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Power boundedness in Banach algebras associated with locally compact groups

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Abstract. Let G be a locally compact group and $B(G)$ the Fourier–Stieltjes algebra of G. Pursuing our investigations of power bounded elements in $B(G)$, we study the extension property for power bounded elements and discuss the structure of closed sets in the coset ring of G which appear as 1-sets of power bounded elements. We also show that L^1-algebras of noncompact motion groups and of noncompact IN-groups with polynomial growth do not share the so-called power boundedness property. Finally, we give a characterization of power bounded elements in the reduced Fourier–Stieltjes algebra of a locally compact group containing an open subgroup which is amenable as a discrete group.

Introduction. An element a of a Banach algebra A is said to be *power bounded* if $\sup_{n\in\mathbb{N}} \|a^n\| < \infty$. Power bounded elements in Banach algebras, especially power bounded operators on Banach spaces, have been studied by several authors, with emphasis on the impact on spectra [1], [23], [30], [31]. Power boundedness of measures on the real line has first been dealt with in [4] (see also [2] and [5] for related problems). For general locally compact abelian groups G, the most comprehensive work on power boundedness in the measure algebra $M(G)$ and the L^1-algebra $L^1(G)$ is due to Schreiber [34]. Actually, [34] has substantially influenced and inspired our previous investigations [20]–[22] on power boundedness in Fourier and Fourier–Stieltjes algebras of locally compact groups, and also the present study. Naturally, for nonabelian groups the proofs turn out to be much more involved.

In [20, Theorem 4.1] we have shown that if G is an arbitrary locally compact group and u is any power bounded element of $B(G)$, then the sets

$$E_u = \{x \in G : |u(x)| = 1\} \quad \text{and} \quad F_u = \{x \in G : u(x) = 1\}$$

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belong to the closed coset ring $\mathcal{R}_c(G)$ of G. These sets are important in the study of power boundedness in $B(G)$ as is, for instance, demonstrated by the structure theorem [22, Theorem 4.5]. The first purpose of this paper is to address the question of which sets in $\mathcal{R}_c(G)$ arise in this manner from power bounded elements in $B(G)$. As outlined in Section 1, this appears to be a very difficult problem. Our main result (Proposition 1.3) concerns SIN-groups and generalizes the corresponding result for locally compact abelian groups [34, Theorem 6.20].

If H is a closed subgroup of G, then functions in $A(H)$ always extend to functions in $A(G)$, whereas for $B(H)$ and $B(G)$ this is far from being true in general. In Section 2 we study the extension problem for power bounded elements. For an arbitrary locally compact group G, denoting by G_0 the connected component of the identity of G, we show that every power bounded element of $B(G_0)$ admits a power bounded extension in $B(G)$ with the same norm (Theorem 2.5). Also, if D is a discrete subgroup of G, then every power bounded function in $A(D)$ extends to some such function in $A(G)$.

Note that every power bounded element has spectral radius ≤ 1. A Banach algebra A is said to have the *power boundedness property* (pb-property) if conversely each element of A with spectral radius ≤ 1 is power bounded. Improving on a result of [34], we show that if G is an IN-group of polynomial growth, then $L^1(G)$ has the pb-property only if G is compact and abelian (Theorem 3.4). The same conclusion is true for general motion groups, i.e. semidirect products $N \rtimes K$, where N is abelian and K is compact (Theorem 3.3).

Finally, extending [34, Theorem 6.22], we establish in Section 4 a characterization of power bounded elements in $B_\lambda(G)$, the dual space of the reduced group C^*-algebra $C^*_\lambda(G)$, for locally compact groups G which have an open subgroup which is amenable as a discrete group.

Preliminaries. Let G be a locally compact group. The Fourier–Stieltjes algebra and the Fourier algebra of G, $B(G)$ and $A(G)$, have been introduced and extensively studied by Eymard in his seminal article [8]. The space $B(G)$ is the linear span of the set $P(G)$ of all continuous positive definite functions on G and can be identified with the Banach space dual of the group C^*-algebra $C^*(G)$. For $u \in B(G)$ and $f \in L^1(G)$, the pairing is given by $\langle u, f\rangle = \int_G f(x)u(x)\,dx$. With pointwise multiplication and the dual space norm, $B(G)$ is a commutative Banach algebra. Every $u \in B(G)$ is the coefficient function of some unitary representation of G. More precisely, given $u \in B(G)$, there exist a unitary representation π of G and $\xi, \eta \in H(\pi)$, the Hilbert space of π, such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for all $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$ [8, Lemme 2.14].

The Fourier algebra $A(G)$ is the closed ideal of $B(G)$ generated by all compactly supported functions on $B(G)$. When G is abelian and $\widehat{G}$ denotes the dual group of G, then via the Fourier and the Fourier–Stieltjes transforms, $A(G)$ and $B(G)$ are isometrically isomorphic to the convolution algebra $L^1(\widehat{G})$ and the measure algebra $M(\widehat{G})$, respectively. For all this, see [8].

1. On the set $\{E \in \mathcal{R}_c(G) : E = F_u$ for some power bounded $u \in B(G)\}$. For any group H, the *coset ring* $\mathcal{R}(H)$ is the Boolean ring generated by all cosets of subgroups of H. If H is a topological group, then the *closed coset ring* $\mathcal{R}_c(H)$ is defined to be

$$\mathcal{R}_c(H) = \{E \in \mathcal{R}(H) : E \text{ is closed in } H\}.$$

For a locally compact abelian group G, the elements of $\mathcal{R}_c(G)$ have been completely described by Gilbert [11] and Schreiber [35]. Forrest [9] verified that the analogous description is valid for arbitrary locally compact groups G. A subset E of G belongs to $\mathcal{R}_c(G)$ if and only if E is of the form

$$E = \bigcup_{i=1}^{n} \Big(x_i H_i \setminus \bigcup_{j=1}^{n_i} y_{ij} K_{ij} \Big), \tag{1.1}$$

where $x_i, y_{ij} \in G$, H_i is a closed subgroup of G and each K_{ij} is either empty or an open subgroup of H_i, $n, n_i \in \mathbb{N}_0$, $1 \le i \le n$, $1 \le j \le n_i$. Moreover, every compact set in $\mathcal{R}(G)$ is a finite union of cosets of some subgroup of G. For examples, see [11] and [33].

For any locally compact group G, let $\mathcal{E}_{\mathrm{pb}}(G)$ and $\mathcal{F}_{\mathrm{pb}}(G)$ denote the collection of sets of the form E_u and F_u, respectively, where u is a power bounded element of $B(G)$. Then $\mathcal{E}_{\mathrm{pb}}(G) = \mathcal{F}_{\mathrm{pb}}(G)$ since $F_u = E_{\frac{1}{2}(1+u)} = F_{\frac{1}{2}(1+u)}$ [20, Proposition 3.5] and $\mathcal{F}_{\mathrm{pb}}(G) \subseteq \mathcal{R}_c(G)$ [20, Theorem 4.1]. The interesting question of which sets in $\mathcal{R}_c(G)$ belong to $\mathcal{F}_{\mathrm{pb}}(G)$ has been addressed in [34, Section 6] for abelian groups. Even in this case, the problem is far away from admitting a complete solution. In fact, in [34, Remark on p. 421] it is mentioned that it is unknown whether the sets $\mathbb{Z} \cup \alpha\mathbb{Z}$, α irrational, and $\{(z, w) \in \mathbb{T}^2 : z = 1 \text{ or } w = 1\}$ belong to $\mathcal{F}_{\mathrm{pb}}(\mathbb{R})$ or $\mathcal{F}_{\mathrm{pb}}(\mathbb{T}^2)$, respectively.

In this section we extend the result of [34] to SIN-groups. However, before doing so, we briefly discuss amenable groups G to point out that for such G it suffices to consider closed subsets of G with empty interior.

LEMMA 1.1. *Let G be an amenable locally compact group and let $E \in \mathcal{R}_c(G)$.*

(i) *The interior E° and the boundary ∂E of E both belong to $\mathcal{R}_c(G)$.*
(ii) *$E^\circ = F_u$ for some power bounded $u \in B(G)$.*

Proof. By [10, Lemma 2.2], the ideal $I(E) = \{u \in A(G) : u|_E = 0\}$ of $A(G)$ has a bounded approximate identity, $(e_\alpha)_\alpha$ say. Let $v \in B(G) = C^*(G)^*$ be a w^*-cluster point of the net $(e_\alpha)_\alpha$. Then, using twice the fact that multiplication in $B(G)$ is separately w^*-continuous, it follows that $vu = u$ for every $u \in I(E)$ and $v^2 = v$. Moreover, regularity of $A(G)$ implies that $v(x) = 1$ for all $x \in G \setminus E$ and hence $v = 1$ on $\overline{G \setminus E} = G \setminus E^\circ$ by continuity. On the other hand, $v = 0$ on E° since

$$\langle v, f\rangle = \lim_\alpha \langle e_\alpha, f\rangle = \lim_\alpha \int_G e_\alpha(x) f(x)\, dx = 0$$

for every $f \in L^1(G) \subseteq C^*(G)$ with support contained in E°. So E° is closed in G, and it is the support of the idempotent $1_G - v \in B(G)$. Consequently, $E^\circ \in \mathcal{R}_c(G)$ [17] and hence also $\partial E = E \setminus E^\circ \in \mathcal{R}_c(G)$. This proves (i), and (ii) is then clear since $E^\circ = F_{1_G - v}$. ∎

The following corollary shows that for an amenable group G determining $\mathcal{F}_{\mathrm{pb}}(G)$ reduces to identifying all nowhere dense subsets in $\mathcal{F}_{\mathrm{pb}}(G)$.

Corollary 1.2. *Let G be an amenable locally compact group and let $E \in \mathcal{R}_c(G)$. Then $E = F_u$ for some power bounded element $u \in B(G)$ if and only $\partial E = F_v$ for some power bounded $v \in B(G)$.*

Proof. If $E = F_u$, where $u \in B(G)$ is power bounded, then $v = u \cdot 1_{G\setminus E^\circ} \in B(G)$ is power bounded and $F_v = F_u \cap (G \setminus E^\circ) = \partial E$. Conversely, if $\partial E = F_v$ then $u = 1_{E^\circ} + v \cdot 1_{G\setminus E^\circ}$ is power bounded since $u^n = 1_{E^\circ} + v^n 1_{G\setminus E^\circ}$ for all $n \in \mathbb{N}$ and u satisfies $F_u = E^\circ \cup \partial E = E$. ∎

Recall that a locally compact group G is called an *SIN-group* (a group with small invariant neighbourhoods) if there exists a neighbourhood basis $\mathcal{V}$ of the identity such that $x^{-1}Vx = V$ for all $V \in \mathcal{V}$ and all $x \in G$. The class of SIN-groups in particular contains all groups with open centres as well as every group which is compact modulo its centre. An important property of such groups to be exploited later is a certain separation property of positive definite functions.

The arguments used in the proofs of the following proposition and of Corollary 1.4 below are refinements of those employed in the proofs of [19, Proposition 2.2] and [9, Proposition 3.10].

Proposition 1.3. *Let G be an SIN-group and let H and K be closed subgroups of G such that $K \subseteq H$ and K is open in H. Then, given any neighbourhood U of e in G, there exist an invariant neighbourhood V of e such that $V \subseteq U$ and continuous positive definite functions u and v with the following properties:*

(i) *$u(x) = v(x)$ for all $x \in VK$.*
(ii) *$u|_H = 1$, $u = 0$ outside of VH and $v = 0$ outside of VK.*

Proof. Since G is an SIN-group, G and H are unimodular, and hence there exists an invariant measure μ_H on the left coset space G/H so that Weil's formula

$$\int_G f(x)\,dx = \int_{G/H}\int_H f(xh)\,dh\,d\mu_H(xH),$$

$f \in L^1(G)$, holds. Since K is open in H, counting measure on H/K is H-invariant and the assignment

$$g \mapsto \int_{G/H}\Big(\sum_{h\in H/K} g(xhK)\Big)\,d\mu_H(xH),$$

$g \in C_c(G)$, defines a G-invariant measure μ_K on G/K such that Weil's formula holds for G, K and G/K.

Using again the fact that K is open in H, there exists an open set W in G such that $W\cap H = K$. Since G is an SIN-group, we find a symmetric, relatively compact, invariant open neighbourhood V of e such that $V^2 \subseteq U\cap W$. Then $V\cap H = K$ and hence $VK\cap H = K$.

Let T_K denote the map from $L^1(G)$ onto $L^1(G/K,\mu_K)$ defined by

$$T_K f(xK) = \int_K f(xk)\,dk,$$

and similarly define T_H. Choose a nonnegative function f in $L^1(G)$ such that $T_K f = \mu_K(VK/K)^{-1/2}$ on VK/K and $T_K f = 0$ elsewhere. Then, since V is relatively compact, $T_K f \in L^2(G/K,\mu_K)$ and $T_K f$ has norm 1.

Notice next that if $x\in G$ and $h_1,h_2\in H$ are such that $xh_1, xh_2 \in VK$, then

$$h_2^{-1}h_1 \in H\cap (VK)^{-1}VK = H\cap V^2K = K.$$

Thus $xhK\cap VK = \emptyset$ if $x\notin VH$, and if $x\in VH$, then $xH\cap VK$ is a singleton. It follows that

$$\begin{aligned}\mu_K(VK/K) &= \int_{G/H}\Big(\sum_{h\in H/K} 1_{VK/K}(xhK)\Big)\,d\mu_H(xH)\\ &= \int_{VH/H}\Big(\sum_{h\in H/K} 1_{VK/K}(xhK)\Big)\,d\mu_H(xH) = \mu_H(VH/H).\end{aligned}$$

This implies that, for all $x\in G$,

$$\begin{aligned}T_H f(xH) &= \sum_{h\in H/K} T_K f(xhK) = \mu_K(VK/K)^{-1/2}\sum_{h\in H/K} 1_{VK/K}(xhK)\\ &= \mu_H(VH/H)^{-1/2}\sum_{h\in H/K} 1_{VK/K}(xhK)\\ &= \mu_H(VH/H)^{-1/2} 1_{VH/H}(xH).\end{aligned}$$

In particular, $T_H f$ has norm 1 in $L^2(G/H,\mu_H)$.

We now define functions u and v on G by setting, for $x \in G$,

$$u(x) = \int_G f(y) T_H f(x^{-1}yH)\, dy \quad \text{and} \quad v(x) = \int_G f(y) T_K f(x^{-1}yH)\, dy.$$

Denoting by π_H and π_K the representations of G induced from the trivial representations of H and K and realizing π_H and π_K in the Hilbert spaces $L^2(G/H, \mu_H)$ and $L^2(G/K, \mu_K)$, respectively, it is straightforward to verify that

$$u(x) = \langle \pi_H(x) T_H f, T_H f\rangle \quad \text{and} \quad v(x) = \langle \pi_K(x) T_K f, T_K f\rangle$$

for all $x \in G$. Thus u and v are continuous positive definite functions and $u|_H = 1$ and $v|_K = 1$. Since $T_H f$ vanishes on $G/H \setminus (VH/H)$, it is clear that $u = 0$ on $G \setminus VH$, and similarly v vanishes outside of VK. This proves (ii).

For (i), recall that $T_H f(zH) = T_K f(zK)$ whenever $z \in VK$. Thus, since $T_H f$ and $T_K f$ vanish on $G \setminus VH$ and $G \setminus VK$, respectively, and both functions are equal on VK/K, for every $x \in G$ we have

$$u(x) - v(x) = \int_{VH \setminus VK} f(xy) T_H(yH)\, dy = \mu_H(VH/H)^{-1/2} \int_{VH \setminus VK} f(xy)\, dy.$$

Therefore, it remains to show that the latter integral is zero whenever $x \in VK$. Now, by Weil's formula,

$$\int_{VH \setminus VK} f(xy)\, dy = \int_{(VH/K) \setminus (VK/K)} T_K f(xyK)\, d(yK).$$

Let $x \in VK$. If $y = vh$ with $v \in V$ and $h \in H$ is such that $xy \in VK$, then $h \in V^3 K \cap H = K$ and hence $y \in VK$. It follows that $T_K f(xyK) = 0$ for all $y \in VH \setminus VK$, as required. ∎

Corollary 1.4. *Let G, H and K be as in Proposition* 1.3. *In addition, suppose that H is a G_δ-set. Then, given any neighbourhood U of the identity, there exists a power bounded $u \in B(G)$ such that $F_u = H \setminus K$ and $u = 0$ ouside of $U(H \setminus K)$.*

Proof. As in the proof of Proposition 1.3, we first choose an open symmetric neighbourhood V of e in G such that $V^2 \subseteq U$ and $V^2 \cap H = K$. Since H is a G_δ-set, we find a decreasing sequence $V = V_1 \supseteq V_2 \supseteq \cdots$ of open symmetric neighbourhoods of e in G such that $\bigcap_{j=1}^\infty V_j H = H$. By Proposition 1.3, for each $j \in \mathbb{N}$ there exist $u_j, v_j \in P(G)$ with the following properties:

(1) $u_j = v_j$ on $V_j K$;
(2) $u_j|_H = 1$, $u_j = 0$ on $G \setminus V_j H$;
(3) $v_j = 0$ on $G \setminus V_j K$.

Now define $u_\infty, v_\infty \in P(G)$ by

$$u_\infty = \sum_{j=1}^{\infty} 2^{-j} u_j \quad \text{and} \quad v_\infty = \sum_{j=1}^{\infty} 2^{-j} v_j.$$

Then u_∞ and v_∞ satisfy

(i) $u_\infty|_H = 1$, $u_\infty = 0$ outside of VH and $v_\infty = 0$ outside of VK.
(ii) $u_\infty = v_\infty$ on VK.

Item (i) is clear from (1)–(3); also $u_\infty = v_\infty$ on K. Thus, let $x \in VK \setminus K$ and let $j \in \mathbb{N}$ be maximal with the property that $x \in V_j K$. Then $u_i(x) = v_i(x)$ for $i \le j$ and $x \notin V_i$ for any $i > j$ because $x \in V_i H \cap V_j K$ for some such i implies that $x \in H \cap V_i^{-1} V_j K \subseteq H \cap V_j^2 K = K$, which is a contradiction. Consequently, $u_i(x) = v_i(x)$ for $i > j$ and hence $u_\infty(x) = v_\infty(x)$. Thus (ii) holds. Now let $u = u_\infty - v_\infty \in B(G)$. Then, by (i) and (ii), u vanishes on $G \setminus VH$ and on VK.

Observe next that u is power bounded. Indeed, for any $n \in \mathbb{N}$, we have $(u_\infty - v_\infty)^n = 0 = u_\infty^n - v_\infty^n$ on VK and outside of VH, whereas for $x \in VH \setminus VK$,

$$(u_\infty(x) - v_\infty(x))^n = u_\infty(x)^n = u_\infty(x)^n - v_\infty(x)^n.$$

Since u_∞ and v_∞ are power bounded, so is u.

It remains to show that $F_u = H \setminus K$. If $x \in H \setminus K$, then $u_\infty(x) = 1$. On the other hand, since $V_j K \cap H \subseteq K$, we see that $x \notin V_j K$ for every j and hence $v_\infty(x) = 0$. Conversely, let $x \in F_u$. Since $0 \le u_j, v_j \le 1$ for all j, we have $0 \le u_\infty, v_\infty \le 1$, and therefore $1 = u(x) = u_\infty(x) - v_\infty(x)$ implies that $v_\infty(x) = 0$ and $u_j(x) = 1$ for all j. As u_j vanishes on $G \setminus V_j H$, it follows that $x \in \bigcap_{j=1}^{\infty} V_j H = H$. Finally, as $u = 0$ on K, we get $F_u \subseteq H \setminus K$. ∎

Lemma 1.5. *Let G be an SIN-group and H a closed subgroup of G such that H is a G_δ-set. Then there exists $u \in P(G)$ such that $F_u = H$.*

Proof. In [34, Lemma 6.15] is was shown that if H is a closed normal subgroup of a locally compact group G and H is a G_δ-set, then there exists a (decreasing) sequence $(W_n)_n$ of open neighbourhoods of the identity e in G such that $\bigcap_{n=1}^{\infty} W_n = H$. However, inspection of the proof shows that normality of H is not used at all. Now, since G is an SIN-group, for each n we may choose an open symmetric neighbourhood V_n of e such that $V_n H = H V_n$ and $V_n^2 \subseteq W_n$.

Now fix $W = W_n$ and let $V = V_n$, and let $q : G \to G/H$, μ_H and T_H be as in the proof of Proposition 1.3. Choose a nonnegative function $v \in L^1(G)$ such that $T_H v = 0$ on $G/H \setminus q(V)$ and $T_H v = \mu_H(q(V))^{1/2}$ on $q(V)$. Define a function u on G by

$$u(x) = \int_G v(y) T_H v(x^{-1}yH)\, dy, \qquad x \in G.$$

Then $u \in P(G)$ and $u|_H = 1$ (compare the proof of Proposition 1.3). Moreover, $u = 0$ outside of WH. In fact, if $x \in G$ is such that $u(x) \neq 0$, then $T_H v(x^{-1}yH) \neq 0$ for some $y \in VH$ and hence

$$x^{-1} \in VHy^{-1} \subseteq VH(VH)^{-1} = VHV = V^2H \subseteq WH.$$

Finally, for each $n \in \mathbb{N}$, let u_n be as just constructed. Then the function $u = \sum_{n=1}^{\infty} 2^{-n} u_n \in P(G)$ satisfies $F_u = H$ since $\bigcap_{n=1}^{\infty} W_n H = H$. ∎

The reader who is familiar with the notion of a neutral subgroup will have observed that the proof of Lemma 1.5 goes through without any changes when H is a neutral subgroup of any locally compact group G.

Let E and F be closed subsets of a locally compact group G. Following [34, Definition 6.18], we say that E and F are *uniformly separated* if there exists a neighbourhood V of the identity such that $VE \cap F = \emptyset$. Replacing V by a symmetric neighbourhood W of e such that $W^2 \subseteq V$, we may assume that $VE \cap VF = \emptyset$ if necessary. As mentioned in [34, Proposition 6.19], two closed subsets E and F of G are uniformly separated if either E is compact and $E \cap F = \emptyset$, or $E \subseteq C$ and $F \subseteq D$, where C and D are distinct cosets of some closed subgroup of G.

In passing observe that every set of the form F_u for some $u \in B(G)$ is a G_δ-set. In fact,

$$F_u = \bigcap_{n=1}^{\infty} \{x \in G : |u(x) - 1| < 1/n\}.$$

Theorem 1.6. *Let G be an SIN-group and let E be a closed G_δ-set in $\mathcal{R}(G)$. Suppose that $E = \bigcup_{i=1}^{N} E_i$, where each E_i is of the form* (1.1), *and that the E_i are pairwise uniformly separated. Then there exists a power bounded $u \in B(G)$ such that $F_u = E$.*

Proof. To start with, notice that every closed subset of a G_δ-set is also a G_δ-set. Suppose first that $N = 1$, so that E is of the form $E = a(H \setminus \bigcup_{j=1}^{m} b_j K_j)$, where H is a closed subgroup of G, each K_j is either empty or an open subgroup of H, and $a \in G$, $b_j \in H$, $1 \leq j \leq m$. Then, given any invariant neighbourhood U of e in G, by Corollary 1.4, for each $j = 1, \ldots, m$, there exists a power bounded element u_j of $B(G)$ such that $F_{u_j} = H \setminus K_j$ and $u_j = 0$ outside of $U(H \setminus K_j)$. If, for all $1 \leq j \leq m$, $b_j K_j \cap H = \emptyset$, then $E = aH$, in which case by Lemma 1.5 there exists $u \in P(G)$ such that $F_{L_a u} = E$. So we can assume that $H \cap b_j K_j \neq H$

exactly for $1 \leq j \leq r$, $r \leq m$. Then $b_j \in H$ for $1 \leq j \leq r$ and

$$E = a\Big(H \setminus \bigcup_{j=1}^{r} b_j K_j\Big) = \bigcap_{j=1}^{r} ab_j(H \setminus K_j).$$

Let $u = \prod_{j=1}^{r} L_{ab_j} u_j \in B(G)$, where $L_x u(y) = u(x^{-1}y)$ for $x, y \in G$ and any function u on G. Then

$$F_u = \bigcap_{j=1}^{m} F_{L_{ab_j} u_j} = \bigcap_{j=1}^{m} ab_j F_{u_j} = E,$$

and since translating is an isometry of $B(G)$, u is power bounded. Moreover, u vanishes outside of $\bigcup_{j=1}^{m} ab_j U(H \setminus K_j) = UE$.

In the general case, since $E_1, \ldots, E_N$ are pairwise uniformly separated, we find a neighbourhood U of e such that $UE_i \cap UE_k = \emptyset$ for $1 \leq i, k \leq N$, $i \neq k$. In particular, each E_i is a G_δ-set since E is one. By the first part of the proof, there exist power bounded elements $u_1, \ldots, u_N$ of $B(G)$ with $F_{u_i} = E_i$ and $u_i = 0$ on $G \setminus UE_i$. Then $u = u_1 + \cdots + u_N \in B(G)$ vanishes on $G \setminus UE$, and u is power bounded since $u^n = u_1^n + \cdots + u_N^n$ for all $n \in \mathbb{N}$. Clearly, $F_u = \bigcup_{i=1}^{N} F_{u_i} = \bigcup_{i=1}^{N} E_i = E$. ∎

We now show that every compact G_δ-set in $\mathcal{R}(G)$ belongs to $\mathcal{F}_{\mathrm{pb}}(G)$. More precisely, we have

Proposition 1.7. *Let G be an arbitrary locally compact group, E a compact set in $\mathcal{R}(G)$ and U an open set containing E. Then there exists a power bounded $u \in A(G)$ with $E \subseteq E_u$ and* $\operatorname{supp} u \subseteq U$. *If E is a G_δ-set, then we may choose u so that $F_u = E$.*

Proof. We start by considering a compact subgroup K of G and an open neighbourhood U of K. Then there exists a symmetric compact neighbourhood V of e in G such that $KV = VK$ and $KV^2 \subseteq U$. Let $M = KV$ and $v = |M|^{-1}(1_M * 1_M^*)$. Then v is a continuous positive definite function and

$$v(x) = |M|^{-1} \int_G 1_M(xy) 1_M(y)\, dy = \frac{|M \cap x^{-1}M|}{|M|},$$

which is easily verified to be one for $x \in K$ and zero for $x \notin KV$.

Suppose that, in addition, K is a G_δ-set and let $U \supseteq U_1 \supseteq U_2 \supseteq \cdots$ be a sequence of open subsets of G such that $K = \bigcap_{n=1}^{\infty} U_n$. For each U_n, choose v_n as in the preceding paragraph and put $v = \sum_{n=1}^{\infty} 2^{-n} v_n$. Then v is a continuous positive definite function such that $F_v = K$ and $\operatorname{supp} v \subseteq U$. It follows by translation that for any coset aK, $a \in G$, we obtain $v_a \in A(G)$ with $\|v_a\| = 1$, $\operatorname{supp} v_a \subseteq aU$ and $F_{v_a} \subseteq aU$ (and $F_{v_a} = aK$ when aK is a G_δ-set).

Let now C be an arbitrary compact set in $\mathcal{R}(G)$. Then there exist a compact subgroup K of G and $a_1, \ldots, a_n \in G$ such that $C = \bigcup_{i=1}^n a_i K$, a disjoint union. Since the sets $a_i K$ are open in C and pairwise disjoint, there exist pairwise disjoint open sets $U_1, \ldots, U_n$ in G such that $U_i \subseteq U$ and $a_i K \subseteq U_i$. Clearly, if C is a G_δ-set, then so are the sets $a_i K$. For each i, let $v_i \in A(G)$ be as guaranteed by the first part of the proof, and put $v = \sum_{i=1}^n v_i \in A(G)$. Then v is power bounded since $v_i v_j = 0$ for $i \neq j$ and hence $v^k = \sum_{i=1}^n v_i^k$ for all $k \in \mathbb{N}$. Obviously, v has all the other required properties. ∎

Example 1.8. Let G be a locally compact group with the property that every proper closed subgroup of G is compact. Then, given any G_δ-set $E \in \mathcal{R}_c(G)$, there exists a power bounded $u \in B(G)$ with $F_u = E$.

To see this, let $E = \bigcup_{i=1}^n E_i$ where each E_i is as on the right hand side of formula (1.1) and $\neq \emptyset$. If all H_i are compact, then E is compact and the existence of u follows from Proposition 1.7. So we may assume that $H_i = G$ for exactly $1 \leq i \leq m \leq n$. Then, for $1 \leq i \leq m$, the subgroups K_{ij} are open and closed in G and $F = \bigcup_{i=1}^m E_i \in \mathcal{R}(G)$ is open and closed, whence $1_F \in B(G)$.

Let $C = \bigcup_{i=m+1}^n E_i \setminus F$, a compact set in $\mathcal{R}(G)$. Since $G \setminus F$ is an open set containing C, and C, being open in E, is a G_δ-set, by Proposition 1.7 there exists $v \in A(G)$ such that $F_v = C$ and $v = 0$ on F. Now $u = 1_F + v$ satisfies $F_u = E$ and u is power bounded since $1_F v = 0$ and hence $u^n = 1_F + v^n$ for all $n \in \mathbb{N}$.

Locally compact groups as considered in the preceding example do exist. Actually, Ol'shanskiĭ [29] has constructed infinite groups such that every proper subgroup is finite and satisfying various additional conditions, thereby answering a number of open questions in group theory. Building on this construction, Losert [25] in the context of his investigation of the Mautner phenomenon has produced an example of a locally compact group such that every proper closed subgroup is compact and which is an IN-group, but fails to be an SIN-group.

2. Extending power bounded functions. Let G be a locally compact group and H a closed subgroup of G, and for any function u on G let $r(u) = u|_H$ denote the restriction of u to H. Then $r(B(G)) \subseteq B(H)$ and $r(A(G)) \subseteq A(H)$ [8]. McMullen [26] has shown that $r(A(G)) = A(H)$. More precisely, given $v \in A(H)$, there exists $u \in A(G)$ with $u|_H = v$ and $\|u\| = \|v\|$. The problem of when $r(B(G)) = B(H)$ has turned out to be much more delicate and has been studied extensively by several authors. We only mention a few basic results. It was shown independently by Henrichs [15] and Cowling and Rodway [6] that $r(B(G)) = B(H)$ whenever G

has small H-invariant neighbourhoods. If H is normal in G, then $v \in B(H)$ extends to some $u \in B(G)$ if and only if the function $x \mapsto \|x \cdot v - v\|$, where $x \cdot v(h) = v(x^{-1}hx)$ for $h \in H$ and $x \in G$, is continuous at the identity of G [6]. Earlier, it was shown in [24] that if G is an arbitrary locally compact group and G_0 denotes the connected component of the identity, then $B(G)|_{G_0} = B(G_0)$. Our main intention in this section is to show that every power bounded element of $B(G_0)$ extends to some power bounded element of $B(G)$ with the same norm (Theorem 2.5).

LEMMA 2.1. *Let G be a projective limit of groups G/K_α and, for each α, let ν_α denote normalized Haar measure of K_α. Let $u \in B(G)$ and set $u_\alpha = u * \nu_\alpha$. Then $u_\alpha \in B(G)$ and $u = w^*\text{-}\lim_\alpha u_\alpha$.*

Proof. Let π be a unitary representation of G and $\xi, \eta \in H(\pi)$ be such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for all $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$. Then

$$u_\alpha(x) = \int_{K_\alpha} \langle \pi(xk)\xi, \eta\rangle \, d\nu_\alpha(k) = \langle \pi(x)\pi(\nu_\alpha)\xi, \eta\rangle,$$

and hence $u_\alpha \in B(G)$ and $\|u_\alpha\| \le \|\pi(\nu_\alpha)\xi\| \cdot \|\eta\| \le \|\xi\| \cdot \|\eta\| = \|u\|$. Since the net $(u_\alpha)_\alpha$ is norm bounded, for $u = w^*\text{-}\lim_\alpha u_\alpha$ it suffices to verify that $\langle u_\alpha, f\rangle \to \langle u, f\rangle$ for all $f \in C_c(G)$. Now fix such an f, choose a compact neighbourhood C of $\operatorname{supp} f$ and let $\epsilon > 0$ be given. Since f is uniformly continuous, there exists a symmetric neighbourhood V of the identity such that $\operatorname{supp} f \cdot V \subseteq C$ and $|f(xy) - f(x)| \le \epsilon/|C|$ for all $x \in G$ and $y \in V$. For large enough α, we then have $K_\alpha \subseteq V$, and hence $|(f * \nu_\alpha)(x) - f(x)| \le \epsilon/|C|$ for all $x \in G$. Thus

$$\|f * \nu_\alpha - f\|_1 = \int_C |(f * \nu_\alpha)(x) - f(x)| \, dx \le \epsilon,$$

and this implies that

$$\begin{aligned} |\langle u_\alpha, f\rangle - \langle u, f\rangle| &= |\langle u, f * \nu_\alpha\rangle - \langle u, f\rangle| \le \|u\| \cdot \|f * \nu_\alpha - f\|_{C^*(G)} \\ &\le \|u\| \cdot \|f * \nu_\alpha - f\|_1 \le \epsilon\|u\|. \end{aligned}$$

This shows that $\langle u_\alpha, f\rangle \to \langle u, f\rangle$ for every $f \in C_c(G)$ ∎

Let H be a σ-compact closed subgroup of G and K a compact normal subgroup of G such that HK is open in G. Let ν and μ be normalized Haar measures of $H \cap K$ and K, respectively, and let $\widetilde{\nu}$ and $\widetilde{\mu}$ be Haar measures on $H/H \cap K$ and HK/K, respectively, so that Weil's formula holds for the pairs $(H, H \cap K)$ and (HK, K). For any topological space X, let $C^b(X)$ denote the space of all bounded complex-valued continuous functions on X.

LEMMA 2.2. *Retain the above situation and notation. Let $u \in C^b(H)$ and define $v \in C^b(HK)$ by $v(hk) = (u * \nu)(h)$ for $h \in H$, $k \in K$. Then, for* *any* $f \in C_c(HK)$,

$$\langle v, f\rangle = \langle u * \nu, f|_H\rangle.$$

Proof. Note first that v is well defined because $u * \nu$ is constant on cosets of $H \cap K$. Let $\phi : H/H \cap K \to HK/K$ denote the group isomorphism defined by $h(H \cap K) \to hK$, $h \in H$. Then ϕ is a homeomorphism since HK is closed in G and H is σ-compact [16, Theorem 5.29]. Let ϕ also denote the associated isomorphism between $C^b(H/H \cap K)$ and $C^b(HK/K)$. Moreover, let $T_K : C_c(HK) \to C_c(HK/K)$ and $T_{H\cap K} : C_c(H) \to C_c(H/H \cap K)$ denote the usual homomorphism. Let ω denote the left invariant measure on the coset space $K/H \cap K$, so that Weil's formula

$$\int\limits_K g(x)\, d\mu(x) = \int\limits_{K/K\cap H} \Big(\int\limits_{H\cap K} g(xk)\, d\nu(k)\Big)\, d\omega(x(H \cap K))$$

holds for all $g \in C(K)$. Then $\omega(K/H \cap K) = 1$, and for any $f \in C_c(HK)$ and $x \in H$ we have

$$\begin{aligned} T_K f(xK) &= \int\limits_{K/H\cap K} \Big(\int\limits_{H\cap K} f(xlk)\, d\nu(k)\Big)\, d\omega(l(H \cap K)) \\ &= \int\limits_{K/H\cap K} \phi(T_{H\cap K} f)(xlk)\, d\omega(l(H \cap K)) \\ &= \int\limits_{K/H\cap K} \phi(T_{H\cap K} f)(xk)\, d\omega(l(H \cap K)) = \phi(T_{H\cap K} f)(xK). \end{aligned}$$

This formula in turn implies

$$\begin{aligned} \langle u * \nu, f|_H\rangle &= \int\limits_{H/H\cap K} \int\limits_{H\cap K} v(h) f(hk)\, d\nu(k)\, d\widetilde{\nu}(h(H \cap K)) \\ &= \int\limits_{H\cap K} v(h) T_{H\cap K} f(h(H \cap K))\, d\widetilde{\nu}(h(H \cap K)) \\ &= \int\limits_{HK/K} v(x)\phi(T_{H\cap K} f)(xK)\, d\widetilde{\mu}(xK) \\ &= \int\limits_{HK/K} v(x) T_K f(xK)\, d\widetilde{\mu}(xK) = \int\limits_{HK} v(x) f(x)\, dx = \langle v, f\rangle, \end{aligned}$$

where we have used the fact that $\widetilde{\mu}$ is the image of $\widetilde{\nu}$ under the map ϕ. ∎

Suppose that G is a projective limit of groups G/K_α, and H is a σ-compact closed normal subgroup of G such that HK_α is open in G for each α. Let ν_α denote normalized Haar measure on $H \cap K_\alpha$, and for $u \in C^b(H)$ let v_α be defined on HK_α by $v_\alpha(xk) = (u * \nu_\alpha)(x)$ for $x \in H$ and $k \in K_\alpha$. Moreover, let $C_{c,\alpha}(G)$ denote the set of all functions in $C_c(G)$ which are constant on cosets of K_α.

LEMMA 2.3. *Let $u \in B(H)$ be such that $w^*\text{-}\lim_{n\to\infty} u^n = 0$. For every $n \in \mathbb{N}$, there exists $w_n \in B(G)$ with the following properties:*

(1) *$w_n|_H = u^n$ and $\|w_n\| = \|u^n\|$.*
(2) *For each α and $f \in C_{c,\alpha}(G)$, $\langle w_n, f\rangle = \langle v_\alpha^n, f|_{HK_\alpha}\rangle$.*

Proof. We apply Lemma 2.2 to u^n in place of u. Since ν_α is normalized and translation invariant, it is easy to check that $u^n * \nu_\alpha = (u * \nu_\alpha)^n$. Let $v_{n,\alpha}$ on HK_α be associated to $u^n * \nu_\alpha$ as in Lemma 2.2. Then $v_{n,\alpha} = v_\alpha^n$; indeed, for $x \in H$ and $k \in K_\alpha$, we have

$$v_{n,\alpha}(xk) = (u^n * \nu_\alpha)(x) = (u * \nu_\alpha)(x)^n = v_\alpha(xk)^n.$$

Then, by Lemma 2.2, for all $f \in C_c(G)$,

$$\langle \widetilde{v}_{\alpha^n}, f\rangle = \langle u^n * \nu_\alpha, f|_H\rangle.$$

Let now $F = \bigcup_\alpha C_{c,\alpha}(G)$. Then F is a linear subspace of $C_c(G)$. In fact, if $f_i \in C_{c,\alpha_i}(G)$, $i = 1, 2$, and $\alpha \geq \alpha_1, \alpha_2$, then $K_\alpha \subseteq K_{\alpha_1} \cap K_{\alpha_2}$ and hence $f_1, f_2 \in C_{c,\alpha}(G)$. Furthermore, F is dense in $C_c(G)$ with respect to the L^1-norm (compare the proof of Lemma 2.1). Note next that if $f \in C_{c,\alpha}(G)$ and $\beta \geq \alpha$, then $\langle \widetilde{v}_{\alpha^n}, f\rangle = \langle \widetilde{v}_{\beta^n}, f\rangle$. Indeed,

$$\langle \widetilde{v}_{\alpha^n}, f\rangle = \langle u^n * \nu_\alpha, f|_H\rangle = \langle u^n, f|_H * \nu_\alpha\rangle = \langle u^n, f|_H\rangle,$$

and similarly for β. We can therefore define a linear functional φ_n on F by

$$\langle \varphi_n, f\rangle = \langle \widetilde{v}_{\alpha^n}, f\rangle, \qquad f \in C_{c,\alpha}(G).$$

We claim that $\|\widetilde{v}_{\alpha^n}\| \leq \|u^n\|$ for each α. To see this, let π be a unitary representation of H and $\xi, \eta \in H(\pi)$ be such that $u^n(x) = \langle \pi(x)\xi, \eta\rangle$ for all $x \in H$ and $\|u^n\| = \|\xi\| \cdot \|\eta\|$. Then, for $x \in H$ and $k \in K_\alpha$,

$$\begin{aligned} v_\alpha^n(xk) = (u^n * \nu_\alpha)(x) &= \int\limits_{H\cap K_\alpha} u^n(xl)\, d\nu_\alpha(l) = \int\limits_{H\cap K_\alpha} \langle \pi(x)\pi(l)\xi, \eta\rangle\, d\nu_\alpha(l) \\ &= \Big\langle \pi(x) \int\limits_{H\cap K_\alpha} \pi(l)\xi\, d\nu_\alpha(l), \eta\Big\rangle, \end{aligned}$$

which implies that

$$\|\widetilde{v}_{\alpha^n}\| = \|v_\alpha^n\| \leq \|\eta\| \cdot \Big\| \int\limits_{H\cap K_\alpha} \pi(l)\xi\, d\nu_\alpha(l)\Big\| \leq \|\eta\| \cdot \|\xi\| = \|u^n\|.$$

Then, for every $f \in F$,

$$|\langle \varphi_n, f\rangle| \leq \|\widetilde{v}_{\alpha^n}\| \cdot \|f\|_{C^*} \leq \|u^n\| \cdot \|f\|_{c^*}.$$

Since F is dense in $C^*(G)$, φ_n extends to a bounded linear functional on $C^*(G)$, and hence there exists $w_n \in B(G)$ with $\langle \varphi_n, f\rangle = \langle w_n, f\rangle$ for all $f \in C^*(G)$. Clearly, by definition of φ_n, (2) holds for w_n, and $\|w_n\| =$

$\|\varphi\| \le \|u^n\|$. It remains to observe that $w_n|_H = u^n$. By Lemma 2.1,

$$\langle u^n, f|_H\rangle = \lim_\alpha \langle u^n * \nu_\alpha, f|_H\rangle = \lim_\alpha \langle \widetilde{v}_{\alpha^n}, f\rangle$$

for every $f \in C_c(G)$ and, on the other hand, $\langle w_n, f\rangle = \langle \widetilde{v}_{\alpha^n}, f\rangle$ for every $f \in C_{c,\beta}(G)$ and all $\alpha \ge \beta$. Since the set of all $f|_H$, $f \in \bigcup_\beta C_{c,\beta}(G)$, is dense in $C_c(H)$, it follows that $w_n|_H = u^n$. ∎

LEMMA 2.4. *Let $u \in B(H)$ be such that w^*-$\lim_{n\to\infty} u^n = 0$, let $w_n \in B(G)$, $n \in \mathbb{N}$, be as in Lemma 2.3 and set $w = w_1$. Then w^*-$\lim_{n\to\infty} w^n = 0$.*

Proof. As in the proof of Lemma 2.2, let μ_α be normalized Haar measure of K_α. We first show by induction that $w_n * \mu_\alpha = w^n * \mu_\alpha$ for all $n \in \mathbb{N}$. Suppose that this equation holds for n and let $g \in C_c(G)$. Then, since $g * \mu_\alpha \in C_{c,\alpha}(G)$, by Lemma 2.3,

$$\begin{aligned}
\langle w_{n+1} * \mu_\alpha, g\rangle &= \langle w_{n+1}, g * \mu_\alpha\rangle = \langle \widetilde{v}_{\alpha^{n+1}}, g * \mu_\alpha\rangle = \langle \widetilde{v}_{\alpha^n}, \widetilde{v}_\alpha(g * \mu_\alpha)\rangle\\
&= \langle w_n, \widetilde{v}_\alpha(g * \mu_\alpha)\rangle = \langle w_n, [\widetilde{v}_\alpha(g * \mu_\alpha)] * \mu_\alpha\rangle\\
&= \langle w_n * \mu_\alpha, \widetilde{v}_\alpha(g * \mu_\alpha)\rangle = \langle w^n * \mu_\alpha, \widetilde{v}_\alpha(g * \mu_\alpha)\rangle\\
&= \langle \widetilde{v}_\alpha, (w^n * \mu_\alpha)(g * \mu_\alpha)\rangle = \langle w, (w^n * \mu_\alpha)(g * \mu_\alpha)\rangle\\
&= \langle (w * \mu_\alpha)(w^n * \mu_\alpha), g * \mu_\alpha\rangle = \langle w^{n+1} * \mu_\alpha, g * \mu_\alpha\rangle\\
&= \langle w^{n+1} * \mu_\alpha, g\rangle.
\end{aligned}$$

Now $w_n * \mu_\alpha = w^n * \mu_\alpha$ for all α means that the bounded linear functionals $g \mapsto \langle w_n, g\rangle$ and $g \mapsto \langle w^n, g\rangle$ agree on $F = \bigcup_\alpha C_{c,\alpha}(G)$. Since F is dense in $C^*(G)$, it follows that $w_n = w^n$ for all $n \in \mathbb{N}$. Therefore, we only have to show that w^*-$\lim_{n\to\infty} w_n = 0$. Since the sequence $(w_n)_n$ is norm bounded and F is dense in $C^*(G)$, it suffices to verify that $\langle w_n, f\rangle \to 0$ for $f \in C_{c,\alpha}(G)$. By Lemma 2.2,

$$\langle w_n, f\rangle = \langle \widetilde{v}_{\alpha^n}, f\rangle = \langle v_\alpha^n, f|_{HK_\alpha}\rangle = \langle u^n * \nu_\alpha, f|_H\rangle = \langle u^n, f|_H * \nu_\alpha\rangle,$$

which converges to zero since w^*-$\lim_{n\to\infty} u^n = 0$. ∎

We are now ready to prove the extension result mentioned at the outset of this section.

THEOREM 2.5. *Let G be any locally compact group and G_0 its connected component of the identity. Then every power bounded element u of $B(G_0)$ extends to some power bounded element w of $B(G)$ with the same norm. If w^*-$\lim_{n\to\infty} u^n = 0$, then w can be found such that w^*-$\lim_{n\to\infty} w^n = 0$.*

Proof. Let u be a power bounded element of $B(G_0)$. By [22], either there exist a complex number λ of absolute value 1 and a character γ of G_0 such that $u(x) = \lambda\gamma(x)$ for all $x \in G_0$, or u satisfies w^*-$\lim_{n\to\infty} u^n = 0$.

In the first case, by [24, Proposition 1.1], γ extends to some continuous positive definite function σ on G, and so $\lambda\sigma$ is the required extension of u. Alternatively, instead of using [24], we could proceed as follows. Since

$|u(x)| = 1$ for all $x \in G_0$, by [21, Proposition 4.5] either u is constant or $w^*\text{-}\lim_{n\to\infty} u^n = 0$. Therefore, to establish the theorem, we have to show that if $w^*\text{-}\lim_{n\to\infty} u^n = 0$, then there exists $w \in B(G)$ such that $w|_{G_0} = u$ and $w^*\text{-}\lim_{n\to\infty} w^n = 0$.

Since G/G_0 is totally disconnected, we can choose an open subgroup L of G containing G_0 such that L/G_0 is compact. If u extends to some element v of $B(L)$ such that $w^*\text{-}\lim_{n\to\infty} v^n = 0$, then the trivial extension of v to G has the same property. Therefore we may assume that G/G_0 is compact.

Now G, being almost connected, is a projective limit of Lie groups G/K_α [27]. Then, for each α, G/G_0K_α is a compact and totally disconnected Lie group, so finite. Thus Lemma 2.4 applies with $H = G_0$ and yields the existence of some $w \in B(G)$ such that $w|_{G_0} = u$ and $w^*\text{-}\lim_{n\to\infty} w^n = 0$. ■

Remark 2.6. Suppose that u is a power bounded element of $A(G_0)$. Then the element w of $B(G)$ constructed in Theorem 2.5 belongs to $A(G)$. To see this, let $H = G_0$ and let $f, g \in L^2(H)$ be such that $u(x) = (f * \widetilde{g})(x)$ for all $x \in H$. As before, let ν_α and μ_α denote the normalized Haar measures of $H \cap K_\alpha$ and K_α, respectively. Then $f * \nu_\alpha$ and $\widetilde{g} * \nu_\alpha$ are in $L^2(H)$ and

$$(u * \nu_\alpha)(x) = [(f * \nu_\alpha) * (\widetilde{g} * \nu_\alpha)](x)$$

for all $x \in H$. So $u * \nu_\alpha \in A(H)$ and $\|u * \nu_\alpha\| \le \|u\|$. This implies that

$$v_\alpha(xk) = (u * \nu_\alpha)(x) = (f_1 * \widetilde{g}_1)(xk), \qquad x \in H,\, k \in K_\alpha,$$

where $f_1, g_1 \in L^2(HK_\alpha)$ are defined by $f_1(xk) = (f * \nu_\alpha)(x)$ and similarly for g_1. Thus $\widetilde{v}_\alpha \in A(G)$ and $\|\widetilde{v}_\alpha\| \le \|u\|$ for every α. Recall next that $\langle w * \mu_\alpha, f\rangle = \langle \widetilde{v}_\alpha, f\rangle$ for all $f \in C_c(G)$, and hence $w * \mu_\alpha \in A(G)$ and $\|w * \mu_\alpha\| \le \|u\|$. Finally, let $g \in C^*(G)$ and $\epsilon > 0$ be given. Since $\bigcup_\alpha C_{c,\alpha}(G)$ is dense in $C^*(G)$, there exist α_0 and $f \in C_{c,\alpha_0}(G)$ with $\|g - f\|_{C^*} \le \epsilon$. Then $f \in C_{c,\alpha}(G)$ for all $\alpha \ge \alpha_0$ and hence, as $\langle w, f\rangle = \langle w, f * \mu_\alpha\rangle = \langle w * \mu_\alpha, f\rangle$,

$$\begin{aligned}|\langle w, g\rangle - \langle w * \mu_\alpha, g\rangle| &\le |\langle w, g\rangle - \langle w, f\rangle| + |\langle w * \mu_\alpha, f\rangle - \langle w * \mu_\alpha, g\rangle| \\ &\le 2\|u\| \cdot \|g - f\|_{C^*} \le 3\epsilon\|u\|.\end{aligned}$$

This shows that $w * \mu_\alpha \to w$ in $B(G)$ and hence $w \in A(G)$.

For locally compact abelian groups the following result was shown in [34, Lemma 6.27].

Theorem 2.7. *Let G be an arbitrary locally compact group and D a discrete subgroup of G. Then every power bounded element u of $A(D)$ extends to some power bounded $v \in A(G)$. If G is first countable, then v can be found to satisfy $E_v = E_u$.*

Proof. First, choose any $u_1 \in A(G)$ extending u. As E_{u_1} is compact, we can select a relatively compact open subset U of G such that $E_{u_1} \subseteq U$. Since

D is discrete and $D \cap U$ is finite, there exist symmetric open neighbourhoods W and V of e with the following properties:

(1) $W^2 \cap D = \{e\}$ and $dW \subseteq U$ for all $d \in D \cap U$;
(2) $UV \cap D = U \cap D$.

Next choose $u_2 \in A(G)$ such that $u_2|_U = 1$, $u_2 = 0$ on $G \setminus UV$ and $0 \leq u_2 \leq 1$, and a continuous positive definite function u_3 with $u_3(e) = 1$ and $u_3 = 0$ on $G \setminus W$. Note that if G is first countable, we can require that $E_{u_3} = \{e\}$.

Now define $v \in A(G)$ by

$$v(x) = u_1(x)[1 - u_2(x)] + \sum_{d \in D \cap U} u(d) u_3(d^{-1}x), \qquad x \in G.$$

The functions $u_3(d^{-1}\cdot)$, $d \in D \cap U$, are pairwise orthogonal. In fact, if $u_3(d_j^{-1}x) \neq 0$, $j = 1, 2$, then $d_j^{-1}x \in W$ and hence $d_1^{-1}d_2 \in W^2 \cap D = \{e\}$. Since $\|u_3\| = u_3(e) = 1$, it follows that $w = \sum_{d \in D \cap U} u(d) u_3(d^{-1}\cdot) \in A(G)$ is power bounded. Notice that also $u_1[1 - u_2]$ and w are orthogonal. Indeed, if, for some $x \in G$, $u_1(x)[1 - u_2(x)] \neq 0$ and $w(x) \neq 0$, then $x \notin U$ and $x \in dW$ for some $d \in D \cap U$, which is impossible. Consequently, v is power bounded once $u_1[1 - u_2]$ has been shown to be power bounded. There exists a compact subset C of G containing U such that $|u_1(x)| \leq 1/2$ for all $x \notin C$. Since $U \supseteq E_{u_1}$ and $C \setminus U$ is compact, it follows that $\sup_{x \notin U} |u_1(x)| < 1$ and hence $\sup_{x \in G} |u_1(x)[1 - u_2(x)]| < 1$ because $u_2 = 1$ on U. This implies that $u_1(1 - u_2)$ is power bounded.

It remains to verify that $v|_D = u$. Suppose first that $x \in D \cap UV = D \cap U$. Then $u_2(x) = 1$ and, as we have seen above, $u_3(d^{-1}x) \neq 0$ exactly for $d = x$, whence $v(x) = u(x)$. If $x \in D \setminus UV$, then $u_2(x) = 0$ and $d^{-1}x \notin W$ for all $d \in D \cap U$. Thus $v(x) = u_1(x) = u(x)$. This completes the proof of the existence of v.

Finally, if $E_{u_3} = \{e\}$ then $E_v = E_u$. Indeed, if $x \in E_v \cap U$ then we have $u_1(x)[1 - u_2(x)] = 0$ and, since $u_3(d^{-1}x) \neq 0$ for at most one d in $D \cap U$, we obtain $|u_3(d^{-1}x)| = 1$, so $x = d \in D$. Similarly, if $x \notin U$ then $|u_1(x)[1 - u_2(x)]| \leq |u_1(x)| < 1$ and, for all $d \in D \cap U$, $d^{-1}x \notin W$, so that $u_3(d^{-1}x) = 0$. ∎

3. The power boundedness property for some L^1-group algebras. Let H be a locally compact group and Γ a compact subgroup of the group $\mathrm{Aut}(H)$ of topological automorphisms of H which contains the inner automorphisms of H (see [16]). Let $E(H, \Gamma)$ denote the set of all continuous positive definite functions α on H such that $\alpha(e) = 1$ and

$$\alpha(x)\alpha(y) = \int_{\Gamma} \alpha(x\gamma(y))\, d\gamma, \qquad x, y \in H,$$

where $d\gamma$ is normalized Haar measure on Γ. Equip $E(H,\Gamma)$ with the topology of uniform convergence on compact subsets of H. Let

$$L^1_\Gamma(H) = \{f \in L^1(H) : f \circ \gamma = f \text{ for all } \gamma \in \Gamma\}.$$

For $\alpha \in E(H,\Gamma)$ and $f \in L^1_\Gamma(H)$, let

$$\varphi_\alpha(f) = \int_H f(x)\alpha(x)\,dx.$$

Then the map $\alpha \mapsto \varphi_\alpha$ is a homeomorphism between $E(H,\Gamma)$ and the spectrum $\Delta(L^1_\Gamma(H))$ of $L^1_\Gamma(H)$ [28, Theorem 5.8 and Proposition 4.8]. If Γ is a closed subgroup of $\Gamma_1 \subseteq \mathrm{Aut}(H)$, then there is a continuous map $\alpha \mapsto \alpha^\sharp$ from $E(H,\Gamma)$ onto $E(H,\Gamma_1)$ given by $\alpha^\sharp(x) = \int_{\Gamma_1} a(\gamma(x))\,d\gamma$ [28, Proposition 4.9].

Lemma 3.1. *Let Γ be a closed subgroup of* $\mathrm{SO}(d)$*, $d \geq 2$, and let $A = L^1_\Gamma(\mathbb{R}^d)$. Then A does not have the power boundedness property.*

Proof. Towards a contradiction, assume that A has the pb-property, and let $B = L^1_{\mathrm{SO}(d)}(\mathbb{R}^d)$ denote the subalgebra of A consisting of all $\mathrm{SO}(d)$-invariant, i.e. radial, functions. Then B has the pb-property. Indeed, if $f \in B$ and $\alpha \in E(\mathbb{R}^d,\Gamma)$, then, since f is $\mathrm{SO}(d)$-invariant,

$$\begin{aligned}\widehat{f}(\alpha) &= \int_{\mathbb{R}^d} \alpha(x)\Big(\int_{\mathrm{SO}(d)} f(\beta(x))\,d\beta\Big)\,dx = \int_{\mathrm{SO}(d)} \Big(\int_{\mathbb{R}^d} \alpha(x) f(\beta(x))\,dx\Big)\,d\beta \\ &= \int_{\mathbb{R}^d} f(x)\Big(\int_{\mathrm{SO}(d)} \alpha(\beta^{-1}(x))\,d\beta\Big)\,dx = \int_{\mathbb{R}^d} f(x)\alpha^\sharp(x)\,dx = \widehat{f}(\alpha^\sharp),\end{aligned}$$

and hence $r_B(f) = r_A(f)$, and therefore $\|f^n\|_1 \leq C < \infty$ for all $n \in \mathbb{N}$ since A has the pb-property. Now, it is not difficult to construct a radial function $f \in L^1(\mathbb{R}^d)$ satisfying $\widehat{f}|_{S^{d-1}} = 1$ and $|\widehat{f}(y)| < 1$ for all $y \in \mathbb{R}^d = \widehat{\mathbb{R}^d}$, $\|y\| \neq 1$. Thus f is power bounded and hence

$$S^{d-1} = \{y \in \mathbb{R}^d : \widehat{f}(y) = 1\} \in \mathcal{R}_c(\mathbb{R}^d)$$

by [20, Theorem 4.1]. However, $S^{d-1} \notin \mathcal{R}_c(\mathbb{R}^d)$ for $d \geq 2$ because otherwise, denoting by $p : \mathbb{R}^d \to \mathbb{R}$ the projection $y = (y_1,\ldots,y_n) \mapsto y_1$, we would have $[-1,1] = p(S^{d-1}) \in \mathcal{R}_c(\mathbb{R})$ [35, Theorem 1.3], which is a contradiction. ∎

Let G be a semidirect product $G = N \rtimes K$ where K is compact with normalized Haar measure μ. Let $f \in L^1(N)$ be K-invariant, and define $g \in L^1(G)$ by $g(x,k) = f(x)$, $x \in N$, $k \in K$. Then (convolution product), for all $n \in \mathbb{N}$,

$$g^n(x,k) = f^n(x), \quad x \in N,\ k \in K.$$

Suppose the formula holds for n. Then, since $\mu(K)=1$,

$$\begin{aligned} g^{n+1}(x,k) &= \int_N \int_K g^n((x,k)(y,l)^{-1})g(y,l)\,d\mu(l)\,dy \\ &= \int_N \int_K g^n(x(kl^{-1})\cdot y^{-1}, kl^{-1})g(y,l)\,d\mu(l)\,dy \\ &= \int_N \int_K f^n(x(kl^{-1})\cdot y^{-1})f(y)\,d\mu(l)\,dy = f^{n+1}(x). \end{aligned}$$

In particular, $\|f^n\|_1 = \|g^n\|_1$ and $r_{L^1(G)}(g) = r_{L^1(N)}(f)$.

Corollary 3.2. *Let $G = V \rtimes K$, where V is a nontrivial vector group and K is a compact group. Then $L^1(G)$ cannot have the power boundedness property.*

Proof. Notice first that if V is a vector group with scalar product $\langle\cdot,\cdot\rangle$ and Γ is a compact group of topological automorphisms of V then, after replacing $\langle\cdot,\cdot\rangle$ by the new scalar product

$$\langle\xi,\eta\rangle_\Gamma = \int_\Gamma \langle\gamma(\xi),\gamma(\eta)\rangle\,d\gamma, \qquad \xi,\eta\in V,$$

we can assume that Γ acts through orthogonal transformations on V.

Let A denote the subalgebra of $L^1(N)$ consisting of all K-invariant functions, and let B be the set of all $g\in L^1(G)$ arising from functions in A as above. Then, by the above calculations, if B has the pb-property, then so does A. By Lemma 3.1, this is impossible whenever $\dim V \geq 2$. Thus, if $L^1(G)$ has the pb-property, then $V=\mathbb{R}$ and hence K acts either trivially or as SO(1) on V. In any case, G is a locally compact group all of whose irreducible representations are finite-dimensional. However, for such a group G, by [34, Theorem 8.8], $L^1(G)$ has the pb-property only if G is compact (and abelian). This contradicts $V\neq\{0\}$. ∎

The preceding corollary is a main step in establishing the following theorem for general motion groups.

Theorem 3.3. *Let G be a semidirect product $G = N\rtimes K$, where N is an abelian locally compact group and K is a compact group. Then $L^1(G)$ has the power boundedness property* (*if and*) *only if G is compact and abelian.*

Proof. By [34, Corollary 8.3], we only have to show that N is compact. Now N is the direct product of a vector group V and a group H which contains a compact open subgroup C [18, Theorem 2]. Since $N\in[\mathrm{SIN}]_G$, by [13, Theorem 0.1] we can assume that V, H and C are all normal in G. As $L^1(G/C)$ has the pb-property, we can assume that H is discrete. Now $V\rtimes K$ is an open subgroup of G and hence $L^1(V\rtimes K)$ has the pb-property.

Corollary 3.2 shows that V is trivial. Therefore, we are left with the case that N is discrete and abelian.

Consider the subalgebra $L^1(N^t \rtimes K)$ of $L^1(G)$ and, towards a contradiction, assume that N^t is infinite. As $L^1(N^t \rtimes K)$ has the pb-property, by [34, Lemma 6.2] it suffices to produce a K-invariant function f in $L^1(N^t)$ such that $|\widehat{f}(\gamma)| = 1$ for all $\gamma \in \widehat{N^t}$ and $\widehat{f}$ has infinite range. Now such an f can be constructed as in the proof of [34, Lemma 8.5] observing that every finite subset of N^t is contained in a finite K-invariant set and hence in a finite normal subgroup of G [32, Theorem 4.32]. This contradiction shows that N^t is finite. Thus, passing to G/N^t, we can assume that N is torsion-free.

Towards a contradiction, assume that N is nontrivial and fix $x \in N$, $x \neq e$. Then $K(x)$, the K-orbit of x, is finite and the subgroup of N generated by $K(x)$ is isomorphic to $\mathbb{Z}^m$ for some $m \in \mathbb{N}$ and normal in G. The centralizer

$$C_K(\mathbb{Z}^m) = \{k \in K : k(y) = y \quad \text{for all } y \in \mathbb{Z}^m\}$$

has finite index in K. Let $M = \mathbb{Z}^m \rtimes C_K(\mathbb{Z}^m)$. Then M is open in G and M is the direct product of $\mathbb{Z}^m$ and $C_K(\mathbb{Z}^m)$. Since $L^1(M)$ has the pb-property, so does $L^1(\mathbb{Z}^m)$, contradicting [34, Lemma 8.5]. This completes the proof of Theorem 3.3. ∎

Before proceeding, we recall that a locally compact group G is *of polynomial growth* if given any relatively compact open subset U of G, there exist constants $c, d > 0$ such that $|U^n| \leq cn^d$ for all $n \in \mathbb{N}$. Nilpotent locally compact groups are of polynomial growth [14, Théorème II.4], and conversely, if G is a finitely generated discrete group of polynomial growth, then G has a nilpotent subgroup of finite index [12, Main Theorem].

The following theorem improves [34, Theorem 8.8]. We remind the reader that a locally compact group G is called an *IN-group* if G has a compact neighbourhood of the identity which is invariant under the inner automorphisms of G.

Theorem 3.4. *Let G be an IN-group of polynomial growth. If $L^1(G)$ has the power boundedness property, then G is compact and abelian.*

Proof. We first assume G is an SIN-group. Then, by [13, Theorem 2.13], G has an open normal subgroup of the form $V \times K$, where V is a vector group and K is a compact group. If $L^1(G)$ has the pb-property, then so does $L^1(V) = L^1(V \times K/K)$ and hence V must be trivial. Then, passing to G/K and observing that $L^1(G/K)$ has the pb-property, we can henceforth assume that G is discrete.

Let H be any finitely generated subgroup of G. Then, being finitely generated and having polynomial growth, H has a nilpotent subgroup M

of finite index. Since that an abelian group A with $L^1(A)$ having the pb-property is compact, a straightforward induction argument shows that M is finite and hence so is H. Then H must be abelian, and as this applies to any finitely generated subgroup H of G, it follows that G is abelian. Finally, since $L^1(G)$ has the pb-property, we conclude that G is compact [34, Theorem 8.6].

Now let G be an IN-group. Then G has a compact normal subgroup C such that G/C is an SIN-group [13, Theorem 2.5]. Since $L^1(G/C)$, being isometrically isomorphic to a subalgebra of $L^1(G)$, has the pb-property, G/C and hence G is compact. Finally, [34, Corollary 8.3] implies that G is also abelian. ∎

It is expected that for much larger classes of locally compact groups G, if $L^1(G)$ has the power boundedness property, then G must be compact and abelian. However, this appears to be a very difficult problem even for nilpotent locally compact groups.

4. Power bounded elements in $B_\lambda(G)$. In this entire section, for any locally compact group G, G_d denotes the group G made discrete. Moreover, λ_G (or simply λ if G is understood) stands for the left regular representation of G (respectively, $L^1(G)$ or $C^*(G)$) on $L^2(G)$. Let $C^*_\lambda(G)$ denote the reduced C^*-algebra of G, i.e. the norm closure of $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$.

The following lemma and its proof are a fairly straightforward adaptation of the equivalence of conditions (i) and (ii) in [34, Lemma 6.21]. We include the proof for completeness.

Lemma 4.1. *Let G be a discrete group and $E \in \mathcal{R}(G)$. For $u \in B(G)$, the following are equivalent:*

(i) *$u^n 1_E \to 0$ in the w^*-topology of $B(G) = C^*(G)^*$.*
(ii) *$u^n 1_E \to 0$ in the strong operator topology of $M(A(G))$.*

Proof. For (i)⇒(ii), if $w^*\text{-}\lim_{n\to\infty} u^n 1_E = 0$, then there exists a constant $C > 0$ such that $\|u^n 1_E\|_{B(G)} \le C$ for all $n \in \mathbb{N}$. Let $v \in A(G)$ and $\epsilon > 0$ be given. There exists $w \in A(G)$ with finite support such that $\|w - v\|_{A(G)} \le \epsilon/2C$, say $w = \sum_{j=1}^m c_j \delta_{x_j}$. It follows that

$$\|u^n 1_E w\|_{A(G)} = \Big\| \sum_{j=1}^m c_j (u^n 1_E)(x_j) \delta_{x_j} \Big\|_{A(G)} \le \sum_{j=1}^m |c_j| \cdot |\langle u^n 1_E, \delta_{x_j}\rangle| \le \epsilon/2$$

for n large enough, and hence

$$\|u^n 1_E v\|_{A(G)} \le \epsilon/2 + \|u^n 1_E\|_{B(G)} \|v - w\|_{A(G)} \le \epsilon.$$

Conversely, if (ii) holds, then the multiplication operators $M_{u^n 1_E}$, $n \in \mathbb{N}$, of $A(G)$ are pointwise bounded, and hence $\|M_{u^n 1_E}\| \le C$ for some constant

$C < \infty$ and all n by the uniform boundedness principle. Let $f \in C^*(G)$ and $\epsilon > 0$ be given, and choose $g = \sum_{j=1}^m c_j \delta_{x_j}$ such that $\|f - g\|_{C^*(G)} \leq \epsilon/C$. Since $\|M_{u^n 1_E}\| = \|u^n 1_E\|_{B(G)}$, it follows that for all $n \in \mathbb{N}$,

$$\begin{aligned} |\langle u^n 1_E, f\rangle| &\leq \|u^n 1_E\|_{B(G)} \|f - g\|_{C^*(G)} + \Big|\sum_{j=1}^m c_j \langle u^n 1_E, \delta_{x_j}\rangle\Big| \\ &\leq \epsilon/2 + \sum_{j=1}^m |c_j| \cdot \|u^n 1_E \delta_{x_j}\|_\infty \leq \epsilon/2 + \sum_{j=1}^m |c_j| \cdot \|u^n 1_E \delta_{x_j}\|_{A(G)}. \end{aligned}$$

Since the latter sum converges to 0 as $n \to \infty$, the proof is complete. ∎

COROLLARY 4.2. *Let G be a discrete group, $E \in \mathcal{R}(G)$ and $u \in B_\lambda(G)$. Consider the following two conditions:*

(1) *$u^n 1_E \to 0$ in the w^*-topology of $B_\lambda(G) = C^*_\lambda(G)^*$.*
(2) *$u^n 1_E \to 0$ in the strong operator topology of $M(A(G))$.*

Then (1) *implies* (2), *and the converse holds whenever G is amenable.*

Proof. Let $q : C^*(G) \to C^*_\lambda(G)$ denote the quotient homomorphism and suppose that (1) holds. Since $B_\lambda(G)$ is an ideal in $B(G)$, $u^n 1_E \in B_\lambda(G)$ and hence $\langle u^n 1_E, T\rangle = \langle u^n 1_E, q(T)\rangle \to 0$ for every $T \in C^*(G)$. So (2) follows from the implication (i)⇒(ii) of Lemma 4.1.

Conversely, if G is amenable, then $C^*(G) = C^*_\lambda(G)$ and thus (2)⇒(1) is a consequence of (ii)⇒(i) of Lemma 4.1. ∎

Let G be a discrete group and, for any subset E of G, let $C^*_\lambda(E) = \overline{\langle \lambda(x) : x \in E\rangle}$, the norm closure in $C^*_\lambda(G)$ of the linear span of all operators $\lambda(x)$, $x \in E$.

LEMMA 4.3. *If $E \in \mathcal{R}(G)$, then $C^*_\lambda(E) = 1_E \cdot C^*_\lambda(G)$.*

Proof. Given $T \in C^*_\lambda(G)$, choose $S_n \in \langle \lambda(x) : x \in G\rangle$, $n \in \mathbb{N}$, such that $S_n \to T$ in $C^*_\lambda(G)$. Then, for each n,

$$|\langle 1_E \cdot S_n - 1_E \cdot T, u\rangle| \leq \|S_n - T\| \cdot \|u\| \cdot \|1_E\|$$

for all $u \in A(G)$. This implies that $1_E \cdot T \in C^*_\lambda(E)$.

Conversely, let $T \in C^*_\lambda(E)$ and choose $S_n \in \langle \lambda(x) : x \in E\rangle$, $n \in \mathbb{N}$, with $\|S_n - T\| \to 0$. Then $S_n = 1_E \cdot S_n$ and hence

$$\|T - 1_E \cdot T\| \leq \|T - S_n\| + \|1_E \cdot S_n - 1_E \cdot T\| \leq 2 \cdot \|T - S_n\| \to 0,$$

so that $T = 1_E \cdot T$, as required. ∎

COROLLARY 4.4. *Let G be a discrete group, $E \in \mathcal{R}(G)$ and $u \in B_\lambda(G)$. Then the following are equivalent:*

(1) *$u^n 1_E \to 0$ in the w^*-topology of $B_\lambda(G)$.*
(2) *For each $T \in C^*_\lambda(E)$, $\langle u^n, T\rangle \to 0$ as $n \to \infty$.*

Proof. The statement follows from $C^*_\lambda(E) = 1_E \cdot C^*_\lambda(G)$ (Lemma 4.3) and $\langle u^n, 1_E \cdot T\rangle = \langle u^n 1_E, T\rangle$ for all $T \in C^*_\lambda(G)$. ∎

Proposition 4.5. *Let G be a discrete group and $u \in B_\lambda(G)$. Then u is power bounded if and only if* (i) *and* (ii) *hold, where:*

(i) *$\|u\|_\infty \le 1$, $E_u \in \mathcal{R}(G)$ and $u|_{E_u}$ is piecewise affine.*
(ii) *The sequence $(u^n)_n$ satisfies condition* (2) *of Corollary* 4.4 *with $E = G \setminus E_u$.*

Proof. If u is power bounded, then $\|u\|_\infty \le 1$, $E_u \in \mathcal{R}(G)$ by [20, Theorem 4.1] and $u|_{E_u}$ is piecewise affine [20, Lemma 3.1]. To show (ii), note that $1_F u$ is power bounded and, for each $x \in G$,

$$\langle (1_F u)^n, \lambda(x)\rangle = 1_F(x)u(x)^n \to 0$$

since $|u(x)| < 1$ for $x \in F$. As the sequence $((1_F u)^n)_n$ is norm bounded, it follows that $\langle (1_F u)^n, T\rangle \to 0$ and hence $\langle u^n, 1_F \cdot T\rangle \to 0$ for every $T \in C^*_\lambda(G)$. So (ii) is a consequence of $C^*_\lambda(E) = 1_F \cdot C^*_\lambda(G)$ (Lemma 4.3).

Conversely, suppose that (i) and (ii) hold. Then $1_F u^n \to 0$ in the w^*-topology of $B_\lambda(G)$. It follows from the uniform boundedness principle that the sequence $(1_F u^n)_n$ is norm bounded. Now, with $v = 1_{E_u} u$, $(1_F u)v = 0$ and hence $u^n = (1_F u)^n + v^n$ for all n. It therefore suffices to verify that v is also power bounded. This can be done by using precisely the same arguments as in the proof of [21, Theorem 3.4]. ∎

For any locally compact group G, let $C^*_\delta(G)$ denote the norm-closure in $\mathcal{B}(L^2(G))$ of the linear span of all operators $\lambda(x)$, $x \in G$.

Theorem 4.6. *Let G be a locally compact group which contains an open subgroup H such that H_d is amenable and let $u \in B_\lambda(G)$. Then u is power bounded if and only if* (i) *and* (ii) *hold:*

(i) *$\|u\|_\infty \le 1$ and there exist pairwise disjoint sets $F_1, \dots, F_n \in \mathcal{R}_c(G)$ such that $E_u = \bigcup_{j=1}^n F_j$, closed subgroups H_j of G and $a_j \in G$ such that $F_j \subseteq a_j H_j$, and characters γ_j of H_j and $\alpha_j \in \mathbb{T}$ such that $u(x) = \alpha_j \gamma_j(a_j^{-1}x)$ for all $x \in F_j$, $1 \le j \le n$.*
(ii) *For each $T \in C^*_\delta(G \setminus E_u)$, $\langle u^n, T\rangle \to 0$ as $n \to \infty$.*

Proof. We apply Proposition 4.5 to G_d. Let $i : G_d \to G$ denote the identity map and recall that, for every $u \in B(G)$, $\|u \circ i\|_{B(G_d)} = \|u\|_{B(G)}$ by [8, Theorem 2.20].

By [7, Theorem 2.5], λ_{G_d} extends to a $*$-homomorphism ϕ from $C^*_\delta(G)$ onto $C^*_{\lambda_{G_d}}(G_d) = C^*_\delta(G_d)$, and the existence of the subgroup H guarantees (actually, is equivalent to) that ϕ is an isomorphism [3, Theorem 1]. Hence we also have $B_\lambda(G) \circ i \subseteq B_\lambda(G_d)$.

Now let $u \in B_\lambda(G)$ be power bounded. Then (i) holds for u by [21, Theorem 3.2]. Since $u \circ i \in B_{\lambda_{G_d}}(G_d)$ is power bounded, by Proposition 4.5 we have

$$\langle u^n, 1_{G \setminus E_u} \cdot T \rangle = \langle u^n 1_{G \setminus E_u}, T \rangle \to 0$$

for every $T \in C^*_{\lambda_{G_d}}(G_d) = C^*_\delta(G)$. Thus (ii) is valid as well.

Conversely, suppose that (i) and (ii) hold for u. Then $u \circ i \in B_\lambda(G_d)$ since $B_\lambda(G) \circ i \subseteq B_\lambda(G_d)$, and $u \circ i$ satisfies condition (i) of Proposition 4.5. Moreover, for each $x \in G \setminus E_u$, by (ii),

$$\langle (u \circ i)^n, \lambda_{G_d}(x) \rangle = u(x)^n = \langle u^n, \lambda_G(x) \rangle \to 0$$

since $\lambda_G(x) \in C^*_\delta(G \setminus E_u)$. So condition (ii) of Proposition 4.5 is also satisfied and hence $u \circ i \in B(G_d)$ is power bounded. As $\|u^n\| = \|u^n \circ i\| = \|(u \circ i)^n\|$ for all n, we deduce that u is power bounded. ∎

From Theorem 4.6 and Corollary 4.2 we immediately conclude

Corollary 4.7. *Let G be a locally compact group such that G_d is amenable and let $u \in B(G)$. Then u is power bounded if and only if* (i) *and* (ii) *hold:*

(i) *$\|u\|_\infty \le 1$, $E_u \in \mathcal{R}(G)$ and $u|_{E_u}$ is as in Theorem* 4.6.
(ii) *For each $v \in A(G_d)$, $\lim_{n\to\infty} \|1_{G \setminus E_u} u^n v\| = 0$.*

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