STUDIA MATHEMATICA 207 (2) (2011)

Growth of (frequently) hypercyclic functions for differential operators

by

HANS-PETER BEISE and JÜRGEN MÜLLER (Trier)

Abstract. We investigate the conjugate indicator diagram or, equivalently, the indicator function of (frequently) hypercyclic functions of exponential type for differential operators. This gives insights into growth conditions for these functions on particular rays or sectors. Our research extends known results in several respects.

1. Introduction. A continuous operator $T : X \to X$, with X a topological vector space, is called *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. Such a vector x is said to be a *hypercyclic vector*. By $\mathcal{HC}(T, X)$, we denote the set of all hypercyclic vectors for T (on X). The operator is called *frequently hypercyclic* if there exists some $x \in X$ such that for every non-empty open set $U \subset X$ the set $\{n : T^n x \in U\}$ has positive lower density. The vector x is called a *frequently hypercyclic vector* in this case and the set of all these vectors will be denoted by $\mathcal{FHC}(T, X)$. We recall that the *lower density* of a discrete set $\Lambda \subset \mathbb{C}$ is defined by

$$\liminf_{r\to\infty} \frac{\#\{\lambda \in \Lambda : |\lambda| \le r\}}{r} =: \underline{\mathrm{dens}}(\Lambda).$$

We are only concerned with spaces consisting of holomorphic functions and therefore the (frequently) hypercyclic vectors are called (frequently) hypercyclic functions in this work.

In [10], G. Godefroy and J. H. Shapiro show that for every non-constant entire function $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ of exponential type, the induced differential operator

$$\varphi(D) : H(\mathbb{C}) \to H(\mathbb{C}), \qquad f \mapsto \sum_{n=0}^{\infty} c_n f^{(n)}, \tag{1.1}$$

2010 *Mathematics Subject Classification*: 30K99, 30D15, 47B38.
Key words and phrases: frequently hypercyclic operators, growth conditions, functions of exponential type, integral transforms.

DOI: 10.4064/sm207-2-1

where $H(\mathbb{C})$ is endowed with the usual topology of locally uniform convergence, is hypercyclic. This result also applies to the case of frequent hypercyclicity as is shown in [7]. Actually, in both articles [10] and [7], the outlined results are given for the case of $H(\mathbb{C}^N)$. The possible rate of growth of the corresponding (frequently) hypercyclic functions has been widely investigated (cf. [4], [5], [7], [8], [9], [11]). It turns out that the level set

$$C_\varphi := \{z : |\varphi(z)| = 1\} \tag{1.2}$$

plays a crucial role in this context. More precisely, under certain additional assumptions, for $\tau_\varphi := \operatorname{dist}(0, C_\varphi)$ there are functions of exponential type τ_φ that belong to $\mathcal{HC}(\varphi(D), H(\mathbb{C}))$, while no function of exponential type less than τ_φ belongs to $\mathcal{HC}(\varphi(D), H(\mathbb{C}))$ (cf. [4]). Moreover, for every $\varepsilon > 0$ there are functions in $\mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ that are of exponential type at most $\tau_\varphi + \varepsilon$ (cf. [7]).

In the following, we abbreviate the exponential function $z \mapsto e^{\alpha z}$ by e_α, for α some complex number. Using the Taylor series of e_α, it is easily seen that the translation operator $f \mapsto f(\cdot + \alpha)$ equals the differential operator $e_\alpha(D)$.

For the translation operator $e_1(D)$ and the ordinary differentiation operator D, more precise growth conditions than for arbitrary φ are known; see [9], [8], [11] and [5]. However, all investigations in this direction have in common that the rate of growth is measured with respect to the *maximum modulus* $M_f(r) := \max_{|z|=r} |f(z)|$ or with respect to the L^p-averages $M_{f,p}(r) := ((2\pi)^{-1} \int_0^{2\pi} |f(re^{it})|^p \, dt)^{1/p}$, where $p \in [1, \infty)$. We extend some of these results by considering growth conditions with respect to rays emanating from the origin.

For the sake of completeness, we recall that an entire function f is said to be *of exponential type* τ if

$$\limsup_{r\to\infty} \frac{\log M_f(r)}{r} =: \tau(f) = \tau,$$

where we set $\log(0) := -\infty$, and f is said to be of exponential type when the above lim sup is not equal to $+\infty$. The *indicator function* of an entire function of exponential type is defined by

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r}, \qquad \theta \in [-\pi, \pi].$$

Let K be a compact convex subset of $\mathbb{C}$. Then

$$H_K(z) := \sup\{\operatorname{Re}(zu) : u \in K\}, \qquad z \in \mathbb{C},$$

is the *support function* of K. Some elementary properties of H_K are collected in the next section. It is known that h_f is determined by the support function

$H_{K(f)}$ of a compact convex set $K(f) \subset \mathbb{C}$, to be more specific, for $z = re^{i\theta}$,

$$rh_f(\theta) = H_{K(f)}(z)$$

(cf. [3]). The set $K(f)$ is called the *conjugate indicator diagram* of f. Note that for $f \equiv 0$, we have $K(f) = \emptyset$. A more direct approach to the conjugate indicator diagram via the so called Borel transform is given at the beginning of the next section.

In this paper we give necessary and sufficient conditions concerning the location and the size of the conjugate indicator diagram of (frequently) hypercyclic functions for differential operators $\varphi(D)$. According to the above relations, this yields information about the growth on particular rays or sectors in terms of the indicator function. Since

$$\max_{\theta \in [-\pi,\pi]} h_f(\theta) = \max_{u \in K(f)} |u| = \tau(f),$$

this also includes information about the possible exponential type. In particular, f is of exponential type zero if and only if $K(f) = \{0\}$.

For $\alpha = \tau e^{i\psi} \in \mathbb{C}$ the indicator function of e_α is given by

$$h_{e_\alpha}(\theta) = \tau \cos(\theta + \psi)$$

and the conjugate indicator diagram is the singleton $\{\alpha\}$. From [6, Theorem 5.4.12] it follows that for an entire function f of exponential type we have $K(f) = \{\alpha\}$ if and only if there is some entire function f_0 of exponential type zero with $f = f_0 e_\alpha$. In that sense, functions which have singleton conjugate indicator diagram are close to the corresponding exponential function. In particular, the indicator functions of f and e_α coincide, which implies that f decreases exponentially in each closed subsector of the half-plane $|\arg(z) + \psi| > \pi/2$ if $\alpha \neq 0$.

Our first result shows that the conjugate indicator diagrams of hypercyclic functions for differential operators are not restricted with respect to their size and shape.

Theorem 1.1. *Let φ be a non-constant entire function of exponential type. Then for every compact convex set $K \subset \mathbb{C}$ that intersects C_φ there exists an $f \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ that is of exponential type with $K(f) = K$.*

Theorem 1.1 implies that for every $\alpha \in C_\varphi$ there exists some f_0 of exponential type zero such that $f = f_0 e_\alpha \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$. Consequently, if C_φ contains the origin, there is a function $f \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ of exponential type zero. For the translation operator $f \mapsto f(\cdot + 1)$ (that is, for $e_1(D)$) a much stronger result is due to S. M. Duyos-Ruiz. She proved that functions $f \in \mathcal{HC}(e_1(D), H(\mathbb{C}))$ can have arbitrary slow transcendental rate of growth, that is, for every $q : [0,\infty) \to [1,\infty)$ such that $q(r) \to \infty$ as r tends to infinity, there is $f \in \mathcal{HC}(e_1(D), H(\mathbb{C}))$ such that $M_f(r) = O(r^{q(r)})$

(cf. [9]). In [8], this result is extended to Hilbert spaces consisting of entire functions of slow growth.

In Section 3 we will introduce a transform that quasi-conjugates differential operators and which enables us to extend the result of S. M. Duyos-Ruiz to the whole class of differential operators in the following sense.

THEOREM 1.2. *Let φ be a non-constant entire function of exponential type and let $\alpha \in C_\varphi$ with $\varphi'(\alpha) \neq 0$. Then for every $q : [0,\infty) \to [1,\infty)$ such that $q(r) \to \infty$ as $r \to \infty$, there is an entire function f_0 with $M_{f_0}(r) = O(r^{q(r)})$ and $f_0 e_\alpha \in \mathcal{HC}(\varphi(D), H(\mathbb{C}))$.*

The above results fail to hold in the case of frequent hypercyclicity. Here, some expansion of the conjugate indicator diagram is required.

THEOREM 1.3. *Let φ be a non-constant entire function of exponential type.*

(1) *If $K \subset \mathbb{C}$ is a compact convex set such that the intersection of K and C_φ contains a continuum, then there is a function $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ of exponential type with $K(f) \subset K$.*
(2) *There is no function $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$ of exponential type and such that $K(f)$ is a singleton.*

In particular, the second part of the above result states that, in contrast to the case of hypercyclicity, a function f of exponential type zero is never frequently hypercyclic for any differential operator $\varphi(D)$ (on $H(\mathbb{C})$). For the case of the translation operator $e_1(D)$ this also follows from the results of [5].

EXAMPLE 1.4. We consider the differentiation operator D.

(1) Let $q : [0,\infty) \to [1,\infty)$ be such that $q(r) \to \infty$ as $r \to \infty$. Then, by Theorem 1.2, there exists a function f_0 such that $M_{f_0}(r) = O(e^{q(r)})$ and $f := f_0 e_1 \in \mathcal{HC}(D, H(\mathbb{C}))$. In particular, this implies the existence of a hypercyclic function for the differentiation operator that is of exponential type 1 and such that $f(z)$ tends to zero exponentially in each closed subsector of the half-plane $\{z : |\arg(z)| > \pi/2\}$.
(2) Let K be the convex hull of $\{e^{i\beta} : |\beta| \leq \alpha\}$ for some $\pi/2 > \alpha > 0$. Then, by Theorem 1.3(1), there exists $f \in \mathcal{FHC}(D, H(\mathbb{C}))$ that is of exponential type with $K(f) \subset K$. This implies the existence of a frequently hypercyclic function for the differentiation operator that is of exponential type 1 and such that $f(z)$ tends to zero exponentially in each closed subsector of $\{z : |\arg(z)| > \pi/2 + \alpha\}$.

The proofs of Theorem 1.1, 1.2 and 1.3 will be given in the following three sections.

2. Hypercyclicity of differential operators. We first introduce some convenient terminology. Let $\Omega \subset \mathbb{C}$ be a domain and K a compact subset of Ω. A cycle Γ in $\Omega \setminus K$ is called a *Cauchy cycle for K in Ω* if $\operatorname{ind}_\Gamma(u) = 1$ for every $u \in K$ and $\operatorname{ind}_\Gamma(w) = 0$ for every $w \in \mathbb{C} \setminus \Omega$. The existence of such a cycle is always guaranteed, and the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

is valid for every $z \in K$ and $f \in H(\Omega)$ (see [17, Theorem 13.5, Theorem 10.35]). By $|\Gamma|$ we denote the trace of Γ and $\operatorname{len}(\Gamma) := \int_a^b |\Gamma'(t)|\, dt$ is the length of Γ. In the following, the complement of K with respect to the extended plane is always a simply connected domain. In this case, Γ may be chosen to be a simple closed path.

For a given compact convex set $K \subset \mathbb{C}$, we denote by $\operatorname{Exp}(K)$ the space of all entire functions f of exponential type that satisfy $K(f) \subset K$. This space naturally appears in the context of analytic functionals (cf. [14], [15], [3]). In what follows, differential operators are mainly considered on $\operatorname{Exp}(K)$, which turns out to be very convenient.

For a function f of exponential type,

$$\mathcal{B}f(z) := \sum_{n=0}^{\infty} f^{(n)}(0)/z^{n+1}$$

is called the *Borel transform* of f. The Borel transform is a holomorphic function on some neighbourhood of infinity that vanishes at infinity. It is known that the conjugate indicator diagram $K(f)$ is the smallest compact convex set such that $\mathcal{B}f$ admits an analytic continuation to $\mathbb{C} \setminus K(f)$, and that the inverse of the Borel transform is given by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{\xi z}\, d\xi$$

where Γ is a Cauchy cycle for $K(f)$ in $\mathbb{C}$ (cf. [6], [3]). This integral formula is known as the *Pólya representation.*

Finally, we make use of the following notation: $\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$, $\mathbb{D} := \{z : |z| < 1\}$ and $\mathbb{T} := \{z : |z| = 1\}$. If $A \subset \mathbb{C}$, then $A^{-1} := \{z : 1/z \in A\}$, where as usual $1/0 := \infty$, $\overline{A}$ is the closure of A, A° is the interior of A and $\operatorname{conv}(A)$ is the convex hull of A. For an open set $\Omega \subset \mathbb{C}_\infty$ the space of functions holomorphic on Ω and vanishing at ∞ (if $\infty \in \Omega$) endowed with the topology of uniform convergence on compact subsets is denoted by $H(\Omega)$. Recall that a function f is said to be *holomorphic at infinity* if $f(1/z)$ is holomorphic at the origin.

For the proof of the next proposition, we refer to the first chapter of [15].

Proposition 2.1. *Let $K \subset \mathbb{C}$ be a compact convex set.*

(1) *For every $n \in \mathbb{N}$,*

$$\|f\|_{K,n} := \sup_{z\in\mathbb{C}} |f(z)| e^{-H_K(z)-n^{-1}|z|}$$

defines a norm $\|\cdot\|_{K,n}$ on $\mathrm{Exp}(K)$, and an entire function f belongs to $\mathrm{Exp}(K)$ if and only if $\|f\|_{K,n} < \infty$ for all n. Moreover, $\mathrm{Exp}(K)$ endowed with the topology induced by the sequence $\{\|\cdot\|_{K,n} : n \in \mathbb{N}\}$ is a Fréchet space.

(2) *The Borel transform*

$$\mathcal{B} = \mathcal{B}_K : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K), \qquad f \mapsto \mathcal{B}f|_{\mathbb{C}_\infty \setminus K},$$

is an isomorphism.

In order to facilitate the calculations concerning $\mathrm{Exp}(K)$, we provide some elementary properties of the support function H_K. Let K_1, K_2 and K be nonempty compact convex subsets of the complex plane. Then

(1) $H_{K_1+K_2} = H_{K_1} + H_{K_2}$,
(2) $H_{\lambda K} = \lambda H_K$ for $\lambda \geq 0$,
(3) $H_{K_1} \leq H_{K_2}$ if and only if $K_1 \subset K_2$

(see, e.g. [3, Proposition 1.3.14]). From $H_{\overline{\mathbb{D}}}(z) = |z|$ it follows in particular that for $\lambda \geq 0$ we have

$$H_{K+\lambda\overline{\mathbb{D}}}(z) = H_K(z) + \lambda|z|.$$

Moreover, the following property of $\mathrm{Exp}(K)$ will be useful.

Lemma 2.2. *Let $K \subset \mathbb{C}$ be a compact convex set and (K_n) a sequence of compact convex supersets of K such that $K_n^\circ \supset K_{n+1}$ and $\bigcap_{n\in\mathbb{N}} K_n = K$. Then $\mathrm{Exp}(K) = \bigcap_{n\in\mathbb{N}} \mathrm{Exp}(K_n)$ set-theoretically and topologically, where the right hand side is endowed with the projective limit topology.*

Proof. The set-theoretic equality is clear. That the spaces also coincide in the topological sense is an immediate consequence of the observation that for a given $l \in \mathbb{N}$, we have $\|\cdot\|_{K_n,j} \leq \|\cdot\|_{K,l}$ for a suitable choice of $n, j \in \mathbb{N}$. ∎

By differentiation of parameter integrals, the Pólya representation yields

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\xi^n e^{\xi z}\, d\xi.$$

Inspired by this formula, we introduce a class of operators on $\mathrm{Exp}(K)$ by replacing ξ^n in the above integral by a function holomorphic on some neighbourhood of K. We define $H(K)$ to be the space of germs of holomorphic functions on K, where $K \subset \mathbb{C}$ is some compact set. To simplify notation, an element of $H(K)$ will always be identified with some of its representatives φ which is defined on an open neighbourhood Ω_φ of K. In the case where

K is convex we always assume Ω_φ to be simply connected (actually we may even suppose Ω_φ to be convex).

Now, for a fixed compact convex set $K \subset \mathbb{C}$ and $\varphi \in H(K)$, we define

$$\varphi(D)f(z) := \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi(\xi)e^{\xi z}\,d\xi \tag{2.1}$$

where Γ is a Cauchy cycle for K in Ω_φ. Obviously, this definition is independent of the choice of Γ. If φ extends to an entire function $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$, the interchange of integration and summation immediately yields

$$\sum_{n=0}^{\infty} c_n f^{(n)}(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi(\xi)e^{\xi z}\,d\xi.$$

Consequently, the operators $\varphi(D)$ from (2.1) are a natural extension of the differential operators in (1.1) and this justifies the notation.

Proposition 2.3. *Let K be a compact convex set in $\mathbb{C}$ and $\varphi \in H(K)$. Then $\varphi(D)$ defined by* (2.1) *is a continuous operator on* $\mathrm{Exp}(K)$.

Proof. For a given positive integer n, we choose Γ such that $|\Gamma| \subset n^{-1}\overline{\mathbb{D}} + K$. Then $H_{\mathrm{conv}(|\Gamma|)} \le H_{K+n^{-1}\overline{\mathbb{D}}}$ and so $\mathrm{Re}(\xi z) - H_K(z) - n^{-1}|z| \le 0$ for all $\xi \in |\Gamma|$ and all $z \in \mathbb{C}$. Hence, $|e^{\xi z - H_K(z) - n^{-1}|z|}| \le 1$ for all $z \in \mathbb{C}$ and all $\xi \in |\Gamma|$. As $\mathcal{B} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism and $|\Gamma|$ is compact in $\mathbb{C}\setminus K$, there is an $m \in \mathbb{N}$ and a constant $C > 0$ such that $\sup\{|\mathcal{B}f(\xi)| : \xi \in |\Gamma|\} \le C\|f\|_{K,m}$. With $M := (2\pi)^{-1}\int_\Gamma |\varphi(\xi)|\,d\xi$, we now obtain

$$\begin{aligned}\|\varphi(D)f\|_{K,n} &= \sup_{z\in\mathbb{C}} \left|\frac{1}{2\pi i}\int_\Gamma \varphi(\xi)\mathcal{B}f(\xi)e^{\xi z}\,d\xi\right| e^{-H_K(z)-n^{-1}|z|} \\ &\le \sup_{z\in\mathbb{C}} \frac{1}{2\pi}\int_\Gamma |\varphi(\xi)|\,|\mathcal{B}f(\xi)|\,|e^{\xi z - H_K(z)-n^{-1}|z|}|\,d\xi \le MC\|f\|_{K,m}.\end{aligned}$$

This proves that $\varphi(D)$ is a continuous self-mapping on $\mathrm{Exp}(K)$. ■

Now, our main result in this section is as follows:

Theorem 2.4. *Let K be a compact convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ non-constant. Then $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \ne \emptyset$ if and only if $\varphi(K)\cap\mathbb{T} \ne \emptyset$. Further, if $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \ne \emptyset$, then the set of all $f \in \mathcal{HC}(\varphi(D), \mathrm{Exp}(K))$ with $K(f) = K$ is residual in* $\mathrm{Exp}(K)$ *in the sense of Baire category.*

Before giving the proof, we establish some auxiliary results for $\mathrm{Exp}(K)$ and $\varphi(D)$.

Proposition 2.5. *Let $K \subset \mathbb{C}$ be a compact convex set.*

(1) *For any $\alpha \in K$, the set $\{Pe_\alpha : P$ polynomial$\}$ is dense in* $\mathrm{Exp}(K)$.
(2) *If A is an infinite subset of K, then* $\mathrm{span}\{e_\alpha : \alpha \in A\}$ *is dense in* $\mathrm{Exp}(K)$.

Proof. Let Σ denote the space of all polynomials. First assume that $0 \in K$. For $f \in \mathrm{Exp}(K)$ we have $\tilde{\mathcal{B}} f := (1/\cdot)\mathcal{B}f(1/\cdot) \in H(\mathbb{C}_\infty \setminus K^{-1})$. Since $\mathcal{B} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, so is $\tilde{\mathcal{B}} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K^{-1})$. Now, Σ is dense in $H(\mathbb{C}_\infty \setminus K^{-1})$ by Runge's theorem and observing that $\tilde{\mathcal{B}}^{-1}(\Sigma) = \Sigma$ shows that Σ is dense in $\mathrm{Exp}(K)$.

Let now K be an arbitrary compact convex set, and let $g = f/e_\alpha$ for some $f \in \mathrm{Exp}(K)$ and $\alpha \in K$. Since $|e^{-\alpha z}| = e^{\mathrm{Re}(-\alpha z)} = e^{H_{\{-\alpha\}}(z)}$ we obtain

$$\begin{aligned}\|f\|_{K,n} &= \sup_{z\in\mathbb{C}} |g(z)\|e^{\alpha z}|e^{-H_K(z)-n^{-1}|z|} = \sup_{z\in\mathbb{C}} |g(z)|e^{-H_K(z)-H_{\{-\alpha\}}(z)-n^{-1}|z|} \\ &= \sup_{z\in\mathbb{C}} |g(z)|e^{-H_{K-\{\alpha\}}(z)-n^{-1}|z|} = \|g\|_{K-\{\alpha\},n},\end{aligned}$$

which shows that $f \mapsto f/e_\alpha$ is an isometric isomorphism from $\mathrm{Exp}(K)$ to $\mathrm{Exp}(K - \{\alpha\})$. Together with the first part, this implies (1).

Without loss of generality, we may assume $0 \notin A$. It is easily seen that $\mathcal{B}e_\alpha = 1/(\cdot - \alpha)$ and thus $\mathcal{B}(\mathrm{span}\{e_\alpha : \alpha \in A\}) = \mathrm{span}\{1/(\cdot - \alpha) : \alpha \in A\}$. Since A has an accumulation point in K, a variant of Runge's theorem (see [13, Theorem 10.2]) implies that $\mathrm{span}\{1/(\cdot - \alpha) : \alpha \in A\}$ is dense in $H(\mathbb{C}_\infty \setminus K)$. As $\mathcal{B} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, this shows (2). ■

A germ $\varphi \in H(K)$ is said to be *zero-free* if there exists a representative φ which is zero-free on some open neighbourhood of K. In this case, we always assume that Ω_φ is so small that φ is zero-free on Ω_φ and thus $1/\varphi \in H(\Omega_\varphi)$.

PROPOSITION 2.6. *Let $K \subset \mathbb{C}$ be a compact convex set and let φ, ψ be in $H(K)$. Then*

$$\varphi(D)\psi(D) = \varphi\psi(D).$$

In particular, if φ is zero-free, then

$$\varphi(D)(1/\varphi)(D) = (1/\varphi)(D)\varphi(D) = \mathrm{id}_{\mathrm{Exp}(K)}$$

and hence $\varphi(D)$ is invertible with $\varphi(D)^{-1} = (1/\varphi)(D)$.

Proposition 2.6 is an immediate consequence of

LEMMA 2.7. *Let K be a compact convex set in $\mathbb{C}$, $f \in \mathrm{Exp}(K)$ and $\varphi \in H(K)$. Then for all $h \in H(\Omega_\varphi)$,*

$$\int_\Gamma \mathcal{B}f(\xi)\varphi(\xi)h(\xi)\,d\xi = \int_\Gamma \mathcal{B}(\varphi(D)f)(\xi)h(\xi)\,d\xi$$

where Γ is a Cauchy cycle for K in Ω_φ.

Proof. It is well known that $E := \mathrm{span}\{e_\alpha : \alpha \in \mathbb{C}\}$ is dense in $H(\mathbb{C})$ (see, e.g., [10, p. 259]), hence in $H(\Omega_\varphi)$ since Ω_φ is simply connected.

We consider the functional

$$\langle \Lambda, h\rangle := \int_{\Gamma} (\mathcal{B}f(\xi)\varphi(\xi) - \mathcal{B}(\varphi(D)f)(\xi))h(\xi)\,d\xi$$

on $H(\Omega_\varphi)$. By the Pólya representation for $\varphi(D)f$,

$$\frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}(\varphi(D)f)(\xi)e^{\xi\alpha}\,d\xi = \varphi(D)f(\alpha) = \frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)\varphi(\xi)e^{\xi\alpha}\,d\xi.$$

Hence $\langle \Lambda, e_\alpha\rangle = 0$ for all $\alpha \in \mathbb{C}$ and consequently $\Lambda|_E = 0$. As E is dense in $H(\Omega_\varphi)$, we have $\Lambda = 0$. ∎

PROPOSITION 2.8. *Let $K \subset \mathbb{C}$ be a compact convex set. Then the set of all $f \in \mathrm{Exp}(K)$ with $K(f) = K$ is residual in* $\mathrm{Exp}(K)$.

Proof. Let $M \subset H(\mathbb{C}_\infty \setminus K)$ be the set of functions that are exactly holomorphic in $\mathbb{C}_\infty \setminus K$, that is, for every $w \in \mathbb{C} \setminus K$ the radius of convergence of the Taylor series with centre w equals $\mathrm{dist}(w, K)$. By a result of V. Nestoridis (see [16, Theorem 4.5]), M is a dense G_δ-set in $H(\mathbb{C}_\infty \setminus K)$. Since $\mathcal{B}^{-1}(M) \subset \{f \in \mathrm{Exp}(K) : K(f) = K\}$ and $\mathcal{B} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, we obtain the assertion. ∎

Proof of Theorem 2.4. Firstly, assume that $\varphi(K) \subset \mathbb{D}$. Let Γ be a Cauchy cycle for K in Ω_φ which is so close to K that $|\varphi| < \delta < 1$ on $|\Gamma|$. Then, by Proposition 2.6, for any $f \in \mathrm{Exp}(K)$ we have

$$|\varphi(D)^n f(0)| = \left|\frac{1}{2\pi i}\int_{\Gamma} \mathcal{B}f(\xi)\varphi(\xi)^n\,d\xi\right| \le \frac{\delta^n}{2\pi}\int_{\Gamma} |\mathcal{B}f(\xi)|\,d\xi \to 0$$

as $n \to \infty$. Consequently, $\varphi(D)$ cannot be hypercyclic on $\mathrm{Exp}(K)$. If $\varphi(K) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, then φ is zero-free, as an element of $H(K)$, and thus, by Proposition 2.6, $\varphi(D)$ is invertible on $\mathrm{Exp}(K)$ with $\varphi(D)^{-1} = (1/\varphi)(D)$. Now, since $(1/\varphi)(K) \subset \mathbb{D}$, we have $\mathcal{HC}((1/\varphi)(D), \mathrm{Exp}(K)) = \emptyset$, and this is equivalent to $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) = \emptyset$ (see [1, Corollary 1.3]).

Let us now assume that $\varphi(K) \cap \mathbb{T} \ne \emptyset$ and that K has non-empty interior. Since φ is non-constant, we deduce that $\varphi(K)$ has non-empty interior, and thus $\mathrm{span}\{e_\alpha : \alpha \in K, |\varphi(\alpha)| > 1\}$ and $\mathrm{span}\{e_\alpha : \alpha \in K, |\varphi(\alpha)| < 1\}$ are dense in $\mathrm{Exp}(K)$ by Proposition 2.5(2). As $\varphi(D)e_\alpha = \varphi(\alpha)e_\alpha$, the Godefroy–Shapiro Criterion (see [1, Corollary 1.10]) yields $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \ne \emptyset$.

As a consequence of Proposition 2.5(1), $\mathrm{Exp}(K)$ is dense in $\mathrm{Exp}(K_n)$ for any compact convex sets $K, K_n \subset \mathbb{C}$ with $K \subset K_n$. Now, using Lemma 2.2 and [12, Corollary 12.19], the first part of the proof yields $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \ne \emptyset$ for general compact convex sets K that satisfy $\varphi(K) \cap \mathbb{T} \ne \emptyset$.

Since $\mathrm{Exp}(K)$ is a Fréchet space, $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \ne \emptyset$ implies that $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K))$ is a dense G_δ-set in $\mathrm{Exp}(K)$ (see [12, Theorem 2.19]).

From Proposition 2.8, we conclude that $\{f \in \operatorname{Exp}(K) : K(f) = K\} \cap \mathcal{HC}(\varphi(D), \operatorname{Exp}(K))$ is residual in $\operatorname{Exp}(K)$. ∎

Proof of Theorem 1.1. As mentioned in the proof of Lemma 2.7, $\operatorname{Exp}(K)$ is dense embedded in $H(\mathbb{C})$ for every non-empty, compact and convex set $K \subset \mathbb{C}$. Now, if φ is an entire function of exponential type, we obtain $\mathcal{HC}(\varphi(D), \operatorname{Exp}(K)) \subset \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ and so Theorem 1.1 is an immediate consequence of Theorem 2.4. ∎

Remark 2.9. Let φ be an entire function of exponential type. The proof of Theorem 2.4 shows that $\varphi(D)^n f(0) \to 0$ as $n \to \infty$ for each entire function f of exponential type satisfying $\varphi(K(f)) \subset \mathbb{D}$. Thus, for such φ actually $\mathcal{HC}(\varphi(D), H(\mathbb{C})) \cap \operatorname{Exp}(K)$ is empty whenever $\varphi(K) \subset \mathbb{D}$, that is, we can exclude hypercyclicity also with respect to the weaker topology of $H(\mathbb{C})$. While Theorem 2.4 states that $\mathcal{HC}(\varphi(D), \operatorname{Exp}(K))$ is empty if $\varphi(K) \subset \mathbb{C}\setminus\overline{\mathbb{D}}$, we do not know whether also $\mathcal{HC}(\varphi(D), H(\mathbb{C})) \cap \operatorname{Exp}(K)$ is empty in the latter case.

3. (Quasi)-conjugacy of differential operators. Let $T : X \to X$ and $S : Y \to Y$ be two continuous operators acting on topological vector spaces X, Y. A useful tool to link the dynamics of such operators is to show that they are (quasi-) conjugate. If one can find a continuous mapping $\Phi : X \to Y$ having dense range and such that $\Phi \circ T = S \circ \Phi$, that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & X \\ \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \Phi} \\ Y & \xrightarrow{\;S\;} & Y \end{array}$$

commutes, then S is said to be *quasi-conjugate* to T (by Φ). If Φ is bijective and Φ^{-1} is continuous, then T and S are said to be *conjugate*.

Proposition 3.1. *If S is quasi-conjugate to T by Φ, then $\Phi(\mathcal{HC}(T, X)) \subset \mathcal{HC}(S, Y)$ and $\Phi(\mathcal{FHC}(T, X)) \subset \mathcal{FHC}(S, Y)$.*

This follows immediately from the definition of quasi-conjugacy (cf. [12, Propositions 2.24 and 9.4]).

In this section, we introduce a transform that quasi-conjugates the operators from Section 2. Let $K \subset \mathbb{C}$ be a compact convex set and $\varphi \in H(K)$. As in the definition of the operators $\varphi(D)$ (cf. 2.1), our starting point is the Pólya representation. For $f \in \operatorname{Exp}(K)$, we set

$$\Phi_\varphi f(z) := \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{\varphi(\xi) z} \, d\xi \tag{3.1}$$

where Γ is a Cauchy cycle for K in Ω_φ. It is clear that this definition is independent of the choice of Γ.

PROPOSITION 3.2. *Let K be a compact convex subset of $\mathbb{C}$ and let $\varphi \in H(K)$ be non-constant. Then, for each $f \in \mathrm{Exp}(K)$, the function $\Phi_\varphi f$ defined by* (3.1) *is an entire function of exponential type with $K(\Phi_\varphi f) \subset \mathrm{conv}(\varphi(K(f)))$. Further*

$$\Phi_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K)))$$

is a continuous operator that has dense range.

Proof. One immediately verifies that $\Phi_\varphi f$ is an entire function. We fix some positive integer n and choose Γ such that $\varphi(|\Gamma|)$ is contained in $\mathrm{conv}(\varphi(K)) + n^{-1}\overline{\mathbb{D}}$. Then

$$H_{\mathrm{conv}(\varphi(|\Gamma|))}(z) \le H_{\mathrm{conv}(\varphi(K))+n^{-1}\overline{\mathbb{D}}}(z) = H_{\mathrm{conv}(\varphi(K))}(z) + \frac{1}{n}|z|$$

and thus

$$\begin{aligned}(3.2)\qquad \|\Phi_\varphi f\|_{\mathrm{conv}(\varphi(K)),n} &= \sup_{z\in\mathbb{C}} \left| \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{\varphi(\xi)z}\, d\xi \right| e^{-H_{\mathrm{conv}(\varphi(K))}(z) - n^{-1}|z|} \\ &\le \frac{\mathrm{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| e^{H_{\mathrm{conv}(\varphi(|\Gamma|))}(z)} e^{-H_{\mathrm{conv}(\varphi(K))}(z) - n^{-1}|z|} \\ &\le \frac{\mathrm{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| < \infty.\end{aligned}$$

As n was arbitrary, this shows that $K(\Phi_\varphi f)$ is contained in $\mathrm{conv}(\varphi(K))$, which in particular implies that $\Phi_\varphi f$ is of exponential type and $\Phi_\varphi f \in \mathrm{Exp}(\mathrm{conv}(\varphi(K)))$.

We proceed to the second assertion. Taking into account that for some $C < \infty$ and $m \in \mathbb{N}$ we have $\sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| \le C\|f\|_{K,m}$ due to the fact that $\mathcal{B} : \mathrm{Exp}(K) \to H(\mathbb{C}_\infty \setminus K)$ is an isomorphism, the continuity of Φ_φ follows from (3.2).

It remains to show that $\Phi_\varphi(\mathrm{Exp}(K))$ is dense in $\mathrm{Exp}(\mathrm{conv}(\varphi(K)))$. Let (K_n) be a sequence of compact convex sets in Ω_φ such that $K_n^\circ \supset K_{n+1}$ and the intersection of these sets is K. As noted above, the Borel transform of e_α is given by $\xi \mapsto 1/(\xi - \alpha)$. Inserting this in (3.1), the Cauchy integral formula yields $\Phi_\varphi(e_\alpha) = e_{\varphi(\alpha)}$ for all α in some K_n. Consequently, for each $n \in \mathbb{N}$,

$$\Phi_\varphi(\mathrm{span}\{e_\alpha : \alpha \in K_n\}) = \mathrm{span}\{e_{\varphi(\alpha)} : \alpha \in K_n\} \subset \mathrm{Exp}(\mathrm{conv}(\varphi(K_n))),$$

which implies that $\Phi_\varphi : \mathrm{Exp}(K_n) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K_n)))$ has dense range by Proposition 2.5(2) and the fact that φ is non-constant. Since $\mathrm{Exp}(K)$ is dense in $\mathrm{Exp}(K_n)$, we deduce that $\Phi_\varphi(\mathrm{Exp}(K))$ is dense in $\mathrm{Exp}(\mathrm{conv}(\varphi(K_n)))$.

Furthermore, we have

$$\bigcap_{n\in\mathbb{N}} \operatorname{conv}(\varphi(K_n)) = \operatorname{conv}(\varphi(K))$$

and hence

$$\bigcap_{n\in\mathbb{N}} \operatorname{Exp}(\operatorname{conv}(\varphi(K_n))) = \operatorname{Exp}(\operatorname{conv}(\varphi(K)))$$

set-theoretically and topologically by Lemma 2.2. It is now obvious that $\Phi_\varphi(\operatorname{Exp}(K))$ is dense in $\operatorname{Exp}(\operatorname{conv}(\varphi(K)))$. ∎

In the formulation of Proposition 3.2, it is necessary to take the convex hull in $\operatorname{Exp}(\operatorname{conv}(\varphi(K)))$, since $\operatorname{Exp}(K)$ is only defined for convex sets K. However, we show that the Borel transform of $\Phi_\varphi f$ actually admits an analytic continuation beyond $\mathbb{C}_\infty \setminus \operatorname{conv}(\varphi(K))$. For that purpose, we introduce a further notation. For a compact set $K \subset \mathbb{C}$, the polynomially convex hull $\widehat{K}$ is defined as the union of K with the bounded components of its complement. Let $K \subset \mathbb{C}$ be a compact convex set, $f \in \operatorname{Exp}(K)$ and $\varphi \in H(K)$. For $w \in \mathbb{C} \setminus \widehat{\varphi(K)}$ we set

$$H_\varphi(w) := \frac{1}{2\pi i} \int_\Gamma \frac{\mathcal{B}f(\xi)}{w - \varphi(\xi)}\, d\xi$$

with Γ a Cauchy cycle for $K \subset \Omega_\varphi$ so near to K that $\varphi(|\Gamma|)$ is contained in a simply connected, compact set $L \supset \widehat{\varphi(K)}$ such that $w \in \mathbb{C} \setminus L$. This definition is independent of the choice of Γ. Since $\varphi(|\Gamma|)$ can be arbitrarily near to $\varphi(K)$, we obtain a function $H_\varphi \in H(\mathbb{C}_\infty \setminus \widehat{\varphi(K)})$.

Proposition 3.3. *The function $H_\varphi \in H(\mathbb{C}_\infty \setminus \widehat{\varphi(K)})$ defines an analytic continuation of $\mathcal{B}(\Phi_\varphi f) \in H(\mathbb{C}_\infty \setminus \operatorname{conv}(\varphi(K)))$.*

Proof. Let Γ_0 be a Cauchy cycle for $\operatorname{conv}(\varphi(K))$ in $\mathbb{C}$. Then we can choose a Cauchy cycle Γ for K in Ω_φ so near to K that $\operatorname{ind}_{\Gamma_0}(\varphi(u)) = 1$ for all $u \in |\Gamma|$. Then

$$\begin{aligned}\frac{1}{2\pi i} \int_{\Gamma_0} H_\varphi(w) e^{wz}\, dw &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) \frac{1}{2\pi i} \int_{\Gamma_0} \frac{e^{wz}}{w - \varphi(\xi)}\, dw\, d\xi \\ &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{\varphi(\xi) z}\, d\xi = \Phi_\varphi f(z)\end{aligned}$$

by the Cauchy integral formula. Considering that $\mathcal{B}_{\operatorname{conv}(\varphi(K))}$ is an isomorphism, we can conclude $H_\varphi|_{\mathbb{C}_\infty \setminus \operatorname{conv}(\varphi(K))} = \mathcal{B}(\Phi_\varphi f)|_{\mathbb{C}_\infty \setminus \operatorname{conv}(\varphi(K))}$. ∎

We now show that Φ_φ commutes with differential operators on $\operatorname{Exp}(K)$. For that purpose, note that by our conventions, if $\varphi \in H(K)$ is zero-free, then Ω_φ is a simply connected domain that contains no zeros of φ. These conditions ensure the existence of $\log \varphi \in H(\Omega_\varphi)$. Moreover, we have to introduce another notion: A germ $\varphi \in H(K)$ is said to be *biholomorphic* if Ω_φ can be chosen so that $\varphi : \Omega_\varphi \to \varphi(\Omega_\varphi)$ is biholomorphic. In this case, we always assume Ω_φ to be so small that this property is ensured.

PROPOSITION 3.4. *Let K be a compact convex subset of $\mathbb{C}$ and let $\varphi \in H(K)$.*

(1) $D : \mathrm{Exp}(\mathrm{conv}(\varphi(K))) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K)))$ *is quasi-conjugate to* $\varphi(D) : \mathrm{Exp}(K) \to \mathrm{Exp}(K)$ *by* Φ_φ.
(2) *If φ is zero-free then*

$$e_1(D) : \mathrm{Exp}(\mathrm{conv}(\log \varphi(K))) \to \mathrm{Exp}(\mathrm{conv}(\log \varphi(K)))$$

is quasi-conjugate to $\varphi(D) : \mathrm{Exp}(K) \to \mathrm{Exp}(K)$ *by* $\Phi_{\log \varphi}$.
(3) *If C is a compact convex subset of $\mathbb{C}$ and $\psi \in H(C)$ is biholomorphic and $\psi(C) \supset \varphi(K)$ then*

$$\psi(D) : \mathrm{Exp}(\mathrm{conv}(\psi^{-1} \circ \varphi(K))) \to \mathrm{Exp}(\mathrm{conv}(\psi^{-1} \circ \varphi(K)))$$

is quasi-conjugate to $\varphi(D) : \mathrm{Exp}(K) \to \mathrm{Exp}(K)$ *by* $\Phi_{\psi^{-1}\circ\varphi}$.

Proof. Let $f \in \mathrm{Exp}(K)$. Interchanging integration and differentiation yields

$$D\Phi_\varphi f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi) e^{\varphi(\xi)z}\, d\xi. \tag{3.3}$$

Invoking Lemma 2.7, we obtain

$$D\Phi_\varphi f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}(\varphi(D)f)(\xi) e^{\varphi(\xi)z}\, d\xi = \Phi_\varphi \varphi(D) f(z).$$

This proves (1).

Now, let φ be zero-free. Again, according to Lemma 2.7, we obtain

$$\begin{aligned} e_1(D)\Phi_{\log\varphi} f(z) &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi) e^{(z+1)\log\varphi(\xi)}\, d\xi \\ &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\varphi(\xi) e^{z\log\varphi(\xi)}\, d\xi \\ &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}(\varphi(D)f)(\xi) e^{z\log\varphi(\xi)}\, d\xi = \Phi_{\log\varphi}\varphi(D) f(z). \end{aligned}$$

This is the assertion in (2).

In order to show (3), we consider an arbitrary $z \in \mathbb{C} \setminus C$ and choose a Cauchy cycle Γ_1 for K in Ω_φ such that $\varphi(|\Gamma_1|) \subset \Omega_{\psi^{-1}}$ and $\psi^{-1} \circ \varphi(|\Gamma_1|)$ is contained in some compact set $L \subset \Omega_\psi$ with $z \in \mathbb{C} \setminus L$. Further, let Γ_2 be a

Cauchy cycle for L in Ω_ψ. Then, by Lemma 2.7 and Proposition 3.3,

$$\begin{aligned}\psi(D)(\Phi_{\psi^{-1}\circ\varphi}f)(z) &= \frac{1}{2\pi i}\int_{\Gamma_2}\mathcal{B}(\Phi_{\psi^{-1}\circ\varphi}f)(w)\psi(w)e^{wz}\,dw\\ &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\mathcal{B}f(\xi)\psi(w)}{w-\psi^{-1}\circ\varphi(\xi)}\,d\xi\,e^{wz}\,dw\\ &= \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{B}f(\xi)\,\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\psi(w)e^{wz}}{w-\psi^{-1}\circ\varphi(\xi)}\,dw\,d\xi\\ &= \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{B}f(\xi)\varphi(\xi)e^{\psi^{-1}\circ\varphi(\xi)z}\,d\xi = \Phi_{\psi^{-1}\circ\varphi}(\varphi(D)f)(z). \quad\blacksquare\end{aligned}$$

Remark 3.5. If $\mathrm{Exp}(K)$ is endowed with the relative topology of $H(\mathbb{C})$, the transform Φ_φ is no longer continuous. Thus, the quasi-conjugacy in Proposition 3.4 is intimately linked with the topology of $\mathrm{Exp}(K)$.

As a first application of the transform introduced, we extend the result of Duyos-Ruiz mentioned in the introduction. For that purpose, a further fact has to be used:

In [8], K. C. Chan and J. H. Shapiro strengthened the result of Duyos-Ruiz. Here, growth of entire functions is measured with respect to a so-called *admissible comparison function*, i.e. an entire function $a(z) = \sum_{n=0}^\infty a_n z^n$ such that $a_n > 0$, $a_{n+1}/a_n \to 0$ as $n \to \infty$ and $(n+1)a_{n+1}/a_n$ is decreasing. For a comparison function a, Chan and Shapiro consider

$$E^2(a) := \left\{ f \in H(\mathbb{C}) : \|f\|_a^2 := \sum_{n=0}^\infty \frac{|f^{(n)}(0)/n!|^2}{a_n^2} < \infty \right\},$$

which is a Hilbert space of entire functions. They prove that the translation $e_\alpha(D)$ is hypercyclic on $E^2(a)$ for every admissible comparison function a and every $\alpha \in \mathbb{C}\setminus\{0\}$ (see [8, Theorem 2.1]). In [8], it is also shown that $f \in E^2(a)$ implies $M_f(r) = O(a(r))$. From the corollary of [8, Theorem 2.1] we can deduce the following:

Theorem (Duyos-Ruiz – Chan and Shapiro). *For every admissible comparison function a there is an $f \in \mathcal{HC}(e_1(D), \mathrm{Exp}(\{0\}))$ such that $M_f(r) = O(a(r))$.*

By means of the transform Φ_φ, we show that this result extends to the operators $\varphi(D)$ as follows:

Theorem 3.6. *Let $K \subset \mathbb{C}$ be a compact convex set and let $\varphi \in H(K)$ be non-constant. Then for every $\alpha \in K$ such that $|\varphi(\alpha)| = 1$ and $\varphi'(\alpha) \neq 0$ and every admissible comparison function a, there is an $f_0 \in \mathrm{Exp}(\{0\})$ such that $M_{f_0}(r) = O(a(r))$ and $f = f_0 e_\alpha \in \mathcal{HC}(\varphi(D), \mathrm{Exp}(K))$.*

Lemma 3.7. *Let $K \subset \mathbb{C}$ be a compact convex set, $\varphi \in H(K)$ and $\alpha \in \mathbb{C}$. Then for every $f \in \operatorname{Exp}(K)$, we have $\varphi(D)f = e_\alpha \varphi_\alpha(D)(f/e_\alpha)$ where $\varphi_\alpha := \varphi(\cdot + \alpha)$.*

Proof. For $\lambda \in K$, we have $\varphi(D)e_\lambda = \varphi(\lambda)e_\lambda$ and hence

$$\varphi(D)e_\lambda = e_\alpha \varphi(\lambda)e_{\lambda-\alpha} = e_\alpha \varphi(\lambda - \alpha + \alpha)e_{\lambda-\alpha},$$

which shows the assertion for $f = e_\lambda$, $\lambda \in K$. Since φ is holomorphic in a neighbourhood of K, we can assume that K has non-empty interior. Then $\operatorname{span}\{e_\lambda : \lambda \in K\}$ is dense in $\operatorname{Exp}(K)$ by Proposition 2.5(2). Further, as outlined in the proof of Theorem 2.5, $f \mapsto f/e_\alpha$ is an isometric isomorphism from $\operatorname{Exp}(K)$ to $\operatorname{Exp}(K - \{\alpha\})$ and we conclude that the above equality extends to all $f \in \operatorname{Exp}(K)$. ∎

Proof of Theorem 3.6. Let $a(z) = \sum_{n=0}^\infty a_n z^n$ be an admissible comparison function. Without loss of generality $a \in \operatorname{Exp}(\{0\})$. By Lemma 3.7 we can assume that $\alpha = 0$ and thus we only have to show the existence of $f \in \mathcal{HC}(\varphi(D), \operatorname{Exp}(\{0\}))$ with $M_f(r) = O(a(r))$, $r > 0$. We define $b(z) := \sum_{n=0}^\infty b_n z^n$ with $b_n := a_n/n!$, which is again an admissible comparison function. Now, as outlined above, the results in [8] yield a function $g \in E^2(b) \cap \mathcal{HC}(e_1(D), \operatorname{Exp}(\{0\}))$. By the definition of E^2 and (b_n),

$$\sum_{n=0}^\infty \frac{|g^{(n)}(0)|^2}{(n!b_n)^2} = \sum_{n=0}^\infty \frac{|g^{(n)}(0)|^2}{a_n^2} < \infty.$$

This implies that $G(z) := \sum_{n=0}^\infty |g^{(n)}(0)|z^n \in E^2(a)$ and hence, as again outlined above, $M_G(r) = O(a(r))$.

Since $\varphi'(0) \neq 0$, φ is biholomorphic as an element of $H(\{0\})$. We can assume that $\varphi(0) = 1$, otherwise, replace e_1 by $\varphi(0)e_1$ in what follows and notice that $g \in \mathcal{HC}(\varphi(0)e_1(D), \operatorname{Exp}(\{0\}))$ (see [1, Corollary 3.3]). Then $f := \Phi_{\varphi^{-1} \circ e_1} g \in \mathcal{HC}(\varphi(D), \operatorname{Exp}(\{0\}))$ by Propositions 3.1 and 3.4. We find some small $\delta > 0$ and $0 < c < \infty$ such that $|\varphi^{-1}(e_1(\xi))| \le c|\xi|$ for all $|\xi| < \delta$. We fix an $r > 0$ with $1/r \le \delta$ and such that for $\Gamma_r : [0, 2\pi) \to \mathbb{C}$, $t \mapsto r^{-1}e^{it}$, we have $e_1(|\Gamma_r|) \subset \Omega_{\varphi^{-1}}$. Now, as $\mathcal{B}g(\xi) = \sum_{n=0}^\infty g^{(n)}(0)/\xi^{n+1}$ on every compact subset of $\mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} M_f(r) &\le \max_{|z|=r} \left| \frac{1}{2\pi i} \int_{\Gamma_r} \mathcal{B}g(\xi) e^{(\varphi^{-1} \circ e_1)(\xi) z} \, d\xi \right| \\ &\le \max_{|z|=r} \sum_{n=0}^\infty |g^{(n)}(0)| \left| \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{(\varphi^{-1} \circ e_1)(\xi) z}}{\xi^{n+1}} \, d\xi \right| \\ &\le \sum_{n=0}^\infty |g^{(n)}(0)| \, r^n e^{\frac{c}{r} r} = e^c G(r). \end{aligned}$$

Thus, $M_f(r) = O(M_G(r)) = O(a(r))$ and this completes the proof. ∎

Proof of Theorem 1.2. By the proof of Theorem 1.1, we have the inclusion $\mathcal{HC}(\varphi(D), \mathrm{Exp}(K)) \subset \mathcal{HC}(\varphi(D), H(\mathbb{C}))$ provided that $\varphi(D)$ extends to a continuous operator on $H(\mathbb{C})$. Now, the assertion of Theorem 1.2 follows from Theorem 3.6 and the observation that for each $q : [0, \infty) \to [1, \infty)$ with $q(r) \to \infty$ as $r \to \infty$, there exists an admissible comparison function a such that $a(r) = O(r^{q(r)})$. ∎

4. Frequent hypercyclicity of differential operators. In this section we apply Φ_φ to extend known results on frequently hypercyclic functions for $e_1(D)$ to the whole class of differential operators $\varphi(D)$ on $\mathrm{Exp}(K)$ as well as on $H(\mathbb{C})$.

In [2], the first author proved the following

THEOREM. *If $K \subset \mathbb{C}$ is a compact convex set that contains two distinct points of the imaginary axis, then $\mathcal{FHC}(e_1(D), \mathrm{Exp}(K)) \neq \emptyset$.*

We can conclude that it is sufficient to require that $e_1(K) \cap \mathbb{T}$ contains a continuum in order to have $\mathcal{FHC}(e_1(D), \mathrm{Exp}(K)) \neq \emptyset$. Similarily, this result holds in the general situation:

THEOREM 4.1. *Let $K \subset \mathbb{C}$ be a compact convex set and let $\varphi \in H(K)$ be non-constant such that $\varphi(K) \cap \mathbb{T}$ contains a continuum. Then we have $\mathcal{FHC}(\varphi(D), \mathrm{Exp}(K))) \neq \emptyset$.*

Proof. Our assumptions ensure the existence of a compact convex set $\tilde{K} \subset K$ such that $\varphi(\tilde{K})$ contains some continuum of $\mathbb{T}$ and φ is biholomorphic as an element of $H(\tilde{K})$. We choose real numbers $a < b$ so that $e^{[ia,ib]} \subset \varphi(\tilde{K})$. The preceding result yields an $f \in \mathcal{FHC}(e_1(D), \mathrm{Exp}([ia, ib]))$, and, by Propositions 3.1 and 3.4, we have

$$\Phi_{\varphi^{-1} \circ e_1} f \in \mathcal{FHC}(\varphi(D), \mathrm{Exp}(\tilde{K})) \subset \mathcal{FHC}(\varphi(D), \mathrm{Exp}(K)). \blacksquare$$

Our next result shows that to some extent the assumptions in Theorem 4.1 are sharp.

THEOREM 4.2. *Let λ be a complex number and let $\varphi \in H(\{\lambda\})$. Then the set $\mathcal{FHC}(\varphi(D), \mathrm{Exp}(\{\lambda\}))$ is empty.*

Proof. According to Theorem 2.4 we can suppose that $|\varphi(\lambda)| = 1$. Then φ is zero-free on a sufficiently small simply connected neighbourhood Ω of λ, which implies the existence of a logarithm function of φ on Ω. Moreover, we may choose a branch so that $\log \varphi(\lambda) = 0$. Then Proposition 3.4(2) shows that $e_1(D) : \mathrm{Exp}(\{0\}) \to \mathrm{Exp}(\{0\})$ is quasi-conjugate to $\varphi(D) : \mathrm{Exp}(\{\lambda\}) \to \mathrm{Exp}(\{\lambda\})$ by $\Phi_{\log \varphi}$. From the results in [5] it follows that $\mathcal{FHC}(e_1(D), H(\mathbb{C})) \cap \mathrm{Exp}(\{0\})$ and thus $\mathcal{FHC}(e_1(D), \mathrm{Exp}(\{0\})$ is empty. Now, according to Proposition 3.1 the assertion follows. ∎

Theorem 1.3(2) is stronger than the previous result since it excludes frequent hypercyclicity with respect to the weaker topology of $H(\mathbb{C})$. Unfortunately, the transform Φ_φ does not preserve (frequent) hypercyclicity with respect to this topology. Thus, some extra argument is required to show Theorem 1.3(2).

Proof of Theorem 1.3. The first part is an immediate consequence of Theorem 4.1 since $\mathcal{FHC}(\varphi(D), \mathrm{Exp}(K)) \subset \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$. Thus, there is only (2) to prove.

We suppose there is some entire function f of exponential type such that $K(f) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$, and $f \in \mathcal{FHC}(\varphi(D), H(\mathbb{C}))$. Then necessarily, $|\varphi(\lambda)| \geq 1$ because otherwise $\varphi(D)^n f(0) \to 0$ as $n \to \infty$ (see Remark 2.9). Hence in some sufficiently small simply connected neighbourhood Ω of λ, the function $\tilde{\varphi} := \varphi/\varphi(\lambda)$ is zero-free, which implies the existence of $\log \tilde{\varphi}$ on Ω with $\log \tilde{\varphi}(\lambda) = 0$. We set $h := \Phi_{\log \tilde{\varphi}} f$. Then $K(h) = \{0\}$ by Proposition 3.2 and, applying Proposition 3.4(2) to $\tilde{\varphi}$, we have

$$h(n) = \frac{1}{\varphi(\lambda)^n} \varphi(D)^n f(0) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.1}$$

By the Casorati–Weierstrass theorem, we can choose $\alpha \in \mathbb{C}$ such that $\varphi(\alpha)$ is close enough to $e^{i\pi}\varphi(\lambda)$ to ensure that for a sufficiently small neighbourhood U of 1 we have

$$\frac{\varphi(\alpha)}{\varphi(\lambda)} U \subset \{z : |\arg(z) - \pi| \leq \pi/4\} \tag{4.2}$$

and $\varphi(\alpha) \neq 0$. Now, by the continuity of $\varphi(D)$ on $H(\mathbb{C})$, for every $\varepsilon > 0$, there are some $r > 0$ and $\delta > 0$ such that for all $g \in H(\mathbb{C})$ that satisfy

$$\sup_{z \in r\overline{\mathbb{D}}} |g(z) - e_\alpha(z)| < \delta, \tag{4.3}$$

we have

$$|\varphi(D)g(0) - \varphi(D)e_\alpha(0)| = |\varphi(D)g(0) - \varphi(\alpha)| < \varepsilon.$$

We assume that $\delta, \varepsilon > 0$ are so small that, whenever g satisfies (4.3),

$$g(0) \in U \quad \text{and} \quad \varphi(D)g(0) \in \varphi(\alpha)U. \tag{4.4}$$

Our assumption implies the existence of a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers with $\underline{\mathrm{dens}}((n_k)_{k\in\mathbb{N}}) > 0$ such that $\sup_{z\in r\overline{\mathbb{D}}} |\varphi(D)^{n_k} f(z) - e_\alpha(z)| < \delta$ for all $k \in \mathbb{N}$. The interpolating property of h in (4.1) combined with (4.4) yields

$$h(n_k) \in \frac{1}{\varphi(\lambda)^{n_k}} U \quad \text{and} \quad h(n_k + 1) \in \frac{\varphi(\alpha)}{\varphi(\lambda)^{n_k+1}} U \quad \text{for all } k \in \mathbb{N}. \tag{4.5}$$

Condition (4.2) implies that the factor $\varphi(\alpha)/\varphi(\lambda)$ rotates U by an angle larger than $\pi/2$ and less than $3\pi/2$. Hence, from (4.5), it follows that for each $k \in \mathbb{N}$ either $\mathrm{Re}(h)$ or $\mathrm{Im}(h)$ has a sign change in $[n_k, n_k + 1]$. The intermediate value theorem yields a sequence $(w_k)_{k\in\mathbb{N}}$ with $w_k \in (n_k, n_k+1)$ and

$$\mathrm{Re}(h(w_k))\mathrm{Im}(h(w_k)) = 0 \quad \text{for all } k \in \mathbb{N}. \tag{4.6}$$

Assuming that the Taylor series of h is given by $\sum_{\nu=0}^{\infty}(h_\nu/\nu!)z^\nu$, we set

$$h_1(z) := \sum_{\nu=0}^{\infty} \frac{\mathrm{Re}(h_\nu)}{\nu!} z^\nu \quad \text{and} \quad h_2(z) := \sum_{\nu=0}^{\infty} \frac{\mathrm{Im}(h_\nu)}{\nu!} z^\nu.$$

The functions h_1, h_2 are of exponential type zero since h is, and thus h_1h_2 is of exponential type zero. Since $\mathrm{Re}(h(x)) = h_1(x)$ and $\mathrm{Im}(h(x)) = h_2(x)$ for every real x, we obtain $h_1h_2(w_k) = 0$ for all $k \in \mathbb{N}$ by (4.6). As $(w_k)_{k\in\mathbb{N}}$ has obviously the same lower density as $(n_k)_{k\in\mathbb{N}}$, we infer that h_1h_2 is a function of exponential type zero having zeros of positive lower density, which is impossible unless it is constantly zero (cf. [6, Theorem 2.5.13]). ∎

Remark 4.3. Theorem 1.3(2) implies Theorem 4.2. To see this, suppose there exists some $f \in \mathcal{FHC}(\varphi(D), \mathrm{Exp}(\{\lambda\})$. Then, by Propositions 3.1 and 3.4,

$$\Phi_\varphi f \in \mathcal{FHC}(D, \mathrm{Exp}(\{\varphi(\lambda)\})) \subset \mathcal{FHC}(D, H(\mathbb{C})),$$

contradicting Theorem 1.3(2). Note that this proof does not use the results from [5].

Remark 4.4. Let φ be an entire function of exponential type. Remark 2.9 shows that $\mathcal{FHC}(\varphi(D), H(\mathbb{C})) \cap \mathrm{Exp}(K)$ is empty whenever $\varphi(K) \subset \mathbb{D}$, and $\mathcal{FHC}(\varphi(D), \mathrm{Exp}(K))$ is empty if $\varphi(K) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Again, we do not know whether $\mathcal{FHC}(\varphi(D), H(\mathbb{C})) \cap \mathrm{Exp}(K)$ is empty in the latter case.

Acknowledgements. We are grateful to the DFG (Deutsche Forschungsgemeinschaft; GZ: MU 1529/5-1) for funding the project from which this work emerged. Moreover, we thank the referee for his careful reading of the first version, his suggestions and corrections, and his patience. His expertise helped improving the presentation significantly.

References

[1] F. Bayart and É. Matheron, *Dynamics of Linear Operators*, Cambridge Univ. Press, Cambridge, 2009.

[2] H.-P. Beise, *Frequently Birkhoff-universal functions*, submitted.

[3] C. A. Berenstein and R. Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, New York, 1995.

[4] L. Bernal-González and A. Bonilla, *Exponential type of hypercyclic entire functions*, Arch. Math. (Basel) 78 (2002), 283–290.

[5] O. Blasco, A. Bonilla, and K.-G. Grosse-Erdmann, *Rate of growth of frequently hypercyclic functions*, Proc. Edinburgh Math. Soc. 53 (2010), 39–59.

[6] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[7] A. Bonilla and K.-G. Grosse-Erdmann, *On a theorem of Godefroy and Shapiro*, Integral Equations Operator Theory 56 (2006), 151–162.
[8] K. C. Chan and J. H. Shapiro, *The cyclic behaviour of translation operators on Hilbert spaces of entire functions*, Indiana Univ. Math. J. 40 (1991), 1421–1449.
[9] S. M. Duyos-Ruiz, *On the existence of universal functions*, Soviet Math. Dokl. 27 (1983), 9–13.
[10] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant cyclic vector manifolds*, J. Funct. Anal. 98 (1991), 229–269.
[11] K.-G. Grosse-Erdmann, *On the universal functions of G. R. MacLane*, Complex Variables Theory Appl. 15 (1990), 193–196.
[12] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer, Berlin, 2011.
[13] D. H. Luecking and L. A. Rubel, *Complex Analysis, a Functional Analysis Approach*, Springer, New York, 1984.
[14] A. Martineau, *Sur les fonctionnelles analytiques et la transformation de Fourier–Borel*, J. Anal. Math. 11 (1963), 1–163.
[15] M. Morimoto, *An Introduction to Sato's Hyperfunctions*, Transl. Math. Monogr. 129, Amer. Math. Soc., 1994.
[16] V. Nestoridis, *Non extendable holomorphic functions*, Math. Proc. Cambridge Philos. Soc. 139 (2005), 351–360.
[17] W. Rudin, *Real and Complex Analysis*, 3rd ed., Mc-Graw-Hill, New York, 1974.

Hans-Peter Beise, Jürgen Müller
FB IV, Mathematics
University of Trier
D-54286 Trier, Germany
E-mail: beise@uni-trier.de
jmueller@uni-trier.de

Received March 4, 2011
Revised version October 1, 2011 (7125)