STUDIA MATHEMATICA 209 (3) (2012)

Open projections in operator algebras II: Compact projections

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Abstract. We generalize some aspects of the theory of compact projections relative to a C^*-algebra, to the setting of more general algebras. Our main result is that compact projections are the decreasing limits of 'peak projections', and in the separable case compact projections are just the peak projections. We also establish new forms of the noncommutative Urysohn lemma relative to an operator algebra, and we show that a projection is compact iff the associated face in the state space of the algebra is weak* closed.

1. Introduction and notation. For us, an *operator algebra* is a closed algebra of operators on a Hilbert space. Selfadjoint operator algebras, or C^*-algebras, are often thought of as a kind of noncommutative topology, and one explicit and important manifestation of this is Akemann's noncommutative topology. He introduced (see e.g. [1, 2, 3]) the open, closed, and compact projections, certain projections in the second dual of the algebra, which generalize open, closed, and compact subsets of a topological space. In [8, 24] the authors and Hay generalized much of the theory of open and closed projections in C^*-algebras, to general operator algebras, thereby initiating a 'noncommutative topology relative to a general operator algebra'. This should be useful in 'noncommutative function theory' in the ways that *peak sets* and related tools have been useful in the study of function spaces (see e.g. [23]). This study was considerably advanced in [12]. In [10] the authors studied the generalization of compact projections to the setting of ternary rings of operators (TROs) (see also e.g. [5, 17, 19]).

2010 *Mathematics Subject Classification*: Primary 46L07, 46L85, 47L30, 46L52; Secondary 32T40, 46H10, 46L05, 46L30.
Key words and phrases: TRO, nonselfadjoint operator algebras, open projection, closed projection, compact projection, peak projection, minimal projection, noncommutative Urysohn lemma, faces, exposed faces, peak projection, semiexposed faces, pure states, quasi-state, state space, hereditary subalgebra, ideals, JB*-triples.

DOI: 10.4064/sm209-3-1

In the present paper we begin to study the analogue of compact projections in the setting of general operator algebras. We do this for the primary reason that this topic is important in its own right, and should have many future applications in 'noncommutative function theory'. To a lesser extent, we hope that our results will eventually find some application in the comparison theory program initiated in our recent paper [11]. Indeed, in the C^*-algebra case (see e.g. [28]), the important topic of Cuntz equivalence and subequivalence is intimately connected to compact projections. One should be aware, though, that direct attempts to apply our results to Cuntz comparison face significant problems. Indeed simple examples, such as variants of the disk algebra, seem to indicate that for Cuntz comparison one would probably mostly need to use compact projections with respect to a containing C^*-algebra, rather than with respect to the algebra itself.

The study of compact projections in the second dual of an operator algebra A turns out to be closely related to the topic of 'Urysohn type lemmas' for A. The noncommutative Urysohn lemma for C^*-algebras was proved by Akemann [1], with later refinements by him and coauthors (see our bibliography), and by L. G. Brown [13]. We begin Section 2 below with a noncommutative Urysohn lemma for nonunital algebras, which involves an ϵ that does not appear in the C^*-algebraic Urysohn lemma. One may not remove this ϵ in general for nonselfadjoint algebras. However if the two projections involved lie in the second dual of A, then the situation is much better: one may remove the ϵ and obtain a much stronger noncommutative Urysohn lemma. We define two variants of the notion of a compact projection which we show are equivalent, one in terms of the set $\{a \in A : \|1-2a\| \leq 1\}$. The latter set, which is written as $\frac{1}{2}\mathfrak{F}_A$, was shown in [12, 11] to often play the role of the positive cone in a C^*-algebra, and so we imagine that in future it will sometimes be important to have the $\frac{1}{2}\mathfrak{F}_A$ formulation. We also show that a projection q in the second dual of A is compact in our sense iff it is compact in the C^*-algebraic sense with respect to a containing C^*-algebra B. This has several consequences, for example q is compact iff it is closed and $qa = aq = q$ for some a in A.

In Section 3 we consider the nonunital case of Hay's *peak projections*. Our main result is that compact projections are the decreasing limits of 'peak projections', and in the separable case are just the peak projections. (There is a point of possible confusion here: if A is nonunital then our main result is quite different from [12, Corollary 2.21], although the latter is one of the ingredients of our result. This is because our 'peak projections' are different (see the last remark in Section 3 below). In the unital case the results essentially coincide, but in the present paper the nonunital case is the only case that is new and of interest (since in the unital case compactness is the same as being closed). One may of course use [12, Corollary 2.21] to get a similar sounding statement in terms of the unitization A^1, but we do not see how to connect this statement easily to our main result here.) We prove in Section 4 that a projection in A^{**} is compact iff the associated face in the state space $S(A)$ of the algebra is weak* closed. Thus compact projections correspond to certain weak* closed faces of $S(A)$. In Section 5 we make some remarks on pure states and minimal projections.

As we have said, in our paper we are generalizing, and are inspired by, results from C^*-algebra theory and their proofs. However for more general algebras there are significant obstacles to be overcome, some of which require deep results from earlier papers of ours and our coauthors. Although we use nothing essential from JB*-triple theory in the present paper, we mention that many results in our paper have matching counterparts with quite different proofs in that theory (see our bibliography below). In particular, [21] contains JB*-triple Urysohn lemmas (see also e.g. [20, 10]), which are considerably deeper than the C^*-algebra variants.

Throughout this paper, A is a fixed operator algebra, and B is a C^*-algebra that contains it. We will also assume throughout that A is *approximately unital*, that is, it has a contractive approximate identity (cai), although this probably is not necessary for many of the results. As we said above we are usually *not* interested in the case where A is unital. We will assume sometimes (in Sections 4 and 5 particularly) that the cai for A is also a cai for B; this is automatic by [9, Lemma 2.1.6] if B is generated by A (that is, there is no proper C^*-subalgebra of B containing A).

We now discuss notation and background, for which it will be helpful for the reader to have easy access to several of the references, particularly [9, 8, 12], for more details beyond what is presented here. For us a *projection* is always an orthogonal projection. We recall that by a theorem due to Ralf Meyer, every operator algebra A has a unique unitization A^1 (see e.g. [9, Section 2.1]). Below, 1 always refers to the identity of A^1 if A has no identity. If A is a nonunital operator algebra represented (completely) isometrically on a Hilbert space H then one may identify the unitization A^1 with the algebra $A + \mathbb{C}I_H$. The second dual A^{**} is also an operator algebra with its (unique) Arens product; this is the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B. Meets and joins in B^{**} of projections in A^{**} remain in A^{**}, since these meets and joins may be computed in the biggest von Neumann algebra contained inside A^{**}. Note that A has a cai iff A^{**} has an identity $1_{A^{**}}$ of norm 1, and then A^1 is sometimes identified with $A + \mathbb{C}1_{A^{**}}$. If A has a cai, then a *state* of A is a functional $\varphi \in \mathrm{Ball}(A^*)$ with $\varphi(e_t) \to 1$ for some (or every) cai (e_t) for A. We write $S(A)$ for the space of states. States extend uniquely to states on the unitization A^1 (see [9, 2.1.19]). If B is a C^*-algebra generated by A, then it is known that any bounded approximate identity (bai) for A is a bai for

$C^*(A)$, and hence states of A are precisely the restrictions to A of states on $C^*(A)$ (see [9, 2.1.19]). It follows that the *quasistate* space $Q(A)$ is weak* compact, where $Q(A) = \{t\varphi : t \in [0,1],\, \varphi \in S(A)\}$. Indeed if $\varphi_t \in S(B)$, $c_t \in [0,1]$, $\varphi \in A^{**}$, and $c_t\varphi_t \to \varphi$ weak* on A, then there is a $\psi \in Q(B)$ and convergent subnets $c_{t_\mu} \to c \in [0,1], \varphi_{t_\mu} \to \psi$ weak* on B. This forces $\varphi = c\psi \in Q(A)$. So $Q(A)$ is weak* compact.

A *hereditary subalgebra* (HSA) of A is a subalgebra D which has a cai and satisfies $DAD \subset D$. For the theory of HSA's in general operator algebras see [8]. These objects are in an order preserving, bijective correspondence with the *open projections* $p \in A^{**}$, by which we mean that there is a net $x_t \in A$ with $x_t = px_tp \to p$ weak*. These are also the open projections p in the sense of Akemann [1, 2] in B^{**}, where B is a C^*-algebra containing A, such that $p \in A^{\perp\perp}$. Indeed the weak* limit of a cai for a HSA is an open projection, and is called the *support projection* of the HSA. Conversely, if p is an open projection in A^{**}, then $pA^{**}p \cap A$ is a HSA in A. If an approximately unital operator algebra A is viewed as a HSA in its unitization, then we will write its support projection as e, or as 1 if there is no danger of confusion.

We recall that a *closed projection* is the 'perp' of an open projection. Suprema (resp. infima) in B^{**} of open (resp. closed) projections in A^{**} remain in A^{**}, by the fact mentioned two paragraphs earlier about meets and joins, together with the C^*-algebraic case of these facts [1, 2].

We will use the notation $\mathfrak{F}_A$ for $\{a \in A : \|1 - a\| \leq 1\}$, and $\frac{1}{2}\mathfrak{F}_A$ for $\{a \in A : \|1 - 2a\| \leq 1\}$, a subset of $\mathrm{Ball}(A)$. In [12] it is proved that elements in $\mathfrak{F}_A$ $\big($resp. $\frac{1}{2}\mathfrak{F}_A\big)$ have nth roots for all $n \in \mathbb{N}$, which are again in $\mathfrak{F}_A$ $\big($resp. $\frac{1}{2}\mathfrak{F}_A\big)$. If $a \in \mathfrak{F}_A$, then $(a^{1/n})$ converges weak* to an open projection which is written as $s(a)$, and this is both the left and the right support projection of a (see [12, Section 2]). In this case $\overline{aAa}$ is a HSA of A, and the support projection of this HSA is $s(a)$, so $\overline{aAa} = s(a)A^{**}s(a) \cap A$.

We recall that a *tripotent* is an element u with $uu^*u = u$. We order tripotents by $u \leq v$ iff $uu^*v = u$. If $x \in \mathrm{Ball}(B)$, define $u(x)$ to be the weak* limit of the sequence $(x(x^*x)^n)$ in B^{**}. This is the largest tripotent in B^{**} satisfying $vv^*x = v$ (see [17]). It is well known that if $\psi \in \mathrm{Ball}(A^*)$, then $\psi(x) = 1$ iff $\psi(u(x)) = 1$ (see e.g. [17, Lemma 3.3(i)]). In fact this is easy to prove in a few lines using spectral theory. The following result, essentially due to Edwards and Rüttimann, has been used elsewhere in our work, and is no doubt very well known (cf. e.g. [24, Proposition 5.5]).

Proposition 1.1. *Suppose x and y lie in the unit ball of a C^*-algebra B. Then $u\big(\frac{x+y}{2}\big) = u(x) \wedge u(y)$ in B^{**} (that is, $u\big(\frac{x+y}{2}\big)$ is the largest tripotent in B^{**} dominated by both $u(x)$ and $u(y)$ in the ordering of tripotents above).*

Proof. In [16, Theorem 4.4] it is proved that there is an order isomorphism from the set of tripotents in $W = B^{**}$ onto a certain set of closed faces of $\mathrm{Ball}(B^*)$. This mapping takes a tripotent u to the face $\{u\}_\prime$ defined by $\{\varphi \in \mathrm{Ball}(B^*) : \varphi(u) = 1\}$.

By the remark above the proposition, $\{u(\frac{x+y}{2})\}_\prime$ is the set of $\psi \in \mathrm{Ball}(A^*)$ with $\psi(\frac{x+y}{2}) = 1$. But this clearly happens iff $\psi(x) = \psi(y) = 1$ that is, iff $\psi \in \{u(x)\}_\prime \cap \{u(y)\}_\prime$. The latter equals $\{u(x) \wedge u(y)\}_\prime$, because of the order isomorphism mentioned in the last paragraph. Hence $\{u(\frac{x+y}{2})\}_\prime = \{u(x) \wedge u(y)\}_\prime$, and so $u(\frac{x+y}{2}) = u(x) \wedge u(y)$ because of the isomorphism in the last paragraph again. ∎

2. Compact projections and the Urysohn lemma. We recall again that, throughout, A is an operator algebra which is approximately unital, although as we said this probably is not necessary for many of the results. Also, B is a C^*-algebra containing A. We will say that a closed projection $q \in A^{**}$ is *compact* in A^{**} if there exists $a \in \mathrm{Ball}(A)$ with $q = qa$. We say that such q is $\mathfrak{F}$-*compact* in A^{**} if the a here may be chosen in $\frac{1}{2}\mathfrak{F}_A$. We will prove below that every compact projection is $\mathfrak{F}$-compact.

Any compact projection q in A^{**} is a compact projection in B^{**}, since in this case it is easy to argue from elementary operator theory that we have $q = qa^* = qaa^*$ for a as above. Clearly any closed projection is compact in A^{**} if A is unital. Any closed projection dominated by a compact projection in A^{**} is compact. If q is a projection in A^{**}, and $qb = q$ for some $b \in \mathrm{Ball}(A)$, then $qu(b) = q$, and $q \leq u(b)$. Here $u(b)$ is the tripotent mentioned in the introduction, namely the weak* limit of $b(b^*b)^n$. However $u(b)$ may not be a projection; soon we will be able to rechoose b so that it is. If q is a compact projection, and if (e_t) is a cai for the HSA supported by $e - q$ (where $e = 1_{A^{**}}$), then $e - e_t \to q$ weak*. Let $y_t = b(e - e_t) \in A$. Then $y_t \to b(e - (e - q)) = bq = q$. We have $y_t \in \mathrm{Ball}(A)$, and $y_t q = b(q - e_t q) = bq = q$.

Theorem 2.1 (A noncommutative Urysohn lemma for approximately unital operator algebras; cf. Theorem 2.24 in [12]). *Let A be an approximately unital operator algebra, a subalgebra of a C^*-algebra B, and let q be a compact projection in A^{**}. Then for any open projection $u \in B^{**}$ with $u \geq q$, and any $\epsilon > 0$, there exists an $a \in \mathrm{Ball}(A)$ with $aq = q$ and $\|a(1-u)\| < \epsilon$ and $\|(1-u)a\| < \epsilon$.*

Proof. Let $q \in A^{\perp\perp}$, let u be an open projection with $u \geq q$, and let $\epsilon > 0$ be given. As above, let (y_t) be a net in $\mathrm{Ball}(A)$ with $y_t q = q$ and $y_t \to q$ weak*. We follow the idea in the last seven lines of the proof of [8, Theorem 6.4] (see also [12, Theorem 2.24]): By the noncommutative

Urysohn lemma [1], there is an $x \in B$ with $q \leq x \leq u$. Then $y_t(1-x) \to q(1-x) = 0$ weak*, and hence weakly in B. Similarly, $(1-x)y_t \to 0$ weakly. By a routine convexity argument in $B \oplus B$, given $\epsilon > 0$ there is a convex combination a of the y_t such that $\|a(1-x)\| < \epsilon$ and $\|(1-x)a\| < \epsilon$. Clearly $aq = q$. Therefore $\|a(1-u)\| = \|a(1-x)(1-u)\| < \epsilon$. Similarly for $\|(1-u)a\| < \epsilon$. ∎

For a C^*-algebra one may remove the ϵ in the last theorem, but this is not necessarily true for more general algebras. However we will see later that things are much better, and the noncommutative Urysohn lemma can be refined, if we assume that the two projections u and q involved lie in A^{**}.

Theorem 2.2. *Let A be an approximately unital operator algebra, a subalgebra of a C^*-algebra B. If q is a projection in A^{**} then the following are equivalent:*

(i) *q is compact in B^{**},*
(ii) *q is a closed projection in $(A^1)^{**}$,*
(iii) *q is compact in A^{**},*
(iv) *q is $\mathfrak{F}$-compact in A^{**}.*

Proof. We may assume that A is nonunital, and that $1_{A^1} = 1_{B^1}$.

(iv)⇒(iii). Obvious.

(iii)⇒(ii). If q is a compact projection in A^{**}, then by the discussion above Theorem 2.1, there exists $y_t \in \mathrm{Ball}(A)$ with $y_t \to q$, and $y_t q = q$. Then $1 - y_t \to 1 - q$, and $(1-y_t)(1-q) = 1 - y_t$, so $1-q$ is open in $(B^1)^{**}$, or equivalently is open in $(A^1)^{**}$ (by [8, Theorem 2.4]). Hence q is closed in $(A^1)^{**}$.

(i)⇔(ii). q being closed in $(A^1)^{**}$, or equivalently in $(B^1)^{**}$ (by [8, Theorem 2.4]), is equivalent to q being compact in B^{**}, by the C^*-algebra case. Note that this implies that q is closed in A^{**} since $e - q = e(1-q)$ is open in $(A^1)^{**}$ where $e = 1_{A^{**}}$, being a commuting product of open projections, hence is open in A^{**}.

(ii)⇒(iv). Consider a projection $q \in A^{**}$ such that q is closed in $(A^1)^{**}$. Then $q^\perp$ is open in $(A^1)^{**}$; let $C = q^\perp (A^1)^{**} q^\perp \cap A^1$. The HSA in A^1 with support projection $q^\perp$. Note that since $eq = q$ we have $(1-e)q^\perp = 1 - e$, and so $f = 1 - e$ is a central minimal projection in $C^{**} = q^\perp (A^1)^{**} q^\perp$. Let D be the HSA in A^1 with support projection $e - q = e(1-q)$. This is an approximately unital ideal in C, indeed $D^{\perp\perp} = eC^{**}$. Note that

$$C^{**}/D^{\perp\perp} \cong C^{**}(q^\perp - q^\perp e) = C^{**} f = \mathbb{C} f \cong \mathbb{C}.$$

The map implementing this isomorphism $C^{**}/D^{\perp\perp} \cong \mathbb{C}f$ is the map $x + D^{\perp\perp} \mapsto xf$. Moreover, the map restricts to an isometric isomorphism $C/D \cong C/D^{\perp\perp} \cong \mathbb{C}f$. This is because if the range of this restriction is not $\mathbb{C}f$

then it is (0), so that $C = D$, which implies the contradiction $e = 1$. By [12, Proposition 6.1], there is an element $d \in \mathfrak{F}_C$ such that $df = 2f$. If $b = d/2 \in \frac{1}{2}\mathfrak{F}_C$, then $bf = f$. We see that $1 - b \in \frac{1}{2}\mathfrak{F}_{A^1}$, and $(1-b)e = 1-b$, so $1 - b \in \frac{1}{2}\mathfrak{F}_A$. Moreover $(1-b)q = q$ since $b \in q^\perp (A^1)^{**} q^\perp$. So q is $\mathfrak{F}$-compact. ∎

From Theorem 2.2(i) it is evident that compact projections in A^{**} have many of the properties of Akemann's compact projections. For example:

Corollary 2.3. *The infimum of any family of compact projections in A^{**} is a compact projection in A^{**}. Also, the supremum of two commuting compact projections in A^{**} is a compact projection in A^{**}.*

Proof. Notice that these infima and suprema may be viewed as infima and suprema of projections in A^{**} or in B^{**}, by remarks in the introduction (particularly the third last paragraph before Proposition 1.1). We prove only the second statement, the first being similar. This supremum may be viewed by Theorem 2.2 as the supremum of two commuting closed projections in $(B^1)^{**}$, which is closed by Akemann's theory, and hence is compact in A^{**} by Theorem 2.2 again. ∎

Corollary 2.4. *Let A be an approximately unital operator algebra, with approximately unital closed subalgebra D. A projection q in $D^{\perp\perp}$ is compact in D^{**} iff q is compact in A^{**}.*

Proof. One direction is obvious. For the other, suppose that q is compact in A^{**}. Then q is compact in B^{**} by Theorem 2.2, hence compact in D^{**} by Theorem 2.2 again, since $D \subset B$. ∎

The following is the analogue of [3, Lemma 2.5].

Corollary 2.5. *Let A be an approximately unital operator algebra. If a projection q in A^{**} is dominated by an open projection p in A^{**}, then q is compact in $pA^{**}p$ (viewed as the second dual of the HSA supported by p), or equivalently, q is compact in A^{**}.*

Theorem 2.6 (Refined noncommutative Urysohn lemma for operator algebras). *Let A be an approximately unital operator algebra. Whenever a compact projection q in A^{**} is dominated by an open projection p in A^{**}, then there exists $b \in \frac{1}{2}\mathfrak{F}_A$ with $q = qb, b = pb$. Moreover, $q \le u(b) \le s(b) \le p$.*

Proof. If $q \le p$ as stated, then by Corollary 2.5 we know q is compact in $D^{**} = pA^{**}p$, where D is the HSA supported by p. By Theorem 2.2 there exists $b \in \frac{1}{2}\mathfrak{F}_D \subset \frac{1}{2}\mathfrak{F}_A$ with $q = qb, b = bp$. Clearly $s(b) \le p$. We are unable to prove the other parts of the last statement yet, however they follow immediately from Corollary 3.3. ∎

Remark. (1) In the case where A is a C^*-algebra, the above represents a proof of Akemann's noncommutative Urysohn lemma which seems simpler than those in the literature (we remark that in this case the appeal to [12, Proposition 6.1] could be replaced by an appeal to the fact that positive elements in a quotient C^*-algebra lift to positive elements of the same norm).

In the setting of JB*-triples it is shown in [19] that a tripotent is compact iff it is closed and 'bounded' in an appropriate sense.

(2) If A is unital and q and p are mutually orthogonal compact projections in A^{**}, then there exists $a \in \mathrm{Ball}(A)$ with $aq = q$ and $ap = -p$, or equivalently, there exists $b \in \frac{1}{2}\mathfrak{F}_A$ with $bq = q$ and $bp = 0$. To see this simply use the formula $a = 2b - 1$ or $b = (a + 1)/2$.

For interest's sake, we give a different proof of our Urysohn lemma if A is a uniform algebra. A (*unital*) *uniform algebra* is a closed unital subalgebra of $C(K)$ for some compact K. An *approximately unital uniform algebra* is a Banach algebra A with cai which is isometrically isomorphic to a subalgebra of a commutative C^*-algebra. It is easy to see that this is the same as an ideal in a unital uniform algebra which has a cai; one can take the unital uniform algebra to be the unitization A^1.

Proposition 2.7. *If A is an approximately unital uniform algebra, if $q \in A^{**}$ is a compact projection in A^{**}, and if $u \in A^{**}$ is an open projection with $u \geq q$, then there exists $a \in \frac{1}{2}\mathfrak{F}_A$ with $aq = q$ and $a = au$. If A is a unital uniform algebra on a compact space K, then the above may be restated in the language of p-sets: if E and F are disjoint p-sets in K for A, then there exists $a \in A$ such that $|1 - 2a(z)| \leq 1$ for all $z \in K$, $a = 1$ on E and $a = 0$ on F.*

Proof. First assume that A is unital, acting on its maximal ideal space M_A. To prove this case, note that the closed sets E, F corresponding to q and $u^{\perp}$ are disjoint p-sets, so $E \cup F$ is a p-set. We follow [14, Proposition 4.1.14]. By a simple Zorn's lemma argument, the ideal $J_E + J_F$, if this is not A, is contained in a proper maximal ideal of A (we recall that J_E is the set of functions in A vanishing on E). This maximal ideal is the kernel of a character, which corresponds to a point $w \in M_A$. Since $f(w) = 0$ for all $f \in J_E$, we must have $w \in E$ (using a well known property of p-sets). Similarly, $w \in F$, so $w \in E \cap F = \emptyset$. This contradiction shows that $J_E + J_F = A$. Writing $1 = f + g$ with $f \in J_E$, $g \in J_F$, we find that $f = 1$ on F and $f = 0$ on E. Let $g = 2f - 1$. By [23, Theorem II.12.5], there exists $h \in \mathrm{Ball}(A)$ with $h = g$ on $E \cup F$. Let $a = (1 + h)/2$; then $a \in \frac{1}{2}\mathfrak{F}_A$, and $aq = q$ and $a = au$.

Now suppose that A is approximately unital. If q is compact in A^{**} then, as we said above, q is a closed projection in $(A^1)^{**}$. By the unital case there exists an $a \in \frac{1}{2}\mathfrak{F}_{A^1}$ with $aq = q$ and $a = au$. Since $a = ae$, where e is the support projection for A in $(A^1)^{**}$, we have $a \in A$, so $a \in \frac{1}{2}\mathfrak{F}_A$. ∎

REMARK. (1) In contrast to the C^*-algebra case, approximately unital operator algebras need not have any compact projections besides 0. Indeed if A is an approximately unital algebra of the type in [12, Section 4], without nontrivial open projections, then A has no nontrivial compact projections.

(2) The Urysohn lemma for a C^*-algebra B may be sharpened to the following, which we have not seen highlighted in the literature: Given projections $q \leq p$ in B^{**}, where q is compact and p is open, there exists $b \in B_+$ and a compact r with $q \leq b \leq s(b) \leq r \leq p$. To prove this, note that if we use the b coming from the usual Urysohn lemma for C^*-algebras, then $u(b)$ and $s(b)$ may be regarded as compact and open sets in K where $C^*(b) = C_0(K)$. By the classical Urysohn lemma there exists a nonnegative $g \in C_c(K)$ which is 1 on the compact set and 0 on the complement of the open set. Then $q \leq g(b) \leq s(g(b)) \leq r \leq s(b) \leq p$ where r corresponds to the compact support of g.

A similar proof gives an analogous Urysohn lemma for TRO's or JB*-triples. However the analogous result for nonselfadjoint operator algebras is false. Indeed this fails for the disk algebra, where closed projections correspond to closed sets in the circle of measure zero, so that there cannot be open p, u and closed q, r with $0 \neq q \leq u \leq r \leq p \neq 1$.

3. Compact projections and peak projections. Let B be a C^*-algebra. Following [24], we call a projection q in B^{**} a *peak projection* if there exists an $x \in \mathrm{Ball}(B)$ such that $xq = q$ and $\varphi(x^*x) < 1$ for all $\varphi \in Q(B)$ such that $\varphi(q) = 0$. As in [24] (see the proof of the next result), this implies that $x^n \to q$ weak*. This forces q to be closed and hence compact (since with respect to the unitization, $1-x^n = (1-x^n)(1-q) \to 1-q$ weak*, so $1-q$ is open, hence $e(1-q) = e-q$ is open, so that q is closed). We also say that x *peaks at* q, or q *is a peak for* x, in this case. If A is a subalgebra of B, and $x \in A$ peaks at q in this sense, then since $x^n \to q$ weak* we clearly have $q \in A^{**}$, and we say that q *is a peak projection for* A, or *is a peak projection in* A^{**}. If a peaks at a projection, then again from this fact about the limit of (x^n) it is easy to see that this projection is the largest projection such that $qa = q$.

The following includes a version of [24, Theorem 5.1] in our setting. In unpublished work [7] with Hay from around 2006, referred to at the end of page 357 of [8], the first author proved a generalization of the following fact to TROs (see also [26]). Here $u(x)$ is the tripotent mentioned in the introduction, namely the weak* limit of $x(x^*x)^n$.

LEMMA 3.1. *Let $x \in \mathrm{Ball}(B)$ for a C^*-algebra B, and let q be a closed projection in B^{**} such that $xq = q$. The following conditions are equivalent:*

(1) *x peaks at q,*
(2) *$\varphi(x^*x(1-q)) < 1$ for all $\varphi \in Q(B)$,*

(3) $\varphi(x^*x(1-q)) < \varphi(1-q)$ *for all* $\varphi \in Q(B)$ *such that* $\varphi(1-q) \neq 0$,
(4) $\varphi(x^*x) < 1$ *for every pure state* φ *of* B *such that* $\varphi(q) = 0$,
(5) $\|px\| < 1$ *for any compact projection* p *in* B^{**} *with* $p \leq 1-q$,
(6) $\|xp\| < 1$ *for any compact projection* $p \leq 1-q$, *and*
(7) $\|xp\| < 1$ *for any minimal projection* $p \leq 1-q$.

These imply that q *equals the weak* limit of* (x^n), *and* q *also equals* $u(x)$, *the weak* limit of* $x(x^*x)^n$. *Conversely, if this weak* limit* $u(x)$ *is a projection, then this projection* $u(x)$ *is compact, indeed* $u(x)x = u(x)$, *and the seven equivalent conditions above hold with* $q = u(x)$.

Proof. Many parts of the proof of [24, Theorem 5.1] carry through verbatim to the nonunital case of the seven numbered conditions, and the rest can be done by going to the unitization and applying [24, Theorem 5.1] as we shall see, and we leave some of this to the reader. For example, we demonstrate in the next paragraph that if (4) holds then it also holds with B replaced by B^1, hence by [24, Theorem 5.1], (1) and (2) for example hold with B replaced by B^1. Since quasistates on B have unique extensions to quasistates on B^1, it follows that (1) and (2) hold as stated.

Thus suppose that (4) holds, and that φ is a pure state on B^1 with $\varphi(q) = 0$. If $\varphi(x^*x) = 1$, then $\psi = \varphi_{|B} \in S(B)$. If $\psi = \frac{1}{2}(\varphi_1 + \varphi_2)$, with $\varphi_i \in Q(B)$, then $\varphi_i(x^*x) = 1$, so that φ_i are states. Hence they have unique extensions to states $\widetilde{\varphi}_i$ on B^1, whose average is clearly φ. Thus $\varphi = \widetilde{\varphi}_i$ and $\psi = \varphi_i$. Hence ψ is pure, and so by hypothesis $\psi(x^*x) = \varphi(x^*x) < 1$, a contradiction. So (4) holds for B^1.

(1)$\Rightarrow$(5). Assume (1) and let p be a compact projection in B^{**} such that $p \leq 1-q$. Suppose $\|px\| = 1$. Then $\|x^*px\| = 1$ and

$$x^*px \leq x^*(1-q)x = x^*x - q.$$

Since p is closed, we see that x^*px is a decreasing limit of terms from B. Thus, x^*px is upper semicontinuous on $Q(B)$. So, it achieves its maximum at a quasistate ψ, which is necessarily a state. Hence,

$$1 = \psi(x^*px) \leq \psi(x^*x - q) = \psi(x^*x) - \psi(q).$$

It follows that $\psi(x^*x) = 1$ and $\psi(q) = 0$. This contradicts (1).

That (1) implies $q = u(x)$, and equals the weak* limit of (x^n), follows as in [24, Lemma 3.6]. One considers the universal representation of B, and observes that since $qx^*xq = q$ and $(1-q)x^*x(1-q)$ is 'completely nonunitary', we have $x^n \to q$ weak* and $(x^*x)^n \to q$ weak*. It follows that $x(x^*x)^n \to xq = q$ weak*, and so $q = r = u(x)$. In the last lines we have not used that q is closed, hence as we saw after the definition of a peak projection, if $xq = q$, then (2) or (3), or the matching condition in the definition of a peak projection, implies that q is closed.

Finally, if the weak* limit $u(x)$ of $x(x^*x)^n$ is a projection q say, then from e.g. [10, Proposition 3.18] and the lines above it, we have $xq = q$ and q is a closed projection. Hence $qx^*x = q$, so q is a compact projection with $q = qx$. Suppose that φ is a state of B annihilating q. In the universal representation of B we can write $\varphi = \langle \cdot\, \zeta, \zeta\rangle$ for a unit vector $\zeta \in H_u$. If $\varphi(x^*x) = 1$, then

$$\langle (1-q)x^*x(1-q), \varphi\rangle = 1,$$

since by Cauchy–Schwarz the other terms in the expansion of the last displayed equation are 0. By the converse to Cauchy–Schwarz, $x^*x\zeta = \zeta$, and $\varphi((x^*x)^n) = 1$. Thus $\varphi(x^*q) = \varphi(q) = 1$, which is a contradiction. So (1) in the lemma holds, hence also (2)–(7). ∎

Corollary 3.2. *If $a \in A$, and q is a projection in A^{**}, then q is a peak for a in A^{**} in the sense above iff q is a peak for a in $(A^1)^{**}$ in the sense of* [24].

Proof. This is obvious from the last proof, or is an exercise. ∎

Corollary 3.3. *If $a \in \frac{1}{2}\mathfrak{F}_A$ then $u(a)$ is a projection, is in A^{**}, and is the peak for a. Indeed $q = u(a)$ satisfies all the equivalent conditions in the last lemma. Also, $u(a) \le s(a)$, and $u(a^n) = u(a^{1/n}) = u(a)$ if $n \in \mathbb{N}$. And $u(a) \neq 0$ iff $\|a\| = 1$.*

Proof. If $a \in \frac{1}{2}\mathfrak{F}_A$ for an approximately unital operator algebra A, then $\|1 - 2a\| \le 1$, and so $a = \frac{1}{2}(1 + y)$ where $y = 2a - 1 \in \mathrm{Ball}(A^1)$. By [8, Proposition 6.7], if p is the support projection of $\overline{(1-a)A^1}$, then $q = 1-p$ is a peak projection for a. So $a^n \to q$ and $(a^*a)^n \to q$ weak* by [24, Lemma 3.6]. So $a(a^*a)^n \to qa = q$ weak*. Hence $q = u(a)$, and this is a projection. Except for the last two statements, the rest follows from Lemma 3.1.

To see that $u(a) \le s(a)$ recall the power series representation from [12], namely $a^{1/n} = \sum_{k=0}^{\infty} \binom{t}{k}(-1)^k(1-a)^k$, where $t = 1/n$. Since $u(a)a = u(a)$ it follows that $u(a)a^{1/n} = u(a)$, and in the limit $u(a)s(a) = u(a)$.

Finally, $u(a^{1/m})$ is the weak* limit of $a^{nm/m}$, which is $u(a)$. Also, $a^m u(a) = u(a)$, and if r is a minimal projection dominated by $1-u(a)$ then $\|ra^m\| \le \|ra\| < 1$. So by Lemma 3.1 we have $u(a^m) = u(a)$. If $\|a\| < 1$ then clearly $(a^*a)^n \to 0$ and $u(a) = 0$. If $\|a\| = 1$ then by a fact above Proposition 1.1 there is a functional with $\psi(u(a)) = 1$, so $u(a) \neq 0$. ∎

We now present our main result:

Theorem 3.4. *If A is an approximately unital operator algebra, then*

(1) *A projection $q \in A^{**}$ is compact iff it is a decreasing limit of peak projections. This is equivalent to q being the infimum of a set of peak projections.*

(2) *If A is a separable approximately unital operator algebra, then the compact projections in A^{**} are precisely the peak projections.*
(3) *A projection in A^{**} is a peak projection in A^{**} iff it is of form $u(a)$ for some $a \in \frac{1}{2}\mathfrak{F}_A$.*

Proof. (2) In the separable case, suppose that $q = qx$ for $x \in \mathrm{Ball}(A)$. By [12, Corollary 2.17 and Proposition 2.22], $1-q = s(w)$ for some $w \in A^1$, and $q = u(z) = u(z)z$ for some $z \in \mathrm{Ball}(A^1)$. Indeed $u(z)$ is a peak projection for z in the sense of [24]. Let $a = zx \in \mathrm{Ball}(A)$. Then $qa = qzx = qx = q$, and for any compact projection $p \leq 1-q$ we have $\|pa\| \leq \|pz\| < 1$ by [24, Theorem 5.1]. So $q = u(a)$ by e.g. Lemma 3.1.

(1) One direction of the first 'iff' is obvious. For the other, let $q \in A^{**}$ be a compact projection with $q = qx$ for some $x \in \mathrm{Ball}(A)$. Then $q \leq u(x)$. Now $1-q$ is an increasing limit of $s(x_t)$ for $x_t \in A^1$ with $\|1-2x_t\| \leq 1$, by [12, Corollary 2.21], so that q is a decreasing weak* limit of the $q_t = s(x_t)^\perp = u(1-x_t)$ (see [12, Proposition 2.22]). We deduce by Proposition 1.1 that $q_t \wedge u(x) = u(1-x_t) \wedge u(x) = u(z)$, where $z = z_t$ is the average of $1-x_t$ and x. So $q_t \geq u(z)$. However a tripotent dominated by a projection in the ordering of tripotents is a projection; thus $u(z)$ is a projection. Also, $q \leq u(z)$ since $q \leq q_t$ and $q \leq u(x)$. Note that $(u(z_t))$ is decreasing, since (q_t) is decreasing. Let $a_t = z_t x \in \mathrm{Ball}(A)$. Then

$$\begin{aligned} u(z_t)a_t &= u(z_t)z_t x = u(z_t)x = u(z_t)u(z_t)^* u(x)u(x)^* x \\ &= u(z_t)u(z_t)^* u(x) = u(z_t), \end{aligned}$$

and for any compact projection $p \leq 1-u(z_t)$ we have $\|pa_t\| \leq \|pz_t\| < 1$ by [24, Theorem 5.1]. So $u(z_t) = u(a_t)$ by e.g. Lemma 3.1. Then $u(a_t) = u(z_t) \searrow q$, since $q \leq u(z_t) \leq q_t \to q$ weak*.

Finally, the 'infimum' statement follows as in [24] from the fact that $u(a) \wedge u(b) = u\big(\frac{a+b}{2}\big)$ (see Proposition 1.1).

(3) One direction is obvious. For the other, let $q = u(b)$ be a peak projection for $b \in \mathrm{Ball}(A)$. This is $\mathfrak{F}$-compact by Theorem 2.2, so there exists $r \in \frac{1}{2}\mathfrak{F}_A$ with $rq = q$. If $d = rb \in \mathrm{Ball}(A)$ then $dq = rq = q$. If φ is a state on B with $\varphi(q) = 0$ then $\varphi(d^*d) \leq \varphi(b^*b) < 1$ by Lemma 3.1. By Lemma 3.1 again, d peaks at q. If $x = dr = rbr$ then a similar argument shows that $x \in \mathrm{Ball}(A)$ and x peaks at q. So $q = u(x)$. Let D be the separable operator algebra generated by x and r. Note that $s(r) \in D^{\perp\perp}$ and indeed $s(r)$ is an identity for $D^{\perp\perp}$ (since $x = rbr$). Thus D is an approximately unital operator algebra. Let $G = C^*(D)$, the C^*-algebra generated by D in B. We now work in D^1 and G^1, and their second duals. The projection $f = 1-s(r)$ is a minimal projection in the center of $(G^1)^{**}$ (note that any cai for D is a cai for G, and hence the support projection $s(r)$ of D is also the support projection of G in $(G^1)^{**}$). Hence f is closed. Of course q is a closed

projection in $(D^1)^{**}$, and $(1-s(r))q = q-s(r)q = 0$ as $qr = q$. Therefore $f+q$ is closed, hence compact. By part (2) of the theorem, $f + q = u(k)$ for some $k \in \mathrm{Ball}(D)$. We have $kq = ku(k)q = u(k)q = q$. Let $h = 2r - 1 \in \mathrm{Ball}(D^1)$. Then $hq = 2rq - q = q$, and $hf = 2rf - f = -f$. Let $a = \frac{1}{2}(hk+1) \in \frac{1}{2}\mathfrak{F}_{D^1}$. Note that

$$\begin{aligned} af &= \tfrac{1}{2}(hkf + f) = \tfrac{1}{2}(hku(k)f + f) = \tfrac{1}{2}(hu(k)f + f) \\ &= \tfrac{1}{2}(hf + f) = \tfrac{1}{2}(-f + f), \end{aligned}$$

so $af = 0$. Hence $a \in D$. We have $aq = \frac{1}{2}(hkq+q) = \frac{1}{2}(hq+q) = q$. Let p be a minimal projection in G^{**} with $p \leq 1 - q$. Then either $pf = 0$ or $pf = p$, or equivalently $p = ps(r)$ or $ps(r) = 0$. Since f is central, $\|ap\|$ equals

$$\begin{aligned} \max\{\|apf\|, \|aps(r)\|\} &= \|aps(r)\| \\ &\leq \tfrac{1}{2}(\|hkps(r)\| + 1) \leq \tfrac{1}{2}(\|kps(r)\| + 1) < 1, \end{aligned}$$

since $\|kps(r)\| < 1$ by Lemma 3.1, because $ps(r)$ is 0 or p, which is closed, and

$$ps(r) \leq (1 - q)s(r) = s(r) - q = 1 - u(k).$$

Thus by Lemma 3.1 again, $u(a) = q$ in D^{**}. Clearly we also have $a \in \frac{1}{2}\mathfrak{F}_A$, and $u(a) = q$ in A^{**}. ∎

COROLLARY 3.5. *Let $a, b \in \frac{1}{2}\mathfrak{F}_A$ for an operator algebra A. If $s(a)$ and $s(b)$ commute then their infimum is of the form $s(c)$ for some $c \in \frac{1}{2}\mathfrak{F}_A$. Similarly, if $u(a)$ and $u(b)$ commute then their supremum is of the form $u(c)$ for some $c \in \frac{1}{2}\mathfrak{F}_A$. That is, the supremum of two commuting peak projections in A^{**} is a peak projection in A^{**}.*

Proof. Consider the separable operator algebra D generated by a and b. Then D^{**} includes $s(a)$ and $s(b)$, and hence also includes $e = s(a) \vee s(b)$. Since e is an identity for D^{**}, D is an approximately unital operator algebra. Clearly $s(a) \wedge s(b)$ is open in D^{**}, since the infimum of two commuting open projections is open (as proved by Akemann), and is in D^{**} by a remark in the 'background and notation' section of our introduction. By [12, Corollary 2.17], it equals $s(c)$ for some $c \in \frac{1}{2}\mathfrak{F}_D \subset \frac{1}{2}\mathfrak{F}_A$.

The second assertion is similar. Define D as above, a separable approximately unital operator algebra. Then $u(a)$, $u(b)$, and $u(a) \vee u(b)$ are in D^{**}. Clearly $u(a) \vee u(b)$ is closed (since the supremum of two commuting closed projections is closed), and

$$a(u(a) \vee u(b)) = au(a)(u(a) \vee u(b)) = u(a)(u(a) \vee u(b)) = u(a) \vee u(b).$$

So u is compact. By Theorem 3.4(2), $u(a) \vee u(b) = u(k)$ for some $k \in \mathrm{Ball}(D)$, and by Theorem 3.4(3) we may assume that $k \in \frac{1}{2}\mathfrak{F}_D \subset \frac{1}{2}\mathfrak{F}_A$. The final assertion is clear from what comes before. ∎

We also point out a simple noncommutative variant of the well known 'Rossi local peak set theorem' from the theory of uniform algebras [23]:

COROLLARY 3.6. *Let A be an approximately unital operator algebra. Suppose that q is a closed projection in A^{**} such that there exists an open projection $u \in A^{**}$ with $u \geq q$, and there exists $a \in \mathrm{Ball}(A)$ with $aq = q$ and $\|ap\| < 1$ for every minimal projection $p \leq u - q$. Then q is a peak projection for A.*

Proof. By Lemma 3.1, q satisfies the conditions for being a peak projection in the second dual of the HSA D supported by u. So there exists $a \in \mathrm{Ball}(D)$ with $q = u(a)$. Hence q is also a peak projection for A. ∎

REMARK. (1) The following illustrates a limitation of our theory. A closed (even compact) projection in A^{**} need not be the infimum of the open projections in A^{**} dominating it, in contrast to the C^*-algebra case (see [24, Proposition 2.3]). Similarly, an open projection in A^{**} need not be the supremum of the closed projections in A^{**} which it dominates. Indeed let A be an approximately unital operator algebra with no nontrivial open projections. See e.g. [12, Section 4]. As in that reference, $(A^1)^{**}$ has only one nontrivial open projection, namely the support projection e of A in $(A^1)^{**}$, and this is clearly not the supremum of the closed projections in A^{**} which it dominates. And $q = 1 - e$ is a closed (even compact) projection in $(A^1)^{**}$, but is not the infimum of the open projections dominating it.

(2) We do not see a simple relationship between peak projections in the sense above, for an approximately unital operator algebra, and peak projections in the sense of [12, Definition 2.20]. Probably the latter should not have been called peak projections. In particular, one cannot say for a peak projection q in the sense of our present paper that $q^{\perp} = s(x)$ for some $x \in \frac{1}{2}\mathfrak{F}_A$ (although this is true if A is separable by [12, Corollary 2.17]). For example, let $B = c_0(I)$ for an uncountable discrete set I, and let $q = (0, 1) \in B \oplus \mathbb{C}$.

4. Compact projections and faces. In this section we generalize some of the facial theory of C^*-algebras from [5] to more general algebras. In [8] we began this; in Theorem 4.1 of that paper it was proved that, for a unital operator algebra A, a projection p is open in A^{**} if and only if $F_p = \{f \in S(A) : f(p) = 0\}$ is weak* closed. It was pointed out that this remains true in the approximately unital case if one instead uses the quasi-state space $Q(A)$, which is weak* compact. Hence, closed projections q correspond to certain weak* closed faces F_{1-q} of $Q(A)$. We will prove that compact projections correspond to certain faces $F_{1-q} \cap S(A)$ of $S(A)$ which are weak* closed in $S(A)$ (or equivalently, in $Q(A)$).

In this section and the next we assume that the cai for A is a cai for the containing C^*-algebra B (this is automatic if A generates B). For a norm one element $x \in A$, we let $\{x\}^{\prime,S(A)}$ denote the weak* closed face $\{f \in S(A) : f(x) = 1\}$ in $S(A)$, and we let $\{x\}^{\prime,A}$ denote the weak* closed face $\{f \in \mathrm{Ball}(A^*) : f(x) = 1\}$ in $\mathrm{Ball}(A^*)$. Such weak* closed faces are said to be *weak* exposed* (compare with [18]). If $x \in A^{**}$, then $\{x\}_{\prime,S(A)}$ denotes $\{f \in S(A) : f(x) = 1\}$ and $\{x\}_{\prime,A}$ denotes $\{f \in \mathrm{Ball}(A^*) : f(x) = 1\}$. Such norm closed faces are said to be *norm exposed*. If A is a C^*-algebra, we write $\{x\}^{\prime,A}$ and $\{x\}_{\prime,A}$ simply as $\{x\}'$ and $\{x\}_{\prime}$. A face of $S(A)$ which is the intersection of a family of sets of the form $\{a\}^{\prime,A}$ for $a \in \mathrm{Ball}(A)$ is said to be *weak* semiexposed*. It is clear that weak* semiexposed faces are weak* closed in $S(A)$. If q is a projection, it is clear that $F_{1-q} \cap S(A) = \{q\}_{\prime,S(A)}$.

The following is a very slight restatement of [17, Lemma 3.3(i)].

PROPOSITION 4.1. *Let $x \in \mathrm{Ball}(A)$, and let $u(x)$ be the weak* limit of the sequence $(x(x^*x)^n)$ in B^{**} for a containing C^*-algebra B. Then $\{u(x)\}_{\prime,B} = \{x\}^{\prime,B}$ and $\{u(x)\}_{\prime,S(B)} = \{x\}^{\prime,S(B)}$. If $u(x)$ is a projection* (*and hence coincides with the weak* limit of (x^n) in $A^{\perp\perp}$ by Lemma* 3.1), *then $\{u(x)\}_{\prime,S(A)} = \{x\}^{\prime,S(A)}$.*

Proof. The first statement was proved in [17, Lemma 3.3(i)], and discussed towards the end of our introduction. The second is immediate from the first; and the third from the second by considering Hahn–Banach extensions of states. ∎

PROPOSITION 4.2. *If A is an approximately unital operator algebra with $a \in \mathrm{Ball}(A)$. Let B be a containing C^*-algebra. Then $u(a) \in B$ is a projection* (*and thus lies in $A^{\perp\perp}$*) *if and only if $\{a\}^{\prime,S(A)} = \{a\}^{\prime,A}$.*

Proof. Suppose that $u(a)$ is a projection. If $f \in \{a\}^{\prime,A}$, then $f(u(a)) = 1$ by Proposition 4.1. Considering the restriction of f to the two-dimensional C^*-algebra $\mathrm{Span}\{u(a), 1\}$ it is clear that $f(1) = 1$, so $f \in S(A)$. Thus $\{a\}^{\prime,S(A)} = \{a\}^{\prime,A}$.

Now suppose that $\{a\}^{\prime,S(A)} = \{a\}^{\prime,A}$. Then $\{a\}^{\prime,S(B)} = \{a\}^{\prime,B}$. Since states are selfadjoint, it follows that $\{a\}^{\prime,B} = \{a^*\}^{\prime,B}$. By Theorem 4.4 of [16] we deduce that $u(a) = u(a^*) = u(a)^*$. So $u(a)$ is a selfadjoint partial isometry, and hence by the spectral theorem equals $p - q$ for mutually orthogonal projections $p, q \in B^{**}$. If $q \neq 0$ there is a state f on B with $f(q) = 1$. Thus $0 \leq f(p) \leq f(1 - q) = 0$. So $f(p) = 0$ and $f(u(a)) = -f(q) = -1$, which by hypothesis forces $-f \in S(B)$. Thus $f = 0$, which is a contradiction. ∎

Most of the following may be deduced from Corollary 4.4 of [17] together with our Theorem 2.2, but we include a direct proof.

PROPOSITION 4.3. *If A is an approximately unital operator algebra and q is a projection in A^{**}, then the following conditions are equivalent:*

(1) *q is compact in A^{**}.*

(2) *The face* $\{q\}_{\prime,S(A)} = F_{1-q} \cap S(A)$ *is weak* semiexposed.*

(3) *The face* $\{q\}_{\prime,S(A)} = F_{1-q} \cap S(A)$ *is weak* closed in* $S(A)$.

Proof. (1)⇒(2). Suppose that q is compact. By Theorem 3.4, q is a decreasing weak* limit of projections $u(a_\mu) \in A^{**}$, with $a_\mu \in \mathrm{Ball}(A)$. If $f \in S(A)$ and $f(u(a_\mu)) = 1$ for all μ, then in the limit we have $f(q) = 1$. Conversely, if f lies in $\{q\}_{\prime,S(A)}$, then

$$1 = f(1) \geq f(u(a_\mu)) \geq f(q) = 1.$$

Hence f lies in $\bigcap\{a_\mu\}^{\prime,S(A)}$. Thus $\bigcap\{a_\mu\}^{\prime,S(A)} = \bigcap\{a_\mu\}^{\prime,A} = \{q\}_{\prime,S(A)}$, and so $\{q\}_{\prime,S(A)}$ is weak* semiexposed.

(2)⇒(3). Obvious.

(3)⇒(1). Considering Hahn–Banach extensions as in the proof of [8, Theorem 4.1], it is clear that $\{q\}_{\prime,S(B)}$ is weak* closed in $S(B)$ for a containing C^*-algebra B. Hence q is compact in B^{**} by Lemma 2.4 of [3], and hence is compact in A^{**} by Theorem 2.2. ∎

Remark. (1) As noted in [8], there may be lots of weak* closed faces of $S(A)$ that are not of the form $\{p\}_{\prime,S(A)}$.

(2) Of course a compact projection q equals $u(a)$ for some $a \in \mathrm{Ball}(A)$ iff $\{q\}_{\prime,S(A)}$ is weak* exposed. Also, this is equivalent to saying that the infimum of the $\{u(a_\mu)\}$ appearing in the proof of (2) in the last result, equals $u(a)$ for some $a \in \mathrm{Ball}(A)$ (since as we discussed above, infima of projections correspond to intersections of the matching faces [17, Corollary 4.4]). We also remark that Theorem 3.4(3) is saying that if $b \in \mathrm{Ball}(A)$ and $\{b\}^{\prime,S(A)} = \{b\}^{\prime,A}$, then $\{b\}^{\prime,S(A)} = \{a\}^{\prime,S(A)}$ for some $a \in \frac{1}{2}\mathfrak{F}_A$.

(3) It is easy to see that the correspondence $\{q\} \mapsto \{q\}_{\prime,S(A)}$ is bijective and order-preserving (by using the C^*-algebra case of this applied to Hahn–Banach extensions of states of A).

Note that a state φ of A achieves its norm at an element a of $\frac{1}{2}\mathfrak{F}_A$ iff there exists a compact projection $q \in A^{**}$ with $\varphi(q) = 1$. To see one direction of this, set $q = u(a)$ and use the fact above Proposition 1.1. For the other direction, if $q \leq u(a)$ for $a \in \frac{1}{2}\mathfrak{F}_A$, then $1 = \varphi(q) \leq \varphi(u(a)) \leq 1$, so that $\varphi(u(a)) = \varphi(a) = 1$. Rephrasing this, we have:

Corollary 4.4. *If* $\varphi \in S(A)$, *then* φ *achieves its norm at an element of* $\frac{1}{2}\mathfrak{F}_A$ *iff there exists a projection* $q \in A^{**}$ *with* $\{q\}_{\prime,S(A)}$ *weak* closed and containing* φ.

Following [5] we have:

Proposition 4.5. *If* A *is an approximately unital operator algebra and* p, q *are projections in* A^{**}, *with* p *open and* q *compact, and* $q \leq p$, *then* $[p,q]_A = \big\{a \in \frac{1}{2}\mathfrak{F}_A : aq = q,\ ap = a\big\}$ *is a (nonempty) norm closed face of* $\frac{1}{2}\mathfrak{F}_A$.

Proof. Clearly $[p,q]_A$ is norm closed, and is nonempty by one of our Urysohn lemmas. That p is open and q compact is only needed to get $[p,q]_A$ nonempty. Thus for the rest of the proof, we may assume that A is unital and $p,q \in A$, by moving to A^{**}. In this case, if $x,y \in \frac{1}{2}\mathfrak{F}_A$ with $\frac{1}{2}(x+y)p = \frac{1}{2}(x+y)$, then $\frac{1}{2}(p^\perp x p^\perp + p^\perp y p^\perp) = 0$. Taking real parts, $\frac{1}{2}(p^\perp(x+x^*)p^\perp + p^\perp(y+y^*)p^\perp) = 0$. Since the real part of an element of $\mathfrak{F}_A$ is positive, we deduce that $p^\perp(x+x^*)p^\perp = 0$. However if $x \in \mathfrak{F}_A$ then it is easy to see that $x^*x \leq x + x^*$, thus $p^\perp x^* x p^\perp \leq p^\perp(x+x^*)p^\perp = 0$. Hence $xp^\perp = 0$, so that $xp = p$. Similarly $yp = p$. By symmetry, replacing all elements by 1 minus the element, we see that $(1-x)(1-q) = 1-x$, or $xq = q$. Similarly $yq = q$. ∎

Of course every $a \in \frac{1}{2}\mathfrak{F}_A$ determines a face $[u(a), s(a)]_A$ in which it lives. The converse of the last result is false, namely a norm closed face of $\frac{1}{2}\mathfrak{F}_A$ need not equal $[q,p]_A$ for some projections q,p in A^{**} with p open and q compact (in contrast to the situation for faces of the positive part of the unit ball in a C^*-algebra [5]). However it might be interesting to characterize such faces.

Remark. See e.g. [15, 22] for the characterizations of norm closed (resp. weak* closed) faces of the unit ball of a JB*-triple (resp. its dual). This is much more difficult than the C^*-algebra case from [5]. See [16] for the (earlier) case of weak* closed (resp. norm closed) faces of the unit ball of a JBW*-triple (resp. its predual). Also, see [27] for the earlier characterization of weak* closed faces of the quasi-state space of a JB-algebra.

5. Pure states and minimal projections. In this section again A is a closed subalgebra of a C^*-algebra B, and we assume that A has a cai which is also a cai for B, so that $1_{B^{**}} = 1_{A^{**}}$. We refer the reader to e.g. [29] for the well known correspondences between pure states on a C^*-algebra B, minimal projections in B^{**}, and maximal left ideals in B.

Proposition 5.1. *A minimal projection in B^{**} which is also in $A^{\perp\perp}$ is compact in A^{**}.*

Proof. This follows from Theorem 2.2(i) and the C^*-algebra case of this result. ∎

Remark. If r is a minimal projection in B^{**} with $r \in A^{\perp\perp}$, then $(1-r)A^{**} \cap A$ is a maximal r-ideal (that is, right ideal with left cai) of A (and equals $\{a \in A : \varphi(a^*a) = 0\}$ where φ is the pure state on B associated with r). This is because any proper r-ideal J of A containing $(1-r)A^{**} \cap A$ must satisfy $(1-r)A^{**} \subset J^{\perp\perp} \subset A^{**}$. The support projection of J is $\geq e-r$, and hence must equal $e-r$ since r is minimal and J is proper. Thus $J = (1-r)A^{**} \cap A$. The converse is false in general: maximal r-ideals need not be associated with minimal projections or with pure states.

As we said in the introduction, states of A are precisely the restrictions to A of states on B (see [9, 2.1.19]). Also every extreme point of $S(A)$ is the restriction to A of a pure state on B. To see this, if ψ is an extreme point of $S(A)$, then the set of Hahn–Banach extensions of ψ is a convex weak* closed subspace of $S(B)$, hence it has an extreme point, which is easily seen to be an extreme point of $S(B)$.

In the following, $\mathrm{oa}(a)$ is the operator algebra generated by a. This has a cai if $a \in \frac{1}{2}\mathfrak{F}_A$ (see [12]).

Proposition 5.2. *Let $a \in \mathrm{Ball}(A)$, and suppose that $u(a)$ is a projection. Then a achieves its norm at a pure state φ of B, and at an extreme point of $S(A)$ (even at $\varphi_{|A}$). Also, there exists a minimal projection r in B^{**} such that $ar = r$. If $B = C^*(\mathrm{oa}(a))$, then there exists a unique pure state ψ of B with $\psi(a) = 1$, and this is a character (homomorphism).*

Proof. Basic functional analysis tells us that there is a functional that achieves its norm at a. By Proposition 4.2, there is a state which achieves its norm at a. The set E of states taking value 1 at a is a weak* closed convex subset of $Q(A)$, so it has an extreme point φ say. If $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ for states φ_k, then $1 = \varphi(a) = \frac{1}{2}(\varphi_1(a) + \varphi_2(a))$, so that $\varphi_k \in E$ and so $\varphi_k = \varphi$. Thus a achieves its norm at an extreme point of $S(A)$. As we said above, φ extends to a pure state on B. If this pure state corresponds to a minimal projection r in B^{**}, then $rar = r$, so that $ar = r$.

Note that $u(a)$ is a minimal, and a central, projection in $C^*(\mathrm{oa}(a))^{**}$. Thus if we define φ_a by $\varphi_a(x)u(a) = xu(a)$ for $x \in C^*(\mathrm{oa}(a))^{**}$, then φ_a is a character and a pure state of $C^*(\mathrm{oa}(a))$, and $\varphi_a(a) = 1$. Since pure states extend, we may extend φ_a to a pure state of B. Conversely, suppose that ψ is a pure state of B such that $\psi(a) = 1$. As we said above Proposition 1.1, this means that $\psi(u(a)) = 1$. As above, there is a minimal projection r in B^{**} such that $\psi(x)r = rxr$ for all $x \in B$. Thus $ru(a)r = r$, so that $r \leq u(a)$. If $B = C^*(\mathrm{oa}(a))$ it follows that $r = u(a)$, so that $\psi = \varphi_a$ on $C^*(\mathrm{oa}(a))$. Thus φ_a is the unique pure state on $C^*(\mathrm{oa}(a))$ with value 1 at a. ∎

Remark. (1) Any pure state on $C^*(\mathrm{oa}(a))$ different from φ_a is also given by a minimal projection $r \in C^*(\mathrm{oa}(a))^{**}$, but this time $ru(a) = 0$ (since $u(a)$ is central and not r), so that $r \leq 1 - u(a)$ and so $\|ar\| < 1$ and $\|ra\| < 1$ by Lemma 3.1.

(2) If $a \in \frac{1}{2}\mathfrak{F}_A$ has norm 1 then $u(a)$ is a projection, so the facts in the last proposition hold. Moreover, the ψ in the statement of that result restricts to an extreme point of $S(\mathrm{oa}(a))$. To see this, note that by the first paragraph of the proof applied to $A = \mathrm{oa}(a)$, there exists $\rho \in \mathrm{ext}(S(\mathrm{oa}(a)))$ with $\rho(a) = 1$. Then ρ extends to a pure state on $C^*(\mathrm{oa}(a))$, which must be φ_a.

If one considers the algebra A of upper triangular 2×2 matrices with the diagonal entries equal to each other, it is clear that there exist two orthogonal minimal projections in $C^*(A) = M_2$ which restrict to the same state on A. From examples like this we see that pure states of $C^*(A)$ need not 'separate points' of A. Also, it seems that a state of an operator algebra need not have a well defined 'support projection' as exists in the C^*-algebra theory. Things seem to be better for states that are extreme points in $S(A)$ (perhaps the q in the next result is similar to a support projection). We note that $Q(A)$ has plenty of extreme points by the Krein–Milman theorem, and these are extreme points of $S(A)$. Indeed this argument shows that every state of A is a weak* limit of convex combinations of extreme points of $S(A)$. We now make some remarks about extreme points of $S(A)$.

First, if $\varphi \in \operatorname{ext}(S(A))$ then φ need not achieve its norm at an element of $\frac{1}{2}\mathfrak{F}_A$, unlike the C^*-algebra case. Indeed if A is a nonunital algebra of the type in [12, Section 4], without nontrivial open projections, then A has no nontrivial compact projections. If there existed $a \in \frac{1}{2}\mathfrak{F}_A$ with $\varphi(a) = 1$, then $\varphi(u(a)) = 1$ as we said above Proposition 1.1, so that $u(a) \neq 0$. Thus $u(a) = e$, however e is compact iff $e \in A$, which gives the contradiction that A is unital. In fact in such examples $\frac{1}{2}\mathfrak{F}_A$ contains no norm 1 elements besides 1 (since $u(a)$ for such an element is nonzero).

Second, it seems unlikely that every $\varphi \in \operatorname{ext}(S(A))$ has a unique pure state extension to B. Note for example that the last part of the last proof in [6] should give rise to an explicit counterexample (and we thank Bill Arveson for a communication on this point). We also thank David Sherman for showing us an example of a pure state on an operator system in M_4 with multiple pure state extensions to the C^*-envelope. However we have:

PROPOSITION 5.3. *Let $\varphi \in \operatorname{ext}(S(A))$.*

(1) *There exists a compact projection q in B^{**} such that a pure state of B extends φ iff the associated minimal projection is dominated by q.*
(2) *If φ has more than one Hahn–Banach extension to B, then there are two pure state extensions of φ to B that are 'mutually orthogonal', that is, their associated minimal projections are mutually orthogonal.*

Proof. Consider the set F of Hahn–Banach extensions of φ to B. This is a face of $S(B)$ which is weak* closed in $S(B)$. Thus by the C^*-algebra theory [5], $F = \{q\}'$, for some compact projection in B^{**}. Let ψ be a pure state in F (see the discussion at the start of this section), with associated minimal projection r. Since r is minimal, by the correspondence between projections and faces [5], $\{r\}'$ is a minimal face. It is weak* closed in $S(A)$ since q is compact by Proposition 4.3. It is also weak* closed in $Q(A)$, and so it has extreme points. By minimality, $\{r\}_{\prime}$ is a singleton, hence just contains ψ. Since $\{\psi\} \subset F$ we obtain $\{r\}_{\prime} \subset \{q\}_{\prime}$, so that $r \leq q$ by the correspondence between compact projections and weak* closed faces [5].

Conversely, if r is a minimal projection dominated by q, then the associated pure state ψ_r is in $\{r\}_{\prime} \subset \{q\}_{\prime} = F$. So ψ_r is an extension of φ.

If F is not a singleton, then since it is weak* closed in $Q(A)$, it is the weak* closed convex hull of its extreme points. Hence F has more than one extreme point. These extreme points are extreme points of $S(A)$, hence are exactly the pure states of B which lie in F. By the above, these correspond to minimal projections dominated by q. Now $qB^{**}q$ is a von Neumann algebra with nontrivial atomic part. In the atomic part there are two different minimal projections, and hence by the structure of atomic von Neumann algebras there are two mutually orthogonal minimal projections. Then consider the associated pure states in F as above. ∎

Closing remark. One can ask if in Theorem 2.1, one may also choose a with $\|1 - 2a\| \leq 1$. In this closing remark we discuss this issue. This is always true iff for every compact projection $q \in A^{**}$, there exists a net (y_t) in $\frac{1}{2}\mathfrak{F}_A$ such that $y_t q = q$, and $y_t \to q$ weak*. To see one direction of this, substitute such y_t into the proof of Theorem 2.1, as in the proof of [12, Theorem 2.24]. For the other direction, proceed as in the proof of [12, Corollary 2.25], but using Theorem 2.1 on the directed set of open $u \geq q$. One obtains $a_u \in \frac{1}{2}\mathfrak{F}_A$ such that $a_u q = q$, and $(1 - a_u)$ is a cai for the ideal in A^1 supported by $1 - q$. Thus $1 - a_u \to 1 - q$ and $a_u \to q$ weak*.

In this connection we remark that by the unital case, there exists a net (y_t) in $\frac{1}{2}\mathfrak{F}_{A^1}$ such that $y_t q = q$, and $y_t \to q$ weak*. We also remark that by [12, Lemma 8.1] there exists a net (y_t) in $\frac{1}{2}\mathfrak{F}_A$ such that $y_t \to q$ weak*.

We saw in Section 3 that there exists a net (a_t) in $\frac{1}{2}\mathfrak{F}_A$ with $u(a_t) \searrow q$. This raises the question: for $a \in \frac{1}{2}\mathfrak{F}_A$ and $q = u(a)$, does there exist a net (y_t) in $\frac{1}{2}\mathfrak{F}_A$ such that $y_t q = q$, and $y_t \to q$ weak*? (Note that we cannot simply define the y_t to be powers of a, since these may leave $\frac{1}{2}\mathfrak{F}_A$.) For this latter question one may assume that A is the operator algebra $\mathrm{oa}(a)$ generated by a, which is commutative. As we said in the last paragraph, there exists a net (y_t) in $\frac{1}{2}\mathfrak{F}_A$ such that $y_t \to q$ weak*. Since $u(a)$ is a minimal and central projection in $(C^*(\mathrm{oa}(a))^{**}$, we have $q y_t = \lambda_t q$ for scalars λ_t which have limit 1 (in fact it is easy to see also that $|1 - 2\lambda_t| \leq 1$).

Acknowledgements. The first author was supported by an extension of grant 0800674 from the National Science Foundation. The second author was supported by Denison University. We thank the referee for his welcome comments, and in particular suggesting some additional references.

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Received September 23, 2011
Revised version March 9, 2012 (7311)