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Vector-valued inequalities for the commutators of fractional integrals with rough kernels

by

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Abstract. Some conditions implying vector-valued inequalities for the commutator of a fractional integral and a fractional maximal operator are established. The results obtained are substantial improvements and extensions of some known results.

1. Introduction. Suppose that $0 < \alpha < n$, $\Omega(x)$ is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^1(S^{n-1})$. Then the *fractional integral operator* $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy,$$

while the related *fractional maximal operator* $M_{\Omega,\alpha}$ is given by

$$M_{\Omega,\alpha}f(x) = \sup_{t>0} \frac{1}{t^{n-\alpha}} \int_{|x-y|<t} |\Omega(x-y)f(y)|\, dy.$$

When $\alpha = 0$, we denote $T_{\Omega,0}$ by T_Ω, and the integral is the Cauchy principal value. The operator $T_{\Omega,\alpha}$ plays an important role in the study of the homogeneous operator T_Ω. For example, recently, Ding and Lu [DL1] applied several results on $T_{\Omega,\alpha}$ to the study of mapping properties for a class of multilinear singular integral operators with homogeneous kernel. If we take $\Omega(y') = 1$, then $T_{1,\alpha}$ is just the Riesz potential I_α, which has been systematically studied by Riesz [R] on $\mathbb{R}^n$ although its one-dimensional version appeared in earlier work of Weyl [W]. This operator plays an important role in analysis, particularly in the study of smoothness properties of functions. See the books by Stein and Weiss [SW] or Grafakos [G] for the basic properties of these operators. The (L^p, L^r) estimate of I_α is the famous Hardy–Littlewood–Sobolev theorem ([HL], [So]):

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Theorem 1.1 (see [HL], [So]). *Let $0 < \alpha < n$. For $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$, there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^n)$,*

$$\|I_\alpha f\|_{L^r} \le C\|f\|_{L^p}.$$

In 1971, Muckenhoupt and Wheeden [MW1] proved the (L^p, L^r) boundedness of $T_{\Omega,\alpha}$ with rough kernel.

Theorem 1.2 (see [MW1]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/r = 1/p - \alpha/n$. Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and it is in $L^q(S^{n-1})$ for some $q > p'$. Let $f \in L^p(\mathbb{R}^n)$. Then there exists a constant C, independent of f, such that*

$$\|T_{\Omega,\alpha} f\|_{L^r} \le C\|f\|_{L^p}.$$

In 1993, Chanillo, Watson and Wheeden [CWW] obtained the weak type $(1, n/(n-\alpha))$ of the fractional integral $T_{\Omega,\alpha}$ with rough kernel.

Theorem 1.3 (see [CWW]). *Let $0 < \alpha < n$ and let $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$. Then for any $\lambda > 0$ and any $f \in L^1$,*

$$|\{x \in \mathbb{R}^n : T_{\Omega,\alpha} f(x) > \lambda\}| \le C\bigg(\frac{1}{\lambda}\|f\|_{L^1}\bigg)^{n/(n-\alpha)},$$

where C is independent of λ and f.

In 2000, Ding and Lu [DL2] proved the (L^p, L^r) boundedness of the fractional maximal operator $M_{\Omega,\alpha}$ and the fractional integral $T_{\Omega,\alpha}$.

Theorem 1.4 (see [DL2]). *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/r=1/p-\alpha/n$, and let Ω be homogeneous of degree zero on $\mathbb{R}^n$. Then for any $f \in L^p(\mathbb{R}^n)$:*

- (i) *if $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, then $\|M_{\Omega,\alpha} f\|_{L^r} \le C\|f\|_{L^p}$,*
- (ii) *if $\Omega \in L^q(S^{n-1})$ and $q > n/(n-\alpha)$, then $\|T_{\Omega,\alpha} f\|_{L^r} \le C\|f\|_{L^p}$,*

where C is independent of f.

In 2011, Chen and Ding [CD2] proved that $T_{\Omega,\alpha}$ is of weak type (p, r) when $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$. Moreover, they applied weak type bounds for $T_{\Omega,\alpha}$ and Marcinkiewicz interpolation to get the following result:

Theorem 1.5 (see [CD2]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/r = 1/p - \alpha/n$, and suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and it is in $L^{n/(n-\alpha)}(S^{n-1})$. Then for any $f \in L^p(\mathbb{R}^n)$,*

$$\|T_{\Omega,\alpha} f\|_{L^r} \le C\|f\|_{L^p},$$

where C is independent of f.

For $0 < \alpha < n$, the *commutator* of the fractional integral $T_{\Omega,\alpha}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$ is defined by

$$[b, T_{\Omega,\alpha}] f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}(b(x) - b(y)) f(y)\, dy.$$

Here $b \in \mathrm{BMO}(\mathbb{R}^n)$ means that

$$\|b\|_{\mathrm{BMO}} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(y)-b_Q|\,dy < \infty$$

with $b_Q = |Q|^{-1}\int_Q b(x)\,dx$. The related commutator of the fractional maximal operator $M_{\Omega,\alpha;b}$ is given by

$$M_{\Omega,\alpha;b}f(x) = \sup_{t>0}\frac{1}{t^{n-\alpha}}\int_{|x-y|<t} |\Omega(x-y)|\,|b(x)-b(y)|\,|f(y)|\,dy.$$

When $\alpha = 0$, we denote $[b, T_{\Omega,0}]$ by $[b, T_\Omega]$, which is defined by

$$[b, T_\Omega]f(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))f(y)\,dy. \tag{1.1}$$

However, Ω needs to have mean value zero in order to define the commutator (1.1) as a principal value. The operator $[b, T_{\Omega,\alpha}]$ plays an important role in the study of the homogeneous operator $[b, T_\Omega]$.

It is well known that the commutator of a fractional integral is very useful in harmonic analysis (see e.g. [CH], [D1], [D2], [D3], [DLP], [ST]). If we take $\Omega(y) = 1$, then $[b, T_{1,\alpha}]$ is just the commutator of $b \in \mathrm{BMO}$ and the Riesz potential I_α, that is,

$$[b, I_\alpha]f(x) = [b, T_{1,\alpha}]f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}(b(x)-b(y))\,dy.$$

In 1982, Chanillo [CH] proved the (L^p, L^r) boundedness of $[b, I_\alpha]$:

THEOREM 1.6 (see [CH]). *Let* $b \in \mathrm{BMO}$, $0 < \alpha < n$, $1 < p < n/\alpha$, *and* $1/p - 1/r = \alpha/n$. *Let* $f \in L^p(\mathbb{R}^n)$. *Then there exists a positive constant* C *independent of* f *such that*

$$\|[b, I_\alpha]f\|_{L^r} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

In 2001, Ding, Lu and Zhang [DLP] gave an example where $[b, I_\alpha]$ is not of weak type $(1, n/(n-\alpha))$, introduced a kind of maximal operator of fractional order associated with the mean Luxemburg norm in the Orlicz space, and using the technique of sharp functions obtained the following result.

THEOREM 1.7 (see [DLP]). *Let* $b \in \mathrm{BMO}$, $0 < \alpha < n$, *and* $\Phi(t) = t(1+\log^+ t)$. *Then there exists a positive constant* C *such that for all* $\lambda > 0$ *and* $f \in L^1(\mathbb{R}^n)$,

$$|\{x\in\mathbb{R}^n : |[b, I_\alpha]f(x)| > \lambda\}|^{(n-\alpha)/n} \le C\Phi\big(\Phi(\|b\|_{\mathrm{BMO}})\big)\|\Phi(f(\cdot)/\lambda)\|_{L^1}\Big\{1 + \frac{\alpha}{n}\log^+\|\Phi(f(\cdot)/\lambda)\|_{L^1}\Big\}.$$

In 1993, Segovia and Torrea [ST] proved the weighted boundedness of commutators for vector-valued integral operators with a pair of weights using the Rubio de Francia extrapolation idea for weighted norm inequalities. As an application, they obtained $(L^p(u^p), L^r(v^q))$ boundedness of fractional integrals when Ω satisfies some smoothness condition. In 1999, Ding and Lu [DL1] extended the result of [ST] to general fractional integrals with rough kernels.

THEOREM 1.8 ([DL1]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/r = 1/p - \alpha/n$. Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^q(S^{n-1})$ for $q > p'$. Let $f \in L^p(\mathbb{R}^n)$. Then for $b \in \mathrm{BMO}$ there exists a constant C independent of f such that*

(i) $\|[b, T_{\Omega,\alpha}]f\|_{L^r} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}$,
(ii) $\|M_{\Omega,\alpha;b}f\|_{L^r} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}$.

Motivated by Theorems 1.4 and 1.5, it is natural to ask whether the size condition on Ω in Theorem 1.8 can be weakened. In this paper, we give a positive answer to this question.

THEOREM 1.9. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/r = 1/p - \alpha/n$, and $\Phi(t) = t(1 + \log^+ t)$. Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Phi(\Omega) \in L^{n/(n-\alpha)}(S^{n-1})$. Let $f \in L^p(\mathbb{R}^n)$. Then for $b \in \mathrm{BMO}$ there exists a constant C independent of f such that*

$$\|M_{\Omega,\alpha;b}f\|_{L^r} \le C\|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}}\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

REMARK 1.10. Since $q > p'$ and $p' > n/(n-\alpha)$, we have

$$L^q(S^{n-1}) \subset L^{\frac{n}{n-\alpha}}(\log^+ L)^{\frac{n}{n-\alpha}}(S^{n-1}).$$

This means that the size condition on Ω in Theorem 1.9 is weaker than that in Theorem 1.8(ii).

In fact, for $\{f_s\}_{s\in\mathbb{Z}} \in L^p(\ell^q)$, as $\sup_{s\in\mathbb{Z}} M_{\Omega,\alpha;b}f_s \le CM_{\Omega,\alpha;b}(\sup_{s\in\mathbb{Z}}|f_s|)$, for $1/r = 1/p - \alpha/n$ we get

$$\Big\|\sup_{s\in\mathbb{Z}} M_{\Omega,\alpha;b}f_s\Big\|_{L^r} \le C\|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}}\|b\|_{\mathrm{BMO}}\Big\|\sup_{s\in\mathbb{Z}}|f_s|\Big\|_{L^p}.$$

By duality,

$$\Big\|\sum_{s\in\mathbb{Z}} M_{\Omega,\alpha;b}f_s\Big\|_{L^r} \le C\|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}}\|b\|_{\mathrm{BMO}}\Big\|\sum_{s\in\mathbb{Z}}|f_s|\Big\|_{L^p}.$$

Interpolating between the two inequalities above, for $1 < q < \infty$ we get

$$\Big\|\Big(\sum_{s\in\mathbb{Z}}(M_{\Omega,\alpha;b}f_s)^q\Big)^{1/q}\Big\|_{L^r} \le C\|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of $\{f_s\}$.

Corollary 1.11. *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, $1/r = 1/p - \alpha/n$, *and* $\Phi(t) = t(1 + \log^+ t)$. *Suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n$ *and* $\Phi(\Omega) \in L^{n/(n-\alpha)}(S^{n-1})$. *Let* $1 < q < \infty$ *and* $\{f_s\} \in L^p(\ell^q)(\mathbb{R}^n)$. *Then for* $b \in \mathrm{BMO}$, *there exists a constant* C *independent of* $\{f_s\}$ *such that*

$$\Big\| \Big(\sum_{s \in \mathbb{Z}} (M_{\Omega,\alpha;b} f_s)^q \Big)^{1/q} \Big\|_{L^r} \le C \|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}} \|b\|_{\mathrm{BMO}} \Big\| \Big(\sum_{s \in \mathbb{Z}} |f_s|^q \Big)^{1/q} \Big\|_{L^p}.$$

Actually, we can apply Corollary 1.11 to get the $(L^p(\ell^q), L^r(\ell^q))$ boundedness of $[b, T_{\Omega,\alpha}]$. In fact, for any fixed $0 < \varepsilon < \min\{\alpha, n - \alpha\}$, we can find $0 < \varepsilon_1 < \varepsilon$ such that

$$\|\Phi(\Omega)\|_{L^{\frac{n}{n-\varepsilon_1-\alpha}}} \le C \|\Omega\|_{L^{\frac{n}{n-\varepsilon-\alpha}}}, \tag{1.2}$$

and for $\{f_s\}_{s\in\mathbb{Z}} \in L^p(l^q)$ (see [DL1]),

$$|[b, T_{\Omega,\alpha}] f_s(x)| \le C[M_{\Omega,\alpha+\varepsilon_1;b} f_s(x)]^{1/2} [M_{\Omega,\alpha-\varepsilon_1;b} f_s(x)]^{1/2}, \tag{1.3}$$

where C depends on n, α, and ε_1. Then by applying the Hölder inequality twice and Corollary 1.11, we get

$$\Big\| \Big(\sum_{s \in \mathbb{Z}} |[b, T_{\Omega,\alpha}] f_s|^q \Big)^{1/q} \Big\|_{L^r} \le C \|\Omega\|_{L^{\frac{n}{n-\varepsilon-\alpha}}} \|b\|_{\mathrm{BMO}} \Big\| \Big(\sum_{s \in \mathbb{Z}} |f_s|^q \Big)^{1/q} \Big\|_{L^p},$$

where C is independent of $\{f_s\}$.

Corollary 1.12. *Let* $0 < \alpha < n$, $1 < p < n/\alpha$, *and* $1/r = 1/p - \alpha/n$. *For any fixed* $\varepsilon > 0$, *suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n$ *and* $\Omega \in L^{n/(n-\varepsilon-\alpha)}(S^{n-1})$. *Let* $1 < q < \infty$ *and* $\{f_s\} \in L^p(\ell^q)(\mathbb{R}^n)$. *Then for* $b \in \mathrm{BMO}$ *there exists a constant* C *independent of* $\{f_s\}$ *such that*

$$\Big\| \Big(\sum_{s \in \mathbb{Z}} |[b, T_{\Omega,\alpha}] f_s|^q \Big)^{1/q} \Big\|_{L^r} \le C \|b\|_{\mathrm{BMO}} \Big\| \Big(\sum_{s \in \mathbb{Z}} |f_s|^q \Big)^{1/q} \Big\|_{L^p}. \tag{1.4}$$

Remark 1.13. Because $q > p'$ and $p' > n/(n - \alpha)$, there exists a constant $\varepsilon > 0$ such that $L^q(S^{n-1}) \subset L^{n/(n-\varepsilon-\alpha)}(S^{n-1})$. This means that the size condition on Ω in Corollary 1.12 is weaker than that in Theorem 1.8(i).

An interesting problem is whether for $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, the commutator $M_{\Omega,\alpha;b}$ or $[b, T_{\Omega,\alpha}]$ is bounded from L^p to L^r for $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$. Recall that in order to prove the (L^p, L^r) boundedness of $M_{\Omega,\alpha}$ or $T_{\Omega,\alpha}$ with $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, Ding and Lu [DL2] have used the Marcinkiewicz interpolation theorem between the weak type $(1, n/(n - \alpha))$ and strong type $(L^{n/\alpha}, L^\infty)$ of $M_{\Omega,\alpha}$ with $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, and Chen and Ding [CD2] have used the Marcinkiewicz interpolation theorem between the weak type $(1, n/(n - \alpha))$ and weak type (p, r) of $T_{\Omega,\alpha}$ with Ω in $L^{n/(n-\alpha)}(S^{n-1})$. Unfortunately, this key technique fails for $M_{\Omega,\alpha;b}$ and $[b, T_{\Omega,\alpha}]$, because they are not of weak type $(1, n/(n - \alpha))$ (see Theorem 1.7). Probably, we need to look for a new method. That is the main difficulty that prevented us from solving this problem completely. More precisely, we are not able to obtain for $M_{\Omega,\alpha;b}$ and $[b, T_{\Omega,\alpha}]$ analogues to Theorems 1.4(i) and 1.5.

For $0 < \alpha < 1$, we can further weaken the size condition on Ω in Corollary 1.12. The main result of this article is:

THEOREM 1.14. *Let $0<\alpha<1$, $1<p<n/\alpha$, and $1/r=1/p-\alpha/n$. Suppose Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^{n/(n-\alpha)}(\log^+ L)^2(S^{n-1})$. Let $1 < q < \infty$ and $\{f_s\} \in L^p(\ell^q)(\mathbb{R}^n)$. Then for $b \in \mathrm{BMO}$ there exists a constant C independent of $\{f_s\}$ such that*

$$\Big\|\Big(\sum_{s\in\mathbb{Z}} |[b, T_{\Omega,\alpha}]f_s|^q\Big)^{1/q}\Big\|_{L^r} \le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}} |f_s|^q\Big)^{1/q}\Big\|_{L^p}.$$

REMARK 1.15. Note that for any $\varepsilon > 0$,

$$L^{\frac{n}{n-\varepsilon-\alpha}}(S^{n-1}) \subset L^{\frac{n}{n-\alpha}}(\log^+ L)^2(S^{n-1}).$$

This means that the size condition on Ω in Theorem 1.14 is weaker than that in Corollary 1.12 for $0 < \alpha < 1$.

REMARK 1.16. Since $M_{\Omega,\alpha;b}$ in Theorem 1.9 is a positive operator, we can get $(L^p(\ell^q), L^r(\ell^q))$ bounds for $M_{\Omega,\alpha;b}$ from (L^p, L^r) bounds for $M_{\Omega,\alpha;b}$ (see Corollary 1.11). Then by (1.3), we can get $(L^p(\ell^q), L^r(\ell^q))$ bounds for $[b, T_{\Omega,\alpha}]$. But the techniques in Corollary 1.12 fail for $[b, T_{\Omega,\alpha}]$ with Ω in $L^{n/(n-\alpha)}(\log^+ L)^2(S^{n-1})$. So we need to look for a new method. In the proof of Theorem 1.14, we use Littlewood–Paley theory, Bony paraproducts, and Fourier transform estimates. These techniques are different from those for fractional integral operators with rough kernels in [CH] and [DL1].

This paper is organized as follows. First, in Section 2, we give some notations and definitions. In Section 3, we prepare some lemmas for the proof of Theorem 1.14. In Section 4, we give the proof of Theorem 1.9. Finally, in Section 5, we prove Theorem 1.14. Throughout this note, the letter "C" will be used to denote positive constants which may be different in different occurrences.

2. Definitions. Firstly, we recall some definitions which will be used in the proof of Theorem 1.14.

Let $\theta \in \mathbb{R}$ and $1 < p < \infty$. The *homogeneous Sobolev space* $L^p_\theta(\mathbb{R}^n)$ is defined as the space of those tempered distributions modulo polynomials, $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, for which the expression $(|\cdot|^\theta \widehat{f})^\vee$ is a function in $L^p(\mathbb{R}^n)$. For distributions f in $L^p_\theta(\mathbb{R}^n)$ we define

$$\|f\|_{L^p_\theta} = \|(|\cdot|^\theta \widehat{f})^\vee\|_{L^p}. \tag{2.1}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function which is supported in the unit ball and satisfies $\varphi(\xi) = 1$ for $|\xi| \le 1/2$. The function $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$ is supported in $\{1/2 \le |\xi| \le 2\}$ and satisfies

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$$

for $\xi \neq 0$. We denote by Δ_j and G_j the convolution operators whose symbols are $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively.

The *paraproduct* of Bony [B] between two functions f, g is defined by

$$\pi_f(g) = \sum_{j\in\mathbb{Z}} (\Delta_j f)(G_{j-3}g).$$

At least formally, we have the Bony decomposition

$$fg = \pi_f(g) + \pi_g(f) + R(f,g)$$

with

$$R(f,g) = \sum_{i\in\mathbb{Z}} \sum_{|k-i|\le 2} (\Delta_i f)(\Delta_k g). \tag{2.2}$$

We recall the definition of A_p and $A(p,q)$ weights for $1 < p, q < \infty$. Let $1 < p < \infty$. A locally integrable positive function w is said to be a *weight of class* A_p if

$$[w]_{A_p} = \sup_{\text{cube}\, Q\in\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty.$$

A locally integrable positive function w on $\mathbb{R}^n$ is said to belong to $A(p,q)$ if

$$\sup_{\text{cube}\, Q\in\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} < \infty.$$

Moreover, the notations "$\wedge$" and "$\vee$" denote the Fourier transform and the inverse Fourier transform, respectively. As usual, for $p \ge 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

We collect the notation to be used throughout this paper:

$$\|\{f_j\}\|_{L^p(\ell^q)} = \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^n} |f(x)|^p\,dx\Big)^{1/p},$$

$$\|f\|_{L^p(w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\Big)^{1/p}.$$

3. Key lemmas. Let us begin with some lemmas, which will be used in the proof of Theorem 1.14. The first one is a direct consequence of Proposition 4.6.4 in [G].

LEMMA 3.1 ([G]). *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\operatorname{supp}(\phi) \subset \{x : 1/2 \le |x| \le 2\}$, *and for* $k \in \mathbb{Z}$ *define the multiplier operator* S_k *by* $\widehat{S_k f}(\xi) = \phi(2^{-k}\xi)\widehat{f}(\xi)$, *and* S_k^2 *by* $S_k^2 f = S_k(S_k f)$. *Let* $1 < p, q < \infty$, $\{f_j\} \in L^p(\ell^q)$, *and let* $\{f_{j,k}\} \in L^p(\ell^q(\ell^2))$. *Then*

(i) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_k f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{f_j\}$;

(ii) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}S_k f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{f_{j,k}\}$;

(iii) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_k^2 f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{f_j\}$;

(iv) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}S_k^2 f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{f_{j,k}\}$.

LEMMA 3.2 ([G]). (a) *Let* $1 \le r < p < \infty$ *and* $w \in L^p$. *Then there exists a constant* $C_1 = C_1(n, r, p, [w]_{A_p})$ *such that for every nonnegative function* g *in* $L^{(p/r)'}(w)$ *there is a function* $G(g)$ *such that*

- (i) $g \le G(g)$,
- (ii) $\|G(g)\|_{L^{p/(p-r)}(w)} \le 2\|g\|_{L^{p/(p-r)}(w)}$,
- (iii) $[G(g)w]_{A_r} \le C_1$.

(b) *Let* $1 < p < r < \infty$ *and* $w \in L^p$. *Then there exists a constant* $C_2 = C_2(n, r, p, [w]_{A_p})$ *such that for every nonnegative function* h *in* $L^{p/(r-p)}(w)$, *there is a function* $H(h)$ *such that*

- (i) $h \le H(h)$,
- (ii) $\|H(h)\|_{L^{p/(r-p)}(w)} \le 2^{r-1}\|h\|_{L^{p/(r-p)}(w)}$,
- (iii) $[H(h)^{-1}w]_{A_r} \le C_2$.

Moreover, both constants $C_1(n, r, p, B)$ *and* $C_2(n, r, p, B)$ *increase as* B *increases.*

LEMMA 3.3. *Suppose that the assumptions of Lemma* 3.1 *hold. Then for* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *we have*

(i) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|[b,S_k]f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of $\{f_j\}$;

(ii) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}[b,S_k]f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of $\{f_{j,k}\}$;

(iii) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|[b,S_k^2]f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of $\{f_j\}$;

(iv) $$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}[b,S_k^2]f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of $\{f_{j,k}\}$.

Proof. We only need to prove (i). Without loss of generality, we may take the dual of $[b,S_j]$ as $[b,S_j]$. In fact, if (i) holds, then there exists a sequence $\{g_j\}\in L^{p'}(\ell^{q'})$ with $\|(\sum_{j\in\mathbb{Z}}|g_j|^{q'})^{1/q'}\|_{L^{p'}}\le 1$ such that

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}[b,S_k]f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}&=\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}[b,S_k]f_{j,k}(x)g_j(x)\,dx\Big|\\&=\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}f_{j,k}(x)[b,S_k]g_j(x)\,dx\Big|.\end{aligned}$$

Then by the Hölder inequality and (i), we get

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\Big|\sum_{k\in\mathbb{Z}}[b,S_k]f_{j,k}\Big|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\\&\quad\le\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|[b,S_k]g_j|^2\Big)^{q'/2}\Big)^{1/q'}\Big\|_{L^{p'}}\\&\quad\le\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\Big\|\Big(\sum_{j\in\mathbb{Z}}|g_j|^{q'}\Big)^{1/q'}\Big\|_{L^{p'}}\\&\quad\le\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|f_{j,k}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

This is (ii).

Now, we prove (i). If we get

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(w)} \le \Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(w)}, \tag{3.1}$$

then since $\{S_kf_j\}_{j,k}$ is a linear operator, Theorem 2.13 in [ABKP] yields (i). So it suffices to prove (3.1).

The proof is divided into three cases. We first consider the case of $1 < q < p < \infty$. It is well known (see [K]) that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_jf|^2\Big)^{1/2}\Big\|_{L^p(w)} \le C\|f\|_{L^p(w)}. \tag{3.2}$$

For $g \in L^{(p/q)'}(w)$, we let $G(|g|)$ be as in Lemma 3.2(a). By Lemma 3.2(a), (3.2), and Hölder's inequality, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(w)} = \Big\|\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}\Big\|^{1/q}_{L^{p/q}(w)} \\
&\quad= \sup_{\|g\|_{L^{(p/q)'}(w)}\le 1}\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j(x)|^2\Big)^{q/2}|g|(x)w(x)\,dx\Big|^{1/q} \\
&\quad\le \sup_{\|g\|_{L^{(p/q)'}(w)}\le 1}\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j(x)|^2\Big)^{q/2}G(|g|)(x)w(x)\,dx\Big|^{1/q} \\
&\quad\le C\sup_{\|g\|_{L^{(p/q)'}(w)}\le 1}\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|f_j(x)|^qG(|g|)(x)w(x)\,dx\Big|^{1/q} \\
&\quad\le C\sup_{\|g\|_{L^{(p/q)'}(w)}\le 1}\Big\|\sum_{j\in\mathbb{Z}}|f_j|^q\Big\|^{1/q}_{L^{p/q}(w)}\|G(|g|)\|^{1/q}_{L^{(p/q)'}(w)} \\
&\quad\le C\sup_{\|g\|_{L^{(p/q)'}(w)}\le 1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(w)}\||g|\|^{1/q}_{L^{(p/q)'}(w)} \\
&\quad\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(w)}.
\end{aligned} \tag{3.3}$$

The argument for $1 < p < q < \infty$ is similar. Given $w \in A_p$ and $(\sum_{j\in\mathbb{Z}}|f_j|^q)^{1/q} \in L^p(\mathbb{R}^n)$, we observe that

$$\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{q-p}{q}} \in L^{\frac{p}{q-p}}(w).$$

Applying Lemma 3.2(b), we have

$$H\Big(\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{q-p}{q}}\Big) \ge \Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{q-p}{q}}$$

and

$$\Big[H\Big(\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{q-p}{q}}\Big)^{-1}w\Big]_{A_r}\le C_2=C_2(n,p,q,[w]_{A_p}).$$

Then applying (3.2) and Hölder's inequality, we have, with $\Sigma:=\sum_{i\in\mathbb{Z}}|f_i|^q$,

$$\Big\|\Big(\Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p(w)}=\Big\|\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}\Big\|^{1/q}_{L^{p/q}(w)} \tag{3.4}$$

$$=\Big\|\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}H(\Sigma^{1-p/q})^{-1}H(\Sigma^{1-p/q})\Big\|^{1/q}_{L^{p/q}(w)}$$

$$\le\Big\|\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}H(\Sigma^{1-p/q})^{-1}\Big\|^{1/q}_{L^1(w)}\|H(\Sigma^{1-p/q})\|^{1/q}_{L^{p/(q-p)}(w)}$$

$$\le\Big(\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{q/2}H(\Sigma^{1-p/q})^{-1}w(x)\,dx\Big)^{1/q}\|H(\Sigma^{1-p/q})\|^{1/q}_{L^{p/(q-p)}(w)}$$

$$\le C\Big(\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}|f_j|^qH(\Sigma^{1-p/q})^{-1}w(x)\,dx\Big)^{1/q}\|\Sigma^{1-p/q}\|^{1/q}_{L^{p/(q-p)}(w)}$$

$$\le C\Big(\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|f_j|^q\Sigma^{p/q-1}w(x)\,dx\Big)^{1/q}\|\Sigma^{1/q}\|^{1-p/q}_{L^p(w)}$$

$$\le C\Big(\int_{\mathbb{R}^n}\Sigma^{p/q}w(x)\,dx\Big)^{1/q}\|\Sigma^{1/q}\|^{1-p/q}_{L^p(w)}$$

$$=C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(w)}.$$

For $p=q$, we get

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j|^2\Big)^{p/2}\Big)^{1/p}\Big\|^p_{L^p(w)} \tag{3.5}$$

$$=\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|S_kf_j(x)|^2\Big)^{p/2}w(x)\,dx$$

$$\le C\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{Z}}|f_j(x)|^2\Big)^{p/2}w(x)\,dx=C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{p/2}\Big)^{1/p}\Big\|^p_{L^p(w)}.$$

Combining (3.3)–(3.5), we get (3.1).

Obviously, (iii) and (iv) also hold by a similar proof to that for (i) and (ii). ∎

Lemma 3.4. *Let* $1 < p, q, r < \infty$, $\{f_j\} \in L^p(\ell^q)$, *and* $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$. *Then for* $1/r = 1/p - \alpha/n$,

$$(3.6)\qquad \Big\|\Big(\sum_{j\in\mathbb{Z}}(M_{\Omega,\alpha}f_j)^q\Big)^{1/q}\Big\|_{L^r} \le C\|\Omega\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{f_j\}$.

Proof. Since $\sup_{j\in\mathbb{Z}} M_{\Omega,\alpha}f_j \le CM_{\Omega,\alpha}(\sup_{j\in\mathbb{Z}}|f_j|)$, for $1/r = 1/p - \alpha/n$ we get

$$(3.7)\qquad \Big\|\sup_{j\in\mathbb{Z}} M_{\Omega,\alpha}f_j\Big\|_{L^r} \le C\|\Omega\|_{L^{\frac{n}{n-\alpha}}}\Big\|\sup_{j\in\mathbb{Z}}|f_j|\Big\|_{L^p}.$$

By duality,

$$(3.8)\qquad \Big\|\sum_{j\in\mathbb{Z}} M_{\Omega,\alpha}f_j\Big\|_{L^r} \le C\|\Omega\|_{L^{\frac{n}{n-\alpha}}}\Big\|\sum_{j\in\mathbb{Z}}|f_j|\Big\|_{L^p}.$$

Interpolating between (3.7) and (3.8), we get (3.6). ∎

Similarly to the proof of Lemma 2.4 in [CD1], applying Lemma 3.4, we get the following result.

Lemma 3.5. *Let* $0 < \alpha < n$, $1 < p, q, r < \infty$, $\{(\sum_k |g_{k,j}|^2)^{1/2}\}_j \in L^p(\ell^q)$, *and* $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$. *Denote* $\sigma_{k,\alpha}(x) = |x|^{\alpha-n}|\Omega(x')|\chi_{\{2^k<|x|\le 2^{k+1}\}}(x)$. *Then for* $1/r = 1/p - \alpha/n$,

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\sigma_{k,\alpha}*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r} \le C\|\Omega\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C *is independent of* $\{g_{k,j}\}$.

Similarly to the proof of the lemma in [H], we get

Lemma 3.6. *For* $0 < \alpha < 1$ *and* $0 < \delta < \infty$, *take* $m_{\alpha,\delta} \in C_0^\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp}(m_{\alpha,\delta}) \subset \{\delta/2 \le |\xi| \le \delta\}$. *Let* $T_{\alpha,\delta}$ *be the multiplier operator defined by*

$$\widehat{T_{\alpha,\delta}f}(\xi) = m_{\alpha,\delta}(\xi)\widehat{f}(\xi).$$

Moreover, for $b \in \mathrm{BMO}$ *and* $k \in \mathbb{N}$, *denote by*

$$T_{\alpha,\delta;b,k}f(x) = T_{\alpha,\delta}\big((b(x)-b(\cdot))^k f\big)(x)$$

the k*th order commutator of* $T_{\alpha,\delta}$. *If for some constants* $0 < \beta$, $\theta < 1$, *and* $-1 < \lambda < 1$, $m_{\alpha,\delta}$ *satisfies*

$$(3.9)\qquad |m_{\alpha,\delta}(\xi)| \le C(2^{-\omega}\delta)^{-\alpha}\min\{\delta^\theta, \delta^{-\beta}\},$$

$$(3.10)\qquad |\nabla m_{\alpha,\delta}(\xi)| \le C(2^{-\omega}\delta)^{-\alpha}\delta^\lambda,$$

then for any fixed $0 < v < 1$ *there exists a constant* $C = C(n, k, v, \alpha, \lambda, \beta) > 0$ *such that*

$$\|T_{\alpha,\delta;\,b,k} f\|_{L^2} \le C(2^{-\omega}\delta)^{-\alpha} \min\{\delta^{\theta v}, \delta^{-\beta v}\} \|b\|_{\mathrm{BMO}}^k \|f\|_{L^2}.$$

LEMMA 3.7 ([CD3]). *For the multiplier* G_k ($k \in \mathbb{Z}$) *defined in Section* 2 *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$|G_k b(x) - G_k b(y)| \le C \frac{|x-y|^\delta 2^{k\delta}}{\delta} \|b\|_{\mathrm{BMO}} \tag{3.11}$$

for any $0 < \delta < 1$, *where* C *is independent of* k *and* δ.

LEMMA 3.8 ([Da]). *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$, *the following properties hold:*

(i) $\Delta_j \Delta_i u \equiv 0$ *if* $|j - i| \ge 2$,
(ii) $\Delta_j (G_{i-3} \Delta_i u) \equiv 0$ *if* $|j - i| \ge 4$.

If we replace Δ_j *with* S_j, *the above equalities also hold.*

4. Proof of Theorem 1.9. The main idea of the proof of Theorem 1.9 is taken from [H]. Without loss of generality, we may assume that b is real-valued and $\|b\|_{\mathrm{BMO}} = C_1$. Define

$$H(b,f)(x) = \sup_{r>0} r^{-n+\alpha} \int_{|x-y|<r} e^{b(x)-b(y)} |f(y)|\, dy.$$

By the John–Nirenberg inequality, there exist positive constants A and B such that for any cube Q,

$$\frac{1}{|Q|} \int_Q \exp\Big(\frac{|b(x) - b_Q|}{A\|b\|_{\mathrm{BMO}}}\Big)\, dx \le B,$$

where b_Q is the mean value of b on the cube Q. Let $C_1 = A \max\{p', r\}$. Straightforward computation shows that for real-valued $b \in \mathrm{BMO}$ with $\|b\|_{\mathrm{BMO}} = C_1$,

$$\frac{1}{|Q|} \int_Q e^{r(b(x)-b_Q)}\, dx \le B, \qquad \frac{1}{|Q|} \int_Q e^{-p'(b(x)-b_Q)}\, dx \le B,$$

and so $e^{b(x)} \in A(p,r)$ ($1 < p, r < \infty$) (the Muckenhoupt weight class) with the $A(p,r)$ constant no more than B. By the weighted inequality for M_α (see [MW2]), where

$$M_\alpha f(x) = \sup_{r>0} r^{-n+\alpha} \int_{|x-y|<r} |f(y)|\, dy,$$

we have

$$\|H(b,f)\|_{L^r} \le C\|f\|_{L^p}. \tag{4.1}$$

Hu [H, p. 15] proved that for $0 < t_1, t_2 < \infty$,

$$t_1 t_2 \le C(t_1 \log(1+t_1) + e^{t_2}) \le C(t_1(1+\log^+ t_1) + e^{t_2}). \tag{4.2}$$

Let $\Phi(t) = t(1+\log^+ t)$ for $t > 0$. Then

$$\|\Phi(\Omega)\|_{L^{\frac{n}{n-\alpha}}(S^{n-1})} \le 1. \tag{4.3}$$

By (4.2), we have

$$\begin{aligned}
M_{\Omega,\alpha;b} f(x) &\le C \sup_{r>0} r^{-n+\alpha} \int\limits_{|x-y|<r} \Phi(\Omega(x-y))|f(y)|\,dy \\
&\quad + C \sup_{r>0} r^{-n+\alpha} \int\limits_{|x-y|<r} e^{|b(x)-b(y)|}|f(y)|\,dy \\
&\le C \sup_{r>0} r^{-n+\alpha} \int\limits_{|x-y|<r} \Phi(\Omega(x-y))|f(y)|\,dy \\
&\quad + C \sup_{r>0} r^{-n+\alpha} \int\limits_{|x-y|<r} e^{b(x)-b(y)}|f(y)|\,dy \\
&\quad + C \sup_{r>0} r^{-n+\alpha} \int\limits_{|x-y|<r} e^{b(y)-b(x)}|f(y)|\,dy \\
&=: I(f)(x) + II(f)(x) + III(f)(x).
\end{aligned}$$

(4.1) shows that

$$\|II(f)\|_{L^r} \le C\|f\|_{L^p}, \qquad \|III(f)\|_{L^r} \le C\|f\|_{L^p}.$$

On the other hand, by Theorem 1.4(i), we get

$$\|I(f)\|_{L^r} \le C\|f\|_{L^p}\|\Phi(\Omega)\|_{L^{n/(n-\alpha)}(S^{n-1})} \le C\|f\|_{L^p}.$$

Therefore,

$$\|M_{\Omega,\alpha;b} f\|_{L^r} \le C\|f\|_{L^p}.$$

This completes the proof of Theorem 1.9.

5. Proof of Theorem 1.14.

Recall that $0 < \alpha < 1$ and

$$[b, T_{\Omega,\alpha}]f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}(b(x)-b(y))f(y)\,dy.$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0 \le \phi \le 1$, $\operatorname{supp}(\phi) \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \phi^4(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$. Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| < 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d \le |\Omega(x')| < 2^{d+1}\}$ for positive integers d. For $d \ge 0$, let

$$\Omega_d(y') = \Omega(y')\chi_{E_d}(y').$$

Then $\Omega(y') = \sum_{d\ge 0} \Omega_d(y')$. Set

$$\sigma_{\alpha,j,d}(x) = \frac{\Omega_d(x')}{|x|^{n-\alpha}} \chi_{\{2^j\le |x|<2^{j+1}\}}(x)$$

for $j\in\mathbb{Z}$. Set

$$m_{\alpha,j,d}(\xi) = \widehat{\sigma_{\alpha,j,d}}(\xi), \quad m^l_{\alpha,j,d}(\xi) = m_{\alpha,j,d}(\xi)\phi(2^{j-l}\xi).$$

Define the operators $T_{\alpha,j,d}$ and $T^l_{\alpha,j,d}$ by

$$\widehat{T_{\alpha,j,d}f}(\xi) = m_{\alpha,j,d}(\xi)\widehat{f}(\xi), \quad \widehat{T^l_{\alpha,j,d}f}(\xi) = m^l_{\alpha,j,d}(\xi)\widehat{f}(\xi).$$

Denote by $[b, T_{\alpha,j,d}]$ and $[b, T^l_{\alpha,j,d}]$ the respective commutators. Define the operator $V_{\alpha,l,d}$ by

$$V_{\alpha,l,d}h(x) = \sum_{j\in\mathbb{Z}} [b, S_{l-j}T^l_{\alpha,j,d}S^2_{l-j}]h(x).$$

Then

$$[b, T_{\Omega,\alpha}]h(x) = \sum_{d\ge 0}\sum_{l\in\mathbb{Z}} V_{\alpha,l,d}h(x).$$

By the Minkowski inequality,

$$\Big\|\Big(\sum_{s\in\mathbb{Z}} |[b, T_{\Omega,\alpha}]f_s|^q\Big)^{1/q}\Big\|_{L^r} \le \sum_{d\ge 0}\sum_{l\in\mathbb{Z}} \Big\|\Big(\sum_{s\in\mathbb{Z}} |V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r}. \tag{5.1}$$

If we can prove that for some $0<\beta<1$,

$$\Big\|\Big(\sum_{s\in\mathbb{Z}} |V_{\alpha,l,d}f_s|^2\Big)^{1/2}\Big\|_{L^2} \le C\|\Omega_d\|_{L^\infty}2^{-\beta|l|}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}} |f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}, \tag{5.2}$$

and for $1<p,r,q<\infty$, $1/r = 1/p - \alpha/n$,

$$\Big\|\Big(\sum_{s\in\mathbb{Z}} |V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r} \le C|l|\,\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}} |f_s|^q\Big)^{1/q}\Big\|_{L^p}, \tag{5.3}$$

where C is independent of l and f, then we get Theorem 1.14. Indeed, taking $q=2$ in (5.3), then interpolating between (5.2) and (5.3), for $1<p,r<\infty$, $1/r=1/p-\alpha/n$, and $0<\theta_1<1$ we get

$$\begin{aligned}\Big\|\Big(\sum_{s\in\mathbb{Z}} |V_{\alpha,l,d}f_s|^2\Big)^{1/2}\Big\|_{L^r} \\ \le C|l|^{1-\theta_1}2^{-\theta_1\beta|l|}\|\Omega_d\|_{L^\infty}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}} |f_s|^2\Big)^{1/2}\Big\|_{L^p}.\end{aligned} \tag{5.4}$$

Next, interpolating between (5.3) and (5.4) again, for $1<p,q,r<\infty$, $1/r=1/p-\alpha/n$, and $0<\theta_2<1$ we get

$$\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r} \le C|l|^{1-\theta_1\theta_2}2^{-\theta_1\theta_2\beta|l|}\|\Omega_d\|_{L^\infty}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^p}. \tag{5.5}$$

Take a large positive integer N such that $N > 2(\beta\theta_1\theta_2)^{-1}$. Then

$$\begin{aligned}&\sum_{d\ge0}\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r}\\&\le \sum_{d\ge0}\sum_{Nd<|l|}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r}+\sum_{d\ge0}\sum_{0\le|l|\le Nd}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r}\\&=: J_1+J_2.\end{aligned}$$

For J_1, using (5.5), we get

$$\begin{aligned}J_1 &\le C\|b\|_{\mathrm{BMO}}\sum_{d\ge0}2^d\sum_{|l|>Nd}2^{-\beta\theta_1\theta_2|l|}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}\\&\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

Finally, by (5.3),

$$\begin{aligned}J_2 &\le C\|b\|_{\mathrm{BMO}}\sum_{d\ge0}\sum_{0<|l|<Nd}|l|2^d(\sigma(E_d))^{\frac{n-\alpha}{n}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}\\&\le C\|b\|_{\mathrm{BMO}}\sum_{d\ge0}Nd^22^d(\sigma(E_d))^{\frac{n-\alpha}{n}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}\\&\le C\|\Omega\|_{L^{\frac{n}{n-\alpha}}(\log^+L)^2}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

Combining the estimates of J_1 and J_2, we get

$$\sum_{d\ge0}\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r} \le C(1+\|\Omega\|_{L^{\frac{n}{n-\alpha}}(\log^+L)^2})\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}. \tag{5.6}$$

From (5.1) and (5.6) it follows that

$$\Big\|\Big(\sum_{s\in\mathbb{Z}}|[b,T_{\Omega,\alpha}]f_s|^q\Big)^{1/q}\Big\|_{L^r} \le C(1+\|\Omega\|_{L^{\frac{n}{n-\alpha}}(\log^+L)^2})\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p},$$

as desired.

It remains to prove (5.2) and (5.3).

The proof of (5.2). Write

$$[b,S_{l-j}^2T_{\alpha,j,d}^lS_{l-j}]f_s = [b,S_{l-j}^2]T_{\alpha,j,d}^lS_{l-j}f_s + S_{l-j}^2[b,T_{\alpha,j,d}^l]S_{l-j}f_s + S_{l-j}^2T_{\alpha,j,d}^l[b,S_{l-j}]f_s.$$

Then

$$\begin{aligned}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^2\Big)^{1/2}\Big\|_{L^2}^2 &= \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}[b,S_{l-j}^2]T_{\alpha,j,d}^lS_{l-j}f_s\Big|^2\Big)^{1/2}\Big\|_{L^2}^2\\ &\quad+\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S_{l-j}^2[b,T_{\alpha,j,d}^l]S_{l-j}f_s\Big|^2\Big)^{1/2}\Big\|_{L^2}^2\\ &\quad+\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S_{l-j}^2T_{\alpha,j,d}^l[b,S_{l-j}]f_s\Big|^2\Big)^{1/2}\Big\|_{L^2}^2\\ &:= I_1+I_2+I_3.\end{aligned}$$

To estimate I_1, I_2 and I_3, we claim that for any fixed $0<\theta<1$, there exists $0<v<1$ such that

$$\|T_{\alpha,j,d}^lh\|_{L^2}\le C2^{-(l-j)\alpha}\|\Omega_d\|_{L^\infty}\min\{2^{\alpha vl},2^{-\theta vl}\}\|h\|_{L^2}, \tag{5.7}$$

$$\|[b,T_{\alpha,j,d}^lh]\|_{L^2}\le C2^{-(l-j)\alpha}\|\Omega_d\|_{L^\infty}\min\{2^{\alpha vl},2^{-\theta vl}\}\|b\|_{\mathrm{BMO}}\|h\|_{L^2}. \tag{5.8}$$

The proofs of (5.7) and (5.8) will be given later.

From Plancherel's theorem, it is easy to see that for $1<p<\infty$,

$$\|f\|_{L^p_{-\alpha}}=\|(|\cdot|^{-\alpha}\widehat{f})^\vee\|_{L^p}=\|I_\alpha f\|_{L^p},$$

and for $1<p<n/\alpha$ and $1/r=1/p-\alpha/n$,

$$\Big\|\Big(\sum_{s\in\mathbb{Z}}|I_\alpha g_s|^2\Big)^{1/2}\Big\|_{L^r}\le C\Big\|\Big(\sum_{s\in\mathbb{Z}}|g_s|^2\Big)^{1/2}\Big\|_{L^p}. \tag{5.9}$$

Now we estimate I_1, I_2 and I_3 separately. By Lemma 3.3(iv) with $q=2$, (5.7) and (5.9), we have

$$\begin{aligned}I_1&=C\|b\|_{\mathrm{BMO}}^2\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\|T_{\alpha,j,d}^lS_{l-j}^2f_s\|_{L^2}^2 \\ &\le C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta v}\}\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-2(l-j)\alpha}\|S_{l-j}f_s\|_{L^2}^2\\ &=C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\sum_{s\in\mathbb{Z}}\Big\|\Big(\sum_{j\in\mathbb{Z}}|2^{-j\alpha}S_jf_s|^2\Big)^{1/2}\Big\|_{L^2}^2\end{aligned} \tag{5.10}$$

$$\begin{aligned}&\le C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\sum_{s\in\mathbb{Z}}\|f_s\|_{L^2_{-\alpha}}^2\\&=C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\sum_{s\in\mathbb{Z}}\|I_\alpha f_s\|_{L^2}^2\\&=C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\Big\|\Big(\sum_{s\in\mathbb{Z}}|I_\alpha f_s|^2\Big)^{1/2}\Big\|_{L^2}^2\\&\le C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}^2.\end{aligned}$$

Similarly, applying Lemma 3.1(iv) with $q=2$, (5.8) and (5.9), we get

$$I_2\le C\|b\|_{\mathrm{BMO}}^2\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}^2. \tag{5.11}$$

Finally, we estimate I_3. By (5.7),

$$\begin{aligned}I_3&\le C\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-2(l-j)\alpha}\int_{\mathbb{R}^n}|S_{l-j}[b,S_{l-j}]f_s(x)|^2\,dx\\&=C\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|2^{-j\alpha}S_j[b,S_j]f_s|^2\Big)^{1/2}\Big\|_{L^2}^2\\&=:C\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}P^2.\end{aligned}$$

If we can prove

$$P\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}, \tag{5.12}$$

then we get

$$I_3\le C\|\Omega_d\|_{L^\infty}^2\min\{2^{2\alpha vl},2^{-2\theta vl}\}\|b\|_{\mathrm{BMO}}^2\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}^2. \tag{5.13}$$

Now we prove (5.12). There exists a sequence $\{g_{j,s}\}\in L^2(\ell^2(\ell^2))$ with $\|(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|g_{j,s}|^2)^{1/2}\|_{L^2}\le 1$ such that

$$\begin{aligned}P&=\Big|\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-j\alpha}\int_{\mathbb{R}^n}S_j[b,S_j]f_s(x)g_{j,s}(x)\,dx\Big|\\&=\Big|\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}[b,S_j]f_s(x)2^{-j\alpha}S_jg_{j,s}(x)\,dx\Big|.\end{aligned}$$

Then by Hölder's inequality and Lemma 3.3(i) with $q=2$, we get

$$\begin{aligned}P&\le C\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|[b,S_j]f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|2^{-j\alpha}S_jg_{j,s}|\Big)^{1/2}\Big\|_{L^{\frac{2n}{n-2\alpha}}}\\&\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|2^{-j\alpha}S_jg_{j,s}|\Big)^{1/2}\Big\|_{L^{\frac{2n}{n-2\alpha}}}.\end{aligned}$$

Let $\theta(\xi) = |\xi|\phi(\xi)$, and let S_j^θ be the Littlewood–Paley operator associated with the bump $\theta(2^{-j}\xi)$; then $2^{-j\alpha}\widehat{S_j g_{j,s}}(\xi) = \theta(2^{-j}\xi)|\xi|^{-\alpha}\widehat{g_{j,s}}(\xi) = \theta(2^{-j}\xi)\widehat{I_\alpha g_{j,s}}(\xi)$. Hence (5.9) implies that

$$\begin{aligned}
P &\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|I_\alpha S_j^\theta g_{j,s}|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n-2\alpha}}}\\
&\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|S_j^\theta g_{j,s}|^2\Big)^{1/2}\Big\|_{L^2}\\
&\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|g_{j,s}|^2\Big)^{1/2}\Big\|_{L^2}\\
&\le C\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^2\Big)^{1/2}\Big\|_{L^{\frac{2n}{n+2\alpha}}},
\end{aligned}$$

which is (5.12). Combining (5.10)–(5.13), we get (5.2).

Now we return to the proof of (5.7) and (5.8). Recall that

$$\widehat{T_{\alpha,j,d}f}(\xi) = m_{\alpha,j,d}(\xi)\widehat{f}(\xi), \quad \widehat{T^l_{\alpha,j,d}f}(\xi) = m^l_{\alpha,j,d}(\xi)\widehat{f}(\xi),$$

where

$$m_{\alpha,j,d}(\xi) = \widehat{\sigma_{\alpha,j,d}}(\xi), \quad m^l_{\alpha,j,d}(\xi) = m_{\alpha,j,d}(\xi)\phi(2^{j-l}\xi).$$

We define $\widehat{\widetilde{T}^l_{\alpha,j,d}h}(\xi) = m^l_{\alpha,j,d}(2^{-j}\xi)\widehat{h}(\xi)$ and denote by $[b,\widetilde{T}^l_{\alpha,j,d}]$ the relevant commutator.

Since

$$m_{\alpha,j,d}(\xi) = \int_{S^{n-1}}\Omega_d(x')\int_{2^j}^{2^{j+1}} e^{-2\pi i r x'\cdot\xi}\frac{dr}{r^{1-\alpha}}\,d\sigma(x'),$$

it is easy to get

$$|m_{\alpha,j,d}(\xi)| \le C\|\Omega_d\|_{L^\infty}2^{\alpha j}, \tag{5.14}$$

$$|\nabla m_{\alpha,j,d}(\xi)| \le C2^{(\alpha+1)j}\|\Omega_d\|_{L^\infty}. \tag{5.15}$$

On the other hand, there exists $0<\alpha<\eta<1$ such that

$$\Big|\int_{2^j}^{2^{j+1}} e^{-2\pi i r x'\cdot\xi}\frac{dr}{r^{1-\alpha}}\Big| \le C2^{\alpha j}|x'\cdot\xi'|^{-\eta}|2^j\xi|^{-\eta}. \tag{5.16}$$

Then by (5.16) we get

$$\begin{aligned}
|m_{\alpha,j,d}(\xi)| &\le C2^{\alpha j}|2^j\xi|^{-\eta}\|\Omega_d\|_{L^\infty}\int_{S^{n-1}}|x'\cdot\xi'|^{-\eta}\,d\sigma(x') \qquad (5.17)\\
&\le C|\xi|^{-\alpha}|2^j\xi|^{-(\eta-\alpha)}\|\Omega_d\|_{L^\infty}.
\end{aligned}$$

Since

$$\operatorname{supp}(m^l_{\alpha,j,d}(2^{-j}\cdot)) \subset \{2^{l-1} \le |\xi| \le 2^{l+1}\}, \tag{5.18}$$

by (5.17) and (5.18) we get

$$\|m^l_{\alpha,j,d}(2^{-j}\cdot)\|_{L^\infty} \le C2^{\alpha(j-l)} \min\{2^{\alpha l}, 2^{-\theta l}\}\|\Omega_d\|_{L^\infty(S^{n-1})}, \tag{5.19}$$

where $\theta = \eta - \alpha$. Since

$$\nabla m^l_{\alpha,j,d}(2^{-j}\xi) = \psi(2^{-l}\xi)\nabla m_{\alpha,j,d}(2^{-j}\xi) + m_{\alpha,j,d}(2^{-j}\xi)\nabla\psi(2^{-l}\xi),$$

by (5.14) and (5.15) we get

$$\|\nabla m^l_{\alpha,j,d}(2^{-j}\cdot)\|_{L^\infty} \le C2^{\alpha(j-l)}2^{|l|}\|\Omega_d\|_{L^\infty(S^{n-1})}. \tag{5.20}$$

From (5.18)–(5.20), using Lemma 3.6 with $\delta = 2^l$, we find that for any fixed $0 < v < 1$,

$$\|\widetilde{T}^l_{\alpha,j,d}h\|_{L^2} \le C\|\Omega_d\|_{L^\infty}2^{-(l-j)\alpha}\min\{2^{\alpha vl}, 2^{-\theta vl}\}\|h\|_{L^2},$$
$$\|[b, \widetilde{T}^l_{\alpha,j,d}]h\|_{L^2} \le C\|\Omega_d\|_{L^\infty}2^{-(l-j)\alpha}\min\{2^{\alpha vl}, 2^{-\theta vl}\}\|b\|_{\mathrm{BMO}}\|h\|_{L^2},$$

where C is independent of j and l. Let $\beta = \min\{\alpha v, \theta v\}$; then by dilation invariance, we get (5.7) and (5.8).

The proof of (5.3). Since $T^l_{\alpha,j,d}S_{l-j} = T_{\alpha,j,d}S^2_{l-j}$ for any $j, l \in \mathbb{Z}$, we may write

$$[b, S^2_{l-j}T^l_{\alpha,j,d}S_{l-j}]f$$
$$= [b, S^2_{l-j}](T_{\alpha,j,d}S^2_{l-j}f) + S^2_{l-j}[b, T_{\alpha,j,d}](S^2_{l-j}f) + S^2_{l-j}T_{\alpha,j,d}([b, S^2_{l-j}]f).$$

Thus,

(5.21)
$$\begin{aligned}\Big\|\Big(\sum_{s\in\mathbb{Z}}|V_{\alpha,l,d}f_s|^q\Big)^{1/q}\Big\|_{L^r} &\le \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}[b, S^2_{l-j}](T_{\alpha,j,d}S^2_{l-j}f_s)\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &\quad+ \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}T_{\alpha,j,d}([b, S^2_{l-j}]f_s)\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &\quad+ \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}[b, T_{\alpha,j,d}](S^2_{l-j}f_s)\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &=: L_1 + L_2 + L_3.\end{aligned}$$

Below we estimate L_i for $i = 1, 2, 3$ separately. By Lemmas 3.3(iv), 3.5 and 3.1(iii), for $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$ we have

$$L_1 \le C\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|T_{\alpha,j,d}S^2_{l-j}f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r}$$
$$\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|S^2_{l-j}f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}$$
$$\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.$$

Similarly,

$$L_2 \le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.$$

Hence, by (5.21), to show (5.3) it remains to estimate L_3. We will apply Bony's paraproducts. Since

$$fg = \pi_f(g) + \pi_g(f) + R(f,g),$$

we have

$$b(x)(T_{\alpha,j,d}S^2_{l-j}f_s)(x)$$
$$= \pi_{(T_{\alpha,j,d}S^2_{l-j}f_s)}(b)(x) + R(b, T_{\alpha,j,d}S^2_{l-j}f_s)(x) + \pi_b(T_{\alpha,j,d}S^2_{l-j}f_s)(x)$$

and

$$bS^2_{l-j}f_s(x) = \pi_{(S^2_{l-j}f_s)}(b)(x) + R(b, S^2_{l-j}f_s)(x) + \pi_b(S^2_{l-j}f_s)(x).$$

Hence

$$\begin{aligned}[b,T_{\alpha,j,d}]S^2_{l-j}f_s(x) &= b(x)(T_{\alpha,j,d}S^2_{l-j}f_s)(x) - T_{\alpha,j,d}(bS^2_{l-j}f_s)(x)\\ &= [\pi_{(T_{\alpha,j,d}S^2_{l-j}f_s)}(b)(x) - T_{\alpha,j,d}(\pi_{(S^2_{l-j}f_s)}(b))(x)]\\ &\quad + [R(b,T_{\alpha,j,d}S^2_{l-j}f_s)(x) - T_{\alpha,j,d}(R(b,S^2_{l-j}f_s))(x)]\\ &\quad + [\pi_b(T_{\alpha,j,d}S^2_{l-j}f_s)(x) - T_{\alpha,j,d}(\pi_b(S^2_{l-j}f_s))(x)].\end{aligned}$$

Thus,

(5.22)

$$\begin{aligned}L_3 &\le \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}\big[\pi_{(T_{\alpha,j,d}S^2_{l-j}f_s)}(b) - T_{\alpha,j,d}(\pi_{(S^2_{l-j}f_s)}(b))\big]\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &\quad + \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}\big[R(b,T_{\alpha,j,d}S^2_{l-j}f_s) - T_{\alpha,j,d}(R(b,S^2_{l-j}f_s))\big]\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &\quad + \Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}\big[\pi_b(T_{\alpha,j,d}S^2_{l-j}f_s) - T_{\alpha,j,d}(\pi_b(S^2_{l-j}f_s))\big]\Big|^q\Big)^{1/q}\Big\|_{L^r}\\ &=: M_1 + M_2 + M_3.\end{aligned}$$

The estimate of M_1. Recall that $\pi_g(f)=\sum_{j\in\mathbb{Z}}(\Delta_j f)(G_{j-3}g)$. By Lemma 3.8(i), we know that $\Delta_i S_k g=0$ for $g\in\mathcal{S}'(\mathbb{R}^n)$ when $|i-k|\geq 3$. Then

$$\begin{aligned}(5.23)\quad &\pi_{(T_{\alpha,j,d}S^2_{l-j}f_s)}(b)(x)-T_{\alpha,j,d}(\pi_{(S^2_{l-j}f_s)}(b))(x)\\ &=\sum_{|i-(l-j)|\leq 2}\{\Delta_i(T_{\alpha,j,d}S^2_{l-j}f_s)(x)(G_{i-3}b)(x)-T_{\alpha,j,d}[(\Delta_i S^2_{l-j}f_s)(G_{i-3}b)](x)\}\\ &=\sum_{|i-(l-j)|\leq 2}[G_{i-3}b,T_{\alpha,j,d}](\Delta_i S^2_{l-j}f_s)(x).\end{aligned}$$

Thus,

$$(5.24)\quad M_1\leq\sum_{|k|\leq 2}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}\big([G_{l-j+k-3}b,T_{\alpha,j,d}](\Delta_{l-j+k}S^2_{l-j}f_s)\big)\Big|^q\Big)^{1/q}\Big\|_{L^r}.$$

Without loss of generality, we may assume $k=0$. By Lemma 3.1(iv),

$$(5.25)\quad M_1\leq C\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|[G_{l-j-3}b,T_{\alpha,j,d}](\Delta_{l-j}S^2_{l-j}f_s)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r}.$$

Note that

$$\begin{aligned}&|[G_{l-j-3}b,T_{\alpha,j,d}](\Delta_{l-j}S^2_{l-j}f_s)(x)|\\ &=\Big|\int_{2^j\leq|x-y|<2^{j+1}}\frac{\Omega_d(x-y)}{|x-y|^{n-\alpha}}(G_{l-j-3}b(x)-G_{l-j-3}b(y))\Delta_{l-j}S^2_{l-j}f_s(y)\,dy\Big|\\ &\leq C\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega_d(x-y)|}{|x-y|^{n-\alpha}}|G_{l-j-3}b(x)-G_{l-j-3}b(y)|\,|\Delta_{l-j}S^2_{l-j}f_s(y)|\,dy.\end{aligned}$$

By Lemma 3.7, for any $0<\delta<1$ we have

$$\begin{aligned}(5.26)\quad &|[G_{l-j-3}b,T_{\alpha,j,d}]\Delta_{l-j}S^2_{l-j}f_s(x)|\\ &\leq C2^{(l-j-3)\delta}\frac{|x-y|^\delta}{\delta}\|b\|_{\mathrm{BMO}}\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega_d(x-y)|}{|x-y|^{n-\alpha}}|\Delta_{l-j}S^2_{l-j}f_s(y)|\,dy\\ &\leq C\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega_d(x-y)|}{|x-y|^{n-\alpha}}|\Delta_{l-j}S^2_{l-j}f_s(y)|\,dy\\ &=C\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}T_{|\Omega|,\alpha,j,d}(|\Delta_{l-j}S^2_{l-j}f_s|)(x),\end{aligned}$$

where

$$T_{|\Omega|,\alpha,j,d}f_s(x)=\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega_d(x-y)|}{|x-y|^{n-\alpha}}f_s(y)\,dy$$

and C is independent of δ and l. Then, by (5.25), (5.26), and Lemmas 3.5

and 3.1(iii), for $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$ we have

$$(5.27)\quad M_1 \le C\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|T_{|\Omega|,\alpha,j,d}(|\Delta_{l-j}S^2_{l-j}f_s|)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r}$$
$$\le C\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|\Delta_{l-j}S^2_{l-j}f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}$$
$$\le C\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|S^2_{l-j}f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}$$
$$\le C\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\frac{2^{l\delta}}{\delta}\|b\|_{\mathrm{BMO}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of l and δ.

The estimate of M_2. By Lemma 3.8(i), we know that for $|k| \le 2$, $\Delta_{i+k}S_{l-j}g = 0$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ when $|i-(l-j)| \ge 5$. Thus

$$\begin{aligned}
&R(b, T_{\alpha,j,d}S^2_{l-j}f_s) - T_{\alpha,j,d}(R(b, S^2_{l-j}f_s))(x)\\
&\quad= \sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)(x)(T_{\alpha,j,d}\Delta_{i+k}S^2_{l-j}f_s)(x)\\
&\qquad - T_{\alpha,j,d}\Big(\sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)(\Delta_{i+k}S^2_{l-j}f_s)\Big)(x)\\
&\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 4}\Big((\Delta_i b)(x)(T_{\alpha,j,d}\Delta_{i+k}S^2_{l-j}f_s)(x)\\
&\qquad\qquad - T_{\alpha,j,d}\big((\Delta_i b)(\Delta_{i+k}S^2_{l-j}f_s)\big)(x)\Big)\\
&\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 4}[\Delta_i b, T_{\alpha,j,d}](\Delta_{i+k}S^2_{l-j}f_s)(x).
\end{aligned}$$

Hence

$$M_2 \le \sum_{|k|\le 6}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big|\sum_{j\in\mathbb{Z}}S^2_{l-j}[\Delta_{l-j+k}b, T_{\alpha,j,d}](\Delta_{l-j+k}S^2_{l-j}f_s)\Big|^q\Big)^{1/q}\Big\|_{L^r}.$$

Without loss of generality, we may assume that $k = 0$. By the equality above and using Lemma 3.1(iv), the inequality $\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty} \le C\|b\|_{\mathrm{BMO}}$ (see [G]), and Lemmas 3.5 and 3.1(iii), for $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$ we have

(5.28)

$$M_2 \le C\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|T_{|\Omega|,\alpha,j,d}(|\Delta_{l-j}S^2_{l-j}f_s|)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r}$$

$$\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|\Delta_{l-j}S_{l-j}^2 f_s|^2\Big)^{q/2}\Big)^{1/r}\Big\|_{L^p}$$
$$\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|S_{l-j}^2 f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}$$
$$\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.$$

The estimate of M_3. By Lemma 3.8(ii), we know $S_j\big(\Delta_i g G_{i-3}h\big)=0$ for $g,h\in\mathcal{S}'(\mathbb{R}^n)$ if $|j-i|\ge 5$. We get

$$\begin{aligned}
&S_{l-j}^2\big(\pi_b(T_{\alpha,j,d}S_{l-j}^2 f_s)-T_{\alpha,j,d}(\pi_b(S_{l-j}^2 f_s))\big)\\
&\quad= S_{l-j}^2\Big(\sum_{i\in\mathbb{Z}}(\Delta_i b)(G_{i-3}T_{\alpha,j,d}S_{l-j}^2 f_s)-T_{\alpha,j,d}\Big(\sum_{i\in\mathbb{Z}}(\Delta_i b)(G_{i-3}S_{l-j}^2 f_s)\Big)\Big)(x)\\
&\quad= \sum_{|i-(l-j)|\le 4}\big\{S_{l-j}^2\big((\Delta_i b)(G_{i-3}T_{\alpha,j,d}S_{l-j}^2 f_s)\big)(x)\\
&\qquad\qquad - S_{l-j}^2 T_{\alpha,j,d}\big((\Delta_i b)(G_{i-3}S_{l-j}^2 f_s)\big)(x)\big\}.
\end{aligned}$$

Thus, from Lemma 3.1(iv), $\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty}\le C\|b\|_{\mathrm{BMO}}$, and Lemmas 3.5 and 3.1(iii), for $1<p<n/\alpha$ and $1/r=1/p-\alpha/n$ we get

(5.29)
$$\begin{aligned}
M_3 &\le C\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|T_{|\Omega|,\alpha,j,d}(|G_{l-j}S_{l-j}^2 f_s|)|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^r}\\
&\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|G_{l-j}S_{l-j}^2 f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\\
&\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}|S_{l-j}^2 f_s|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\\
&\le C\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}.
\end{aligned}$$

By (5.22) and (5.27)–(5.29), we get

$$L_3\le C\max\{2,2^{\delta l}/\delta\}\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}$$

for $l\in\mathbb{Z}$, where C is independent of δ and l. If $l>0$, we take $\delta=1/l$; then

$$L_3\le C|l|\,\|b\|_{\mathrm{BMO}}\|\Omega_d\|_{L^{\frac{n}{n-\alpha}}}\Big\|\Big(\sum_{s\in\mathbb{Z}}|f_s|^q\Big)^{1/q}\Big\|_{L^p}$$

for $l\in\mathbb{Z}$, where C is independent of l. This gives (5.3).

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