$$

$$
\begin{aligned}
&+\frac{s^n}{n!}S(t-s)Cx\\
&\overset{\text{(a)}}{=} A\int_{t-s}^{t}\frac{(t-r)^n}{n!}S(r)Cx\,dr-\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}S(r)Cx\,dr\\
&\quad+\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}\cdot\frac{r^n}{n!}C^2x\,dr+\frac{s^n}{n!}S(t-s)Cx\\
&\overset{\text{(b)}}{=} -\frac{s^n}{n!}A\int_0^{t-s}S(r)Cx\,dr+A\int_{t-s}^{t}\frac{(t-\sigma)^{n-1}}{(n-1)!}\int_0^{\sigma}S(r)Cx\,dr\,d\sigma\\
&\quad-\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}S(r)Cx\,dr+\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}\cdot\frac{r^n}{n!}C^2x\,dr+\frac{s^n}{n!}S(t-s)Cx\\
&\overset{\text{(c)}}{=} \frac{s^n}{n!}\cdot\frac{(t-s)^n}{n!}C^2x+\Big(\int_{t-s}^{t}-\int_0^s\Big)\frac{(t-r)^{n-1}}{(n-1)!}S(r)Cx\,dr\\
&\quad-\int_{t-s}^{t}\frac{(t-r)^{n-1}}{(n-1)!}\cdot\frac{r^n}{n!}C^2x\,dr+\int_0^s\frac{(t-r)^{n-1}}{(n-1)!}\cdot\frac{r^n}{n!}C^2x\,dr\\
&=\frac{1}{(n-1)!}\Big(\int_0^t-\int_0^s-\int_0^{t-s}\Big)(t-r)^{n-1}S(r)Cx\,dr,
\end{aligned}
$$

where the identity (a) follows from (1) by our hypothesis, (b) holds by integration by parts, and (c) holds by applying (1) twice: to the integrands of $\int_{t-s}^{t}$ and $\int_0^{t-s}$.

Hence $\{S(t) : t \in [0,\tau)\}$ is a local n-times integrated C-semigroup. Obviously its generator is an extension of A.

(ii) We only need to show the solution of $C_{k+1}(\tau)$ is unique. Let $v(\cdot)$ be a solution of $C_{k+1}(\tau)$ with initial value x. For $r < t < \tau$, define $u(r) = S(t-r)v(r)$; then

$$
\begin{aligned}
\frac{d}{dr}S(t-r)v(r) &= -S(t-r)Av(r)-\frac{(t-r)^{n-1}}{(n-1)!}Cv(r)\\
&\quad+S(t-r)Av(r)-S(t-r)\frac{r^n}{n!}Cx.
\end{aligned}
$$

Integrating it from 0 to t, we have

$$
0=\int_0^t\frac{(t-r)^{n-1}}{(n-1)!}Cv(r)\,dr+\int_0^t\frac{r^n}{n!}S(t-r)Cx\,dr
$$

$$= -\int_0^t \frac{(t-r)^n}{n!} Cv'(r)\,dr + \int_0^t \frac{r^n}{n!} S(t-r)Cx\,dr$$
$$= \int_0^t \frac{(t-r)^n}{n!}[-Cv'(r) + S(r)Cx]\,dr.$$

The above equation holds for all $t \in [0,\tau)$, hence $Cv'(t) = S(t)Cx$, that is, $v(t) = \int_0^t S(s)x\,ds$ for all $t \in [0,\tau)$. ∎

THEOREM 2.5. *Suppose A is closed and $CA \subseteq AC$. Suppose that $C_{n+1}(\tau)$ is C-wellposed, and $S(\cdot)$ is given by Proposition* 2.3. *Then*:

(a) *$S(t)x = 0$ for all $t \in [0,\tau)$ implies $x = 0$.*
(b) *$S(t)C = CS(t)$ for all $t \in [0,\tau)$.*
(c) *For $x \in D(A)$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*
(d) *$S(t)S(s) = S(s)S(t)$ for all $0 \le s,t < \tau$.*
(e) *Suppose $A = C^{-1}AC$. Then $x \in D(A)$ and $Ax = y$ if and only if*

$$S(t)x = \int_0^t S(s)y\,ds + \frac{t^n}{n!}Cx, \qquad \forall t \in [0,\tau).$$

(f) *$S(t)$ is a local n-times integrated C-semigroup generated by an extension of A, $C^{-1}AC$.*

(g) *Suppose $\varrho_C(A) \neq \emptyset$. Then for all $\lambda \in \varrho_C(A)$,*

$$(\lambda - A)^{-1}CS(t) = S(t)(\lambda - A)^{-1}C, \qquad t \in [0,\tau).$$

Proof. (a) follows from the definition of C-wellposedness.

(b) holds since A commutes with C and the solution is unique.

(c) Let $x \in D(A)$. To see $S(t)x \in D(A)$ with $AS(t)x = S(t)Ax$, define

$$\widetilde{S}(t)x = \int_0^t S(t)Ax\,ds + \frac{t^n}{n!}Cx.$$

Then

$$A\int_0^t \widetilde{S}(s)x\,ds = A\int_0^t \left(\int_0^s S(r)Ax\,dr + \frac{s^n}{n!}Cx\right) ds$$
$$= \int_0^t \left(A\int_0^s S(r)Ax\,dr\right) ds + \frac{t^{n+1}}{(n+1)!}CAx$$
$$= \int_0^t S(s)Ax\,ds - \frac{t^{n+1}}{(n+1)!}CAx + \frac{t^{n+1}}{(n+1)!}CAx$$
$$= \widetilde{S}(t)x - \frac{t^n}{n!}Cx;$$

by the uniqueness of the solution, we have $\widetilde{S}(t)x = S(t)x$. So we also have

$$A\int_0^t S(s)x\,ds = \int_0^t S(s)Ax\,ds;$$

differentiating it with respect to t, and using the closedness of A, we obtain $S(t)x \in D(A)$ with $AS(t)x = S(t)Ax$.

(d) Choose $s \in [0, \tau)$ and $x \in X$. By (c) we have

$$A\int_0^t S(s)S(r)x\,dr = S(s)A\int_0^t S(r)x\,dr = S(s)S(t)x - \frac{t^n}{n!}S(s)Cx,$$

so $u(t) = \int_0^t S(s)S(r)x\,dr$ is a solution of $C_{n+1}(\tau)$ at $CS(s)x$ (since $CS(s)x = S(s)Cx$ by (b)). But Proposition 2.3 implies that so is $\int_0^t S(r)S(s)x\,dr$. Hence, by uniqueness, $\int_0^t S(s)S(r)x\,dr = \int_0^t S(r)S(s)x\,dr$ for all $t \in [0, \tau)$, which implies that $S(s)S(t)x = S(t)S(s)x$.

(e) *Necessity* follows from the definition of $S(t)$ and (c).

Sufficiency. Since

$$S(t)x = \int_0^t S(s)y\,ds + \frac{t^n}{n!}Cx \tag{2}$$

and

$$S(t)x = A\int_0^t S(s)x\,ds + \frac{t^n}{n!}Cx,$$

we have $A\int_0^t S(s)x\,ds = \int_0^t S(s)y\,ds$, which means that $S(t)x \in D(A)$ and $AS(t)x = S(t)y$ as A is closed; also, from (2) we know that $Cx \in D(A)$, and

$$ACx = \frac{n!}{t^n}\Big(AS(t)x - A\int_0^t S(s)y\,ds\Big) = Cy \in R(C),$$

thus $x \in D(A)$.

(f) It follows from (b), (c), and Propositions 2.3 and 2.4 that $S(t)$ is an n-times integrated C-semigroup generated by an extension, B, of A. From the proof of (e), we see that $B \subseteq C^{-1}AC$. Conversely, if $Cx \in D(A)$ and $ACx = Cy \in R(C)$, then

$$S(t)Cx = \int_0^t S(s)Cy\,ds + \frac{t^n}{n!}C^2x;$$

since C is injective and commutes with $S(t)$, it follows that $x \in D(B)$ and $Bx = y$.

(g) Let $\lambda \in \varrho_C(A)$ and $x \in X$. Then

$$A\int_0^t S(s)x\,ds = S(t)x - \frac{t^n}{n!}Cx,$$

so that

$$(\lambda - A)^{-1}CA\int_0^t S(s)x\,ds = (\lambda - A)^{-1}CS(t)x - \frac{t^n}{n!}(\lambda - A)^{-1}C^2x.$$

Since $(\lambda - A)^{-1}C$ commutes with A, we have

$$A\int_0^t (\lambda - A)^{-1}CS(s)x\,ds = (\lambda - A)^{-1}CS(t)x - \frac{t^n}{n!}(\lambda - A)^{-1}C^2x,$$

and thus (g) follows from the uniqueness of the solution. ∎

REMARKS 2.6. Recall that we assumed in Section 1 that C is injective.

(a) If $CA \subseteq AC$, then

$$x \in D(A),\ Ax = y \ \Leftrightarrow\ S(t)x = \int_0^t S(s)y\,ds + \frac{t^n}{n!}Cx \tag{3}$$

implies $A = C^{-1}AC$.

(b) If C commutes with all $S(t)$, then (3) also implies $CA \subseteq AC$.

(c) By Theorem 2.5(d), if $\{S(t) : t \in [0, \tau)\}$ gives the solution of $C_{n+1}(\tau)$, then $S(t)S(s) = S(s)S(t)$ for all $s, t \in [0, \tau)$. On the other hand, if $\{S(t) : t \in [0, \tau)\}$ is a local n-times integrated semigroup then $S(t)S(s) = S(s)S(t)$ for all $s, t \in [0, \tau)$ with $s + t < \tau$; we do not know whether this identity holds for all $s, t \in [0, \tau)$.

(d) If $C_{n+1}(\tau)$ is C-wellposed, then for every $x \in D(A^{n+1})$,

$$T(t)x := \int_0^t S(s)A^{k+1}x\,ds + \frac{t^k}{k!}A^kCx + \ldots + tACx + Cx$$

gives the solution of $C_0(\tau)$ at Cx, where $S(t)$ is given by Proposition 2.3.

(e) We will see in the next section that there exists a local integrated C-semigroup whose generator has empty C-resolvent.

3. Relations to second order Cauchy problems. Consider the second order Cauchy problem

$$(\mathrm{ACP}_2, \tau) \qquad \begin{cases} u''(t) = Au(t) & (-\tau < t < \tau), \\ u(0) = x, \quad u'(0) = y. \end{cases}$$

Let $x, y \in X$. A function $u(t)$ is called a *mild solution* of (ACP_2, τ) at (x, y) if

$$w(t) := \int_0^t (t-s)u(s)\,ds \in D(A)$$

and

$$\frac{d^2}{dt^2} w(t) = Aw(t) + x + ty, \qquad -\tau < t < \tau.$$

(ACP_2, τ) is called *C-wellposed* if it has a unique mild solution for every pair of $x, y \in R(C)$.

A strongly continuous family $\{C(t)\}_{t\in(-\tau,\tau)}$ of operators is called a *local C-cosine function* if $C(0) = C$ and

(4) $$C(t+s)C + C(t-s)C = 2C(s)C(t), \qquad \forall s, t,\ t+s, t-s \in (-\tau, \tau).$$

$C(t)$ is called *nondegenerate* if $C(t)x \equiv 0$ for all $t \in (-\tau, \tau)$ implies $x = 0$.

If $C(t)$ is nondegenerate, then the *generator*, A, is defined by

$$x \in D(A) \text{ and } Ax = y \ \Leftrightarrow\ C(t)x = \int_0^t (t-s)C(s)y\,ds + Cx,\ t \in (-\tau, \tau).$$

We collect the properties of local C-cosine functions in the following.

PROPOSITION 3.1. *Let $\{C(t)\}_{t\in(-\tau,\tau)}$ be a local C-cosine function generated by A. Then*:

(a) *$C(t)C = CC(t)$ for all $t \in (-\tau, \tau)$.*
(b) *$C(-t) = C(t)$ for all $t \in (-\tau, \tau)$.*
(c) *$C(t)C(s) = C(s)C(t)$ for all $t, s \in (-\tau, \tau)$.*
(d) *$C(t)A \subseteq AC(t)$ for all $t \in (-\tau, \tau)$.*
(e) *$C^{-1}AC = A$.*
(f) *$x \in D(A) \Leftrightarrow \frac{d^2}{dt^2}C(t)x|_{t=0}$ exists and is in $R(C)$ and $C''(0)x = ACx = CAx$ and $C'(0)x = 0$.*
(g) *$\int_0^t (t-s)C(s)x\,ds \in D(A)$ and $A\int_0^t (t-s)C(s)x\,ds = C(t)x - Cx$.*

Proof. (a) and (b) are obvious from the definition of a local C-cosine function.

(c) By (b), we can assume that $t, s \geq 0$.

If $t + s < \tau$, we have $C(t)C(s) = C(s)C(t)$ from (4).

If $t+s > \tau$ while $t/2+s < \tau$, then from $2C(t/2)C(t/2) = C(t)C+C^2$, we get $C(t)C = 2C(t/2)C(t/2) - C^2$; since C is injective, we only need to show $C(t/2)C(t/2)C(s) = C(s)C(t/2)C(t/2)$. But this holds since $t/2 + s < \tau$, so $C(t/2)$ commutes with $C(s)$.

Iterating this process proves (c) for all $t, s \in (-\tau, \tau)$.

(d) Let $x \in D(A)$. Then $C(t)x = \int_0^t (t-s)C(s)Ax\,ds + Cx$, which combined with (a) and (c) gives

$$C(t)C(r)x = \int_0^t (t-s)C(s)C(r)Ax\,ds + CC(r)x$$

and hence $C(r)x \in D(A)$ with $AC(r)x = C(r)Ax$.

(e) can be shown similarly to Theorem 2.5(e) and Remark 2.6(a).

(f) We only need to prove the sufficiency. Suppose $C''(0)x = Cy$ and $C'(0)x = 0$, $t \in (-\tau,\tau)$, and h is small enough. Then

$$\frac{1}{4h^2}(C(t+2h) + C(t-2h) - 2C(t))Cx = \frac{1}{2h^2}C(t)(C(2h) - C)x.$$

Hence $C(t)Cx$ is twice differentiable and

$$C''(t)Cx = C(t)C''(0)x = C(t)Cy.$$

Integrating it with respect to t twice, we have

$$C(t)Cx = \int_0^t (t-s)C(s)Cy\,ds + C^2x,$$

which implies that $x \in D(A)$ since C is injective.

The proof of (g) is contained in that of [WW, Proposition 2.4]. ∎

We need the following relations between second order Cauchy problems and cosine functions.

LEMMA 3.2 ([WW]). *Suppose A is closed, $C \in B(X)$ is injective and $C^{-1}AC = A$. Then the following statements are equivalent*:

(a) (ACP_2,τ) *is C-wellposed.*

(b) *There exists a family $\{C(t)\}_{t\in(-\tau,\tau)}$ satisfying*:

(i) $\int_0^t (t-s)C(s)x\,ds \in D(A)$ *and* $t \mapsto A\int_0^t (t-s)C(s)x\,ds$ *is continuous in* $(-\tau,\tau)$.

(ii) $A\int_0^t (t-s)C(s)x\,ds = C(t)x - Cx$ *for all* $t \in (-\tau,\tau)$.

(iii) $C(t)A \subseteq AC(t)$.

(c) *A generates a local C-cosine function $\{C(t)\}_{t\in(-\tau,\tau)}$.*

Now we are in a position to clarify the relations between the second Cauchy problem (ACP_2,τ) and the twice integrated Cauchy problem

$$\widetilde{C}_2(\tau) \qquad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + t\begin{pmatrix} x \\ y \end{pmatrix}, \\ \mathcal{U}(0) = 0, \end{cases}$$

where $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}$ on $E = X \times X$.

THEOREM 3.3. (ACP_2,τ) *is C-wellposed if and only if* $\widetilde{C}_2(\tau)$ *is* $\mathcal{C}$-

wellposed, where $\mathcal{C} := \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$.

Proof. Suppose (ACP_2, τ) is C-wellposed and $C(t)$ is given by Lemma 3.2. For $x, y \in X$, let

$$u_1(t) = \int_0^t (t-s)C(s)x\,ds + \int_0^t \frac{(t-s)^2}{2} C(s)y\,ds,$$

$$u_2(t) = \int_0^t (C(s)-C)x\,ds + \int_0^t (t-s)C(s)y\,ds.$$

Then $\mathcal{U}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ gives the solution of $\widetilde{C}_2(\tau)$ at $\begin{pmatrix} Cx \\ Cy \end{pmatrix}$.

Suppose $\mathcal{U}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ is the solution of

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \quad \mathcal{U}(0) = 0.$$

Then $u_1'(t) = u_2(t)$, $u_2'(t) = Au_1(t)$ with $u_1(0) = u_2(0) = 0$, which means that $u_1''(t) = Au_1(t)$ and $u_1(0) = u_1'(0) = 0$. Hence $u_1(t)$ gives a solution of (ACP_2, τ) at $x = 0$. Since the solution is unique, we have $u_1(t) = u_2(t) = 0$.

Conversely, let $\widetilde{C}_2(\tau)$ be $\mathcal{C}$-wellposed, and suppose $\mathcal{U}(t) = (u_1(t)\ u_2(t))^\top$ is the solution of $\mathcal{U}'(t) = \mathcal{A}\,\mathcal{U}(t) + t(0\ Cx)^\top$, $\mathcal{U}(0) = 0$. Then $u_1''(t) = Au_1(t) + Cx$ gives a mild solution of (ACP_2, τ). The uniqueness of the solution can be proved as above. ∎

From this theorem we can derive a local twice integrated C-semigroup from every local C-cosine function. So the examples in [HH] can serve as examples of local twice integrated C-semigroups. Therefore, we have examples of local integrated C-semigroups whose generator has empty C-resolvent. This is different from the generators of local integrated semigroups as it was shown in [AEK] that every such generator has nonempty resolvent.

4. Extension of solutions. In this section we show that a solution given on a finite interval can always be extended if a loss of regularity is accepted.

THEOREM 4.1. *Let* $\tau > 0$ *and* $k \in \mathbb{N}$. *Assume that* $C_{k+1}(\tau)$ *is* C-*wellposed. Then* $C_{2k+1}(2\tau)$ *is* C^2-*wellposed. Thus, for all* $\tau' > 0$, *there exist* $k', l \in \mathbb{N}$ *such that* $C_{k'}(\tau')$ *is* C^l-*wellposed.*

Proof. Let $\tau_0 < \tau$. All that needs to be shown is that $C_{2k+1}(2\tau_0)$ has a unique solution. Define for $t \in [0, \tau_0)$,

$$T_{2k-m}(t) = \int_0^t \frac{(t-s)^{k-m-1}}{(k-m-1)!} S_k(s)C\,ds, \quad 0 \le m \le k,$$

and for $\tau_0 \le t \le 2\tau_0$,

$$T_{2k}(t) = S_k(\tau_0)S_k(t-\tau_0) + \sum_{m=1}^{k-1}(\tau_0^m T_{2k-m}(t-\tau_0) + (t-\tau_0)^m T_{2k-m}(\tau_0)).$$

Then $T_{2k} : [0, 2\tau_0] \to B(X)$ is strongly continuous. Moreover, the function $v(t) = \int_0^t T_{2k}(s)x\,ds$ is a solution of $C_{2k+1}(2\tau_0)$ at C^2x. The verification is analogous to that of [AEK, Theorem 4.1], so it is omitted.

We must show that the solution of $C_{2k+1}(2\tau)$ is unique. Although we can deduce it from Proposition 2.4 and Theorem 2.5, it can also be derived directly from the C-wellposedness of $C_{k+1}(\tau)$. Let $v(t)$ be a solution of $C_{2k+1}(2\tau)$ with initial value $x = 0$, that is, $v'(t) = Av(t), t \in [0, 2\tau)$ and $v(0) = 0$. Then the restriction of $v(t)$ to $[0, \tau)$ is also a solution of $C_{k+1}(\tau)$ with initial value $x = 0$; by the wellposedness of $C_{k+1}(\tau)$, we have $v(t) \equiv 0$ on $[0, \tau)$. Since $v(\cdot)$ is continuous, $v(\tau) = 0$. Let $w(t) = v(t + \tau)$, $t \in [0, \tau)$. Then w is also a solution of $C_{k+1}(\tau)$ at $x = 0$, and the same reasoning leads to $w(t) \equiv 0$ on $[0, \tau)$, that is, $v(t) \equiv 0$ on $[\tau, 2\tau)$. In sum, $v(t) \equiv 0$ on $[0, 2\tau)$. ■

5. Generation of local integrated C-semigroups. Suppose the Cauchy problem $C_{k+1}(\tau)$ is C-wellposed, and the strongly continuous family $S(t)$ is given by Proposition 2.3. Let $\gamma \in [0, \tau)$, and define the local Laplace transform of S by

$$L_\gamma(\lambda) = \int_0^\gamma e^{-\lambda s} S(s)\,ds, \qquad \lambda \in \mathbb{R}.$$

Note that $L_\gamma(\lambda)$ can be viewed as the Laplace transform of

$$\widetilde{S}(s) = \begin{cases} S(s), & s \le \gamma, \\ 0, & s > \gamma. \end{cases}$$

For $\lambda \in C$ and $t \ge 0$, let

$$g_\gamma(\lambda) = \int_0^\gamma e^{\lambda(\gamma-s)} \frac{s^{k-1}}{(k-1)!}\,ds = \frac{e^{\lambda\gamma}}{\lambda^k} + q_\gamma(\lambda)$$

where

$$q_\gamma(\lambda) = -\frac{1}{\lambda^k} - \frac{\gamma}{\lambda^{k-1}} - \frac{\gamma^2}{2!\lambda^{k-2}} - \ldots - \frac{\gamma^{k-1}}{(k-1)!\lambda}.$$

By the above definition,

$$g_\gamma(0) = \int_0^\gamma \frac{s^{k-1}}{(k-1)!}\,ds = \frac{\gamma^k}{k!}.$$

PROPOSITION 5.1. *Let $\gamma \in [0, \tau)$ and $\lambda \ge 0$. Then $L_\gamma(\lambda)$ satisfies:*

(a) *For every $x \in X$, $L_\gamma(\lambda)x$ is infinitely differentiable with respect to λ, and there exists $M_\gamma > 0$ such that*

$$\left\| \frac{\lambda^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} L_\gamma(\lambda) \right\| \le M_\gamma, \qquad \forall \lambda \ge 0,\ n \in \mathbb{N}.$$

(b) *For every $x \in X$, $L_\gamma(\lambda)x \in D(A)$ and*

$$(\lambda - A)L_\gamma(\lambda)x = e^{-\gamma\lambda}(g_\gamma(\lambda)Cx - S(\gamma)x).$$

(c) $L_\gamma(\lambda)L_\gamma(\mu) = L_\gamma(\mu)L_\gamma(\lambda)$, $L_\gamma(\lambda)C = CL_\gamma(\lambda)$.

(d) *For every $x \in D(A)$, $AL_\gamma(\lambda)x = L_\gamma(\lambda)Ax$.*

Proof. (a) Obviously $L_\gamma(\lambda)$ is infinitely differentiable with

$$\frac{d^{n-1}}{d\lambda^{n-1}} L_\gamma(\lambda)x = (-1)^{n-1} \int_0^\gamma e^{-\lambda s} s^{n-1} S(s)x\, ds,$$

hence

$$\begin{aligned}\left\| \frac{\lambda^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} L_\gamma(\lambda) \right\| &\le \sup_{0 \le s \le \gamma} \|S(s)\| \cdot \frac{\lambda^n}{(n-1)!} \int_0^\infty e^{-\lambda s} s^{n-1}\, ds \\ &= \sup_{0 \le s \le \gamma} \|S(s)\| =: M_\gamma.\end{aligned}$$

(b) Since

$$\begin{aligned} L_\gamma(\lambda) &= \int_0^\gamma e^{-\lambda s} \frac{d}{ds} \int_0^s S(r)\, dr\, ds \\ &= e^{-\lambda\gamma} \int_0^\gamma S(r)\, dr + \lambda \int_0^\gamma e^{-\lambda s} \int_0^s S(r)\, dr\, ds, \end{aligned}$$

by the closedness of A we have $L_\gamma(\lambda)x \in D(A)$ and

$$\begin{aligned} (\lambda - A)L_\gamma(\lambda)x &= \lambda L_\gamma(\lambda)x - e^{-\lambda\gamma}\left[S(\gamma)x - \frac{\gamma^k}{k!}Cx\right] \\ &\quad - \lambda \int_0^\gamma e^{-\lambda s} \left[S(s)x - \frac{s^k}{k!}Cx\right] ds \\ &= -e^{-\lambda\gamma} S(\gamma)x + e^{-\lambda\gamma} \frac{\gamma^k}{k!} Cx + \lambda \int_0^\gamma e^{-\lambda s} \frac{s^k}{k!} Cx\, ds \\ &= -e^{-\lambda\gamma} S(\gamma)x + \int_0^\gamma e^{-\lambda s} \frac{s^{k-1}}{(k-1)!} Cx\, ds \\ &= -e^{-\lambda\gamma} S(\gamma)x + e^{-\lambda\gamma} g_\gamma(\lambda)Cx. \end{aligned}$$

(c) holds since $S(t)$ commutes with $S(s)$ for all $s, t \in [0, \tau)$ by Theorem 2.5.

(d) For $x \in D(A)$, by Theorem 2.5, we have $S(t)x \in D(A)$ with $AS(t)x = S(t)Ax$, so

$$AL_\gamma(\lambda)x = A\int_0^\gamma e^{-\lambda s}S(s)x\,ds = \int_0^\gamma e^{-\lambda s}S(s)Ax\,ds = L_\gamma(\lambda)Ax,$$

which is (d). ∎

We call $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \ge 0\} \subset B(X)$ an *asymptotic C-resolvent* for A if there exists a strongly continuous family $\{V(t) : t \in [0,\tau)\} \subset B(X)$ such that (a), (c) and (d) hold and (b) holds with $S(\gamma)$ replaced by $V(\gamma)$. Now we investigate the converse of Proposition 5.1.

THEOREM 5.2. *Let A be a closed operator. Suppose that A has an asymptotic C-resolvent $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \ge 0\}$. Then the Cauchy problem $C_{k+2}(\tau)$ is C-wellposed.*

Proof. By (a) and the Arendt–Widder theorem [Ar], there exists a Lipschitz continuous operator-valued function $S_\gamma(t)$ such that

$$L_\gamma(\lambda) = \lambda\int_0^\infty e^{-\lambda t}S_\gamma(t)\,dt, \qquad \gamma \in (0,\tau),\ \lambda \ge 0, \tag{5}$$

and $S_\gamma(0) = 0$, $\|S_\gamma(t+h) - S_\gamma(t)\| \le M_\gamma h$.

For $x \in X$, by (b), $L_\gamma(\lambda)x \in D(A)$,

$$AL_\gamma(\lambda)x = \lambda A\int_0^\infty e^{-\lambda t}S_\gamma(t)x\,dt = \lambda^2 A\int_0^\infty e^{-\lambda t}\Big(\int_0^t S_\gamma(s)x\,ds\Big)\,dt;$$

on the other hand, also by (b),

$$\begin{aligned} AL_\gamma(\lambda)x &= -e^{-\gamma\lambda}(g_\gamma(\lambda)Cx - V(\gamma)x) + \lambda L_\gamma(\lambda)x \\ &= \lambda^2\int_0^\infty e^{-\lambda t}S_\gamma(t)x\,dt - \int_0^\gamma e^{-\lambda s}\frac{s^{k-1}}{(k-1)!}Cx\,ds \\ &\quad +\lambda\int_\gamma^\infty e^{-\lambda s}V(\gamma)x\,ds \\ &= \lambda^2\int_0^\infty e^{-\lambda t}S_\gamma(t)x\,dt - \lambda^2\int_0^\infty e^{-\lambda t}f(t)Cx\,dt \\ &\quad +\lambda^2\int_0^\infty e^{-\lambda t}h(t)V(\gamma)x\,dt, \end{aligned}$$

where $f(t)$ is the twofold integral of

$$f_1(t) = \begin{cases} t^{k-1}/(k-1)!, & t < \gamma, \\ 0, & t \ge \gamma, \end{cases}$$

and $h(t)$ is the integral of

$$h_1(t) = \begin{cases} 0, & t < \gamma, \\ 1, & t \geq \gamma. \end{cases}$$

Combining the two identities, we have

$$A\int_0^\infty e^{-\lambda t}\int_0^t S_\gamma(s)x\,ds\,dt = \int_0^\infty e^{-\lambda t}(S_\gamma(t)x - f(t)Cx + h(t)V(\gamma)x)\,dt.$$

By [XL, Chap. 1, Theorem 1.10], $\int_0^t S_\gamma(s)x\,ds \in D(A)$ and

$$A\int_0^t S_\gamma(s)x\,ds = S_\gamma(t)x - f(t)Cx + h(t)V(\gamma)x;$$

in particular, since $f(t) = t^{k+1}/(k+1)!$ and $h(t) = 0$ on $[0,\gamma)$, for $t \in [0,\gamma)$ we have

$$A\int_0^t S_\gamma(s)x\,ds = S_\gamma(t)x - \frac{t^{k+1}}{(k+1)!}Cx, \tag{6}$$

which gives the solution of $C_{k+2}(\gamma)$. Now let $x \in D(A)$. Since $L_\gamma(\lambda)$ commutes with A by the assumption (d), we have $S_\gamma(t)x \in D(A)$ with $AS_\gamma(t)x = S_\gamma(t)Ax$ by (5) and the uniqueness of the Laplace transform. So (3) implies

$$S'_\gamma(t)x = AS_\gamma(t)x + \frac{t^k}{k!}Cx, \qquad \forall x \in D(A); \tag{7}$$

also, by (d), $S_\gamma(t)C = CS_\gamma(t)$.

We define $S(t)$ on $[0,\tau)$ by

$$S(t)x = S_\gamma(t)x \quad \text{ for } t \in [0,\gamma),\ \gamma \in [0,\tau) \text{ and } x \in X.$$

Then $S(t)x$ is well defined and $\{S(t)x : 0 \leq t < \tau\}$ gives a solution of $C_{k+2}(\tau)$. Indeed, by (6) and (7), for $\gamma_1,\gamma_2 \in [0,\tau)$,

$$\begin{aligned}\frac{d}{dr}&S_{\gamma_2}(t-r)\int_0^r S_{\gamma_1}(s)x\,ds \\ &= -S_{\gamma_2}(t-r)A\int_0^r S_{\gamma_1}(s)x\,ds - \frac{(t-r)^k}{k!}C\int_0^r S_{\gamma_1}(s)x\,ds \\ &\quad + S_{\gamma_2}(t-r)A\int_0^r S_{\gamma_1}(s)x\,ds - S_{\gamma_2}(t-r)\frac{r^{k+1}}{(k+1)!}Cx\end{aligned}$$

for $x \in X$ and $0 \leq r \leq t < \min(\gamma_1,\gamma_2)$. Integrating both sides with respect to r from 0 to t, we get

$$0 = \int_0^t \frac{(t-r)^{k+1}}{(k+1)!}(-CS_{\gamma_1}(r)x + S_{\gamma_2}(r)Cx)\,dr$$

for all t, which implies $CS_{\gamma_1}(r)x = S_{\gamma_2}(r)Cx$. Since C is injective, we have $S_{\gamma_1}(t)x = S_{\gamma_2}(t)x$ for $t < \min(\gamma_1, \gamma_2)$. The uniqueness of the solution of $C_{k+2}(\tau)$ can be proved similarly. ■

COROLLARY 5.3. *Suppose A is a densely defined closed operator. Then the Cauchy problem $C_{k+1}(\tau)$ is C-wellposed if and only if A has an asymptotic C-resolvent $\{L_\gamma(\lambda) : \gamma \in [0,\tau),\ \lambda \geq 0\}$.*

Proof. By (5), the Lipschitz continuity of $S_\gamma(t)$ and the denseness of $D(A)$, $S'_\gamma(t)$ can be extended to a bounded linear operator, $T(t)$, on X, such that $T(t)x$ gives the unique solution of $C_{k+1}(\tau)$. ■

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