STUDIA MATHEMATICA 209 (3) (2012)

Bounded operators on weighted spaces of holomorphic functions on the upper half-plane

by

MOHAMMAD ALI ARDALANI (Sanandaj) and
WOLFGANG LUSKY (Paderborn)

Abstract. Let v be a standard weight on the upper half-plane $\mathbb{G}$, i.e. $v : \mathbb{G} \to]0,\infty[$ is continuous and satisfies $v(w) = v(i \operatorname{Im} w)$, $w \in \mathbb{G}$, $v(it) \ge v(is)$ if $t \ge s > 0$ and $\lim_{t\to 0} v(it) = 0$. Put $v_1(w) = \operatorname{Im} w\, v(w)$, $w \in \mathbb{G}$. We characterize boundedness and surjectivity of the differentiation operator $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$. For example we show that D is bounded if and only if v is at most of moderate growth. We also study composition operators on $Hv(\mathbb{G})$.

1. Introduction. A continuous function $v : O \to]0,\infty[$ on an open subset O of $\mathbb{C}$ is called a *weight*. For a function $f : O \to \mathbb{C}$ we consider the weighted sup-norm

$$\|f\|_v = \sup_{z\in O} |f(z)|v(z)$$

and study the space

$$Hv(O) = \{f : O \to \mathbb{C} \text{ holomorphic} : \|f\|_v < \infty\}.$$

In this paper we deal with holomorphic functions on the upper half-plane $\mathbb{G} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.

DEFINITION 1.1. A weight v on $\mathbb{G}$ is called a *standard weight* if

$$v(w) = v(i \operatorname{Im} w),\ w \in \mathbb{G},\ v(is) \le v(it) \text{ when } 0 < s \le t, \text{ and } \lim_{t\to 0} v(it) = 0.$$

EXAMPLE. Let $\alpha, \beta > 0 > \gamma$. Then the functions $v_1(w) = (\operatorname{Im} w)^\beta$, $v_2(w) = \min(v_1(w), 1)$,

$$v_3(w) = \begin{cases} (1-\log(\operatorname{Im} w))^\gamma & \text{if } \operatorname{Im} w < 1, \\ \operatorname{Im} w & \text{if } \operatorname{Im} w \ge 1, \end{cases} \qquad v_4(w) = \log(\operatorname{Im} w + 1),$$

$v_5(w) = (\operatorname{Im} w)^\beta \exp(\alpha \operatorname{Im} w)$ and $v_6(w) = \exp(-\beta/\operatorname{Im} w)$, $w \in \mathbb{G}$, are standard weights.

2010 *Mathematics Subject Classification*: Primary 46E15; Secondary 47B38.
Key words and phrases: differentiation operator, composition operator, holomorphic functions, weighted spaces, upper half-plane.

DOI: 10.4064/sm209-3-2

Observe that for a holomorphic function $f : \mathbb{G} \to \mathbb{C}$, we have $f \in Hv(\mathbb{G})$ if and only if $\sup_{x\in\mathbb{R}} |f(x + it)| = O(1/v(it))$ as $t \to 0$ and $t \to \infty$.

We want to compare the growth of $f \in Hv(\mathbb{G})$ with the growth of f'. We investigate the differentiation operator $Df = f'$ as an operator between $Hv(\mathbb{G})$ and $Hv_1(\mathbb{G})$ where $v_1(w) = \operatorname{Im} w\, v(w)$, $w \in \mathbb{G}$. We also study the growth of $f \circ \varphi$ where $\varphi : O \to \mathbb{G}$ is a holomorphic map and $O \subset \mathbb{C}$ is open, i.e. we deal with composition operators between two different weighted spaces of holomorphic functions on $\mathbb{G}$.

Definition 1.2. Let v be a standard weight on $\mathbb{G}$.

(i) v satisfies *condition* $(\star)$ if there are constants $c, \beta > 0$ such that

$$\frac{v(it)}{v(is)} \le c\left(\frac{t}{s}\right)^{\beta} \quad \text{whenever } 0 < s \le t.$$

(ii) v satisfies *condition* $(\star\star)$ if there are constants $d, \gamma > 0$ such that

$$d\left(\frac{t}{s}\right)^{\gamma} \le \frac{v(it)}{v(is)} \quad \text{whenever } 0 < s \le t.$$

It is easily seen (see [1]) that v satisfies $(\star)$ if and only if

$$\sup_{k\in\mathbb{Z}} \frac{v(i2^{k+1})}{v(i2^k)} < \infty,$$

and v satisfies $(\star\star)$ if and only if

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{v(i2^k)}{v(i2^{n+k})} < 1.$$

In the preceding examples, v_1, v_2, v_3 and v_4 satisfy $(\star)$ and v_1 also satisfies $(\star\star)$. (Note that a weight is necessarily unbounded if it satisfies $(\star\star)$.)

It was shown in [1] that, for a standard weight v with $(\star)$, $Hv(\mathbb{G})$ is isomorphic to l_∞ if and only if v also satisfies $(\star\star)$. Moreover, according to a result of Stanev [8], $Hv(\mathbb{G}) \ne \{0\}$ if and only if there are constants $a, b > 0$ such that $v(it) \le ae^{bt}$, $t > 0$.

In the following we always assume that v is such that $Hv(\mathbb{G}) \ne \{0\}$.

We start with the differentiation operator D where $Df = f'$.

Theorem 1.3. *Let v be a standard weight and put $v_1(w) = \operatorname{Im} w\, v(w)$, $w \in \mathbb{G}$. Then the following are equivalent:*

(i) $DHv(\mathbb{G}) \subset Hv_1(\mathbb{G})$.
(ii) *D is a bounded operator* $Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$.
(iii) *v satisfies* $(\star)$.

As a corollary, v is an essential weight (see Proposition 3.5 below; for the definition see Section 2). We prove Theorem 1.3 in Section 3. In Section 4 we prove the following result.

THEOREM 1.4. *Let v be a standard weight and put $v_1(w) = \operatorname{Im} w\, v(w)$, $w \in \mathbb{G}$. Then the following are equivalent:*

(i) $DHv(\mathbb{G}) = Hv_1(\mathbb{G})$.
(ii) *D is an isomorphism between $Hv(\mathbb{G})$ and $Hv_1(\mathbb{G})$.*
(iii) *v satisfies $(\star)$ and $(\star\star)$.*

We mention again that $(\star)$ and $(\star\star)$ together imply that $Hv(\mathbb{G})$ is isomorphic to l_∞.

There are results for radial weights on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ corresponding to Theorems 1.3 and 1.4 (see [5, 7]). However the preceding theorems cannot be inferred from them by applying a biholomorphic map $\alpha : \mathbb{D} \to \mathbb{G}$. If v is a standard weight on $\mathbb{G}$ then $v \circ \alpha$ is not radial on $\mathbb{D}$, i.e. $v(\alpha(z)) \neq v(|\alpha(z)|)$ on $\mathbb{D}$ in general. For weights on $\mathbb{D}$ of the type $v \circ \alpha$ nothing is known so far.

We also consider composition operators $C_\varphi = f \circ \varphi$ where $\varphi : O \to \mathbb{G}$ is a holomorphic map and $O \subset \mathbb{C}$ is open. As a direct consequence of [4] we obtain (see end of Section 4)

COROLLARY 1.5. *Let v_2 be a weight on O and v_1 a standard weight on $\mathbb{G}$ satisfying condition $(\star)$. Then C_φ is a bounded operator $Hv_1(\mathbb{G}) \to Hv_2(O)$ if and only if*

$$\sup_{z \in O} \frac{v_2(z)}{v_1(\varphi(z))} < \infty,$$

2. The associated weight. For a weight $v : O \to]0, \infty[$ the function

$$\tilde{v}(z) = \inf\{1/|h(z)| : h \in Hv(O),\ \|h\|_v \le 1\}, \quad z \in O,$$

is called *the associated weight*. It is known ([2]) that $\|f\|_v = \|f\|_{\tilde{v}}$ for any $f \in Hv(O)$ and, for any $z \in O$, there is $h \in Hv(O)$ with $\|h\|_v \le 1$ such that $\tilde{v}(z) = 1/|h(z)|$. Moreover, $v(z) \le \tilde{v}(z)$ for all $z \in O$. The weight v is called *essential* if v and $\tilde{v}$ are equivalent.

Now let v be a standard weight on $\mathbb{G}$. It is easily seen that then $\tilde{v}(w) = \tilde{v}(i \operatorname{Im} w)$, $w \in \mathbb{G}$.

LEMMA 2.1. *We have $\tilde{v}(it) \ge \tilde{v}(is)$ whenever $t \ge s$.*

Proof. Take

$$\alpha(z) = \frac{1+z}{1-z} i, \quad z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Then α maps $\mathbb{D}$ onto $\mathbb{G}$, we have

$$\alpha^{-1}(w) = \frac{w - i}{w + i}, \quad w \in \mathbb{G},$$

α^{-1} maps $\Gamma(t) := \{w : \operatorname{Im} w = t\}$ onto

$$\Delta(t) := \left\{z : \left|z - \frac{t}{1+t}\right| = \frac{1}{1+t}\right\} \setminus \{1\}$$

and we obtain $\lim_{\operatorname{Re} w\to\pm\infty} \alpha^{-1}(w) = 1$.

Now fix $t > s > 0$. Then $\Delta(t)$ is a subset of the interior of $\Delta(s)$. Hence for $h \in Hv(\mathbb{G})$ we obtain

$$M(h,t) := \sup_{w\in\Gamma(t)} |h(w)| = \sup_{z\in\Delta(t)} |h(\alpha(z))| \le \sup_{z\in\Delta(s)} |h(\alpha(z))| = M(h,s)$$

in view of the maximum principle. (Take into account that $h\circ\alpha$ is bounded on $\left\{z : \left|z - \frac{s}{1+s}\right| \le \frac{1}{1+s}\right\} \setminus \{1\}$ since h is bounded on $\{w : \operatorname{Im} w \ge s\}$.) Since the translation operator T_x with $(T_x h)(w) = h(w+x)$ is an isometry for any real x, we infer

$$\begin{aligned}\tilde v(is) &= \inf\{1/|k(is)| : k \in Hv(\mathbb{G}),\ \|k\|_v \le 1\}\\ &= \inf\{1/\sup_x |(T_x h)(is)| : h \in Hv(\mathbb{G}),\ \|h\|_v \le 1\}\\ &= \inf\{1/M(h,s) : h \in Hv(\mathbb{G}),\ \|h\|_v \le 1\}\\ &\le \inf\{1/M(h,t) : h \in Hv(\mathbb{G}),\ \|h\|_v \le 1\} = \tilde v(it). \ \blacksquare\end{aligned}$$

Observe that $\gamma_{\tilde v}(t) := \tilde v(it)$ is monotone, hence differentiable a.e. Moreover, the fundamental theorem of integral calculus holds for $\gamma_{\tilde v}$ (see [6]).

Lemma 2.2. *Let v_1 be as in Theorem 1.3. Then with $c = \exp(-3\pi^2/4 - 1/4)$ we have*

$$c t\tilde v(it) \le \tilde{v_1}(it) \le t\tilde v(it) \quad \textit{for all } t > 0.$$

Proof. Fix $t_0 > 0$ and consider $h \in Hv_1(\mathbb{G})$ with $\|h\|_{v_1} \le 1$ and $\tilde{v_1}(it_0) = 1/|h(it_0)|$. Put

$$g(w) = h(w)e^{-\log^2(w/t_0)}$$

where log is the main branch of the complex logarithm. Then g is holomorphic on $\mathbb{G}$. Put $\delta(t) = t^{-1}\exp(-\log^2(t/t_0))$, $t > 0$. Then $\delta(t)$ attains its sup at $t_0\exp(-1/2)$. We have $\sup_{t>0}\delta(t) = \exp(1/4)/t_0$. Moreover, with $w = x + it$,

$$|g(w)| = |h(w)| \exp\left(\left(\arg\left(\frac{w}{t_0}\right)\right)^2 - \log^2\left(\frac{|w|}{t_0}\right)\right)$$

and hence

$$\sup_{x\in\mathbb{R}} |g(x+it)| \le \sup_{x\in\mathbb{R}} |h(x+it)| \exp\left(\pi^2 - \log^2\left(\frac{t}{t_0}\right)\right).$$

This implies

$$\|g\|_v \le \exp(\pi^2) \sup_{t>0} \sup_{x\in\mathbb{R}} (|h(x+it|tv(it)) \sup_{t>0} \frac{\exp(-\log^2(t/t_0))}{t}$$
$$= \frac{\exp(\pi^2+1/4)}{t_0}\|h\|_{v_1} \le \frac{\exp(\pi^2+1/4)}{t_0}.$$

Since $\exp(\pi^2/4)h(it_0) = g(it_0)$ we obtain

$$\tilde{v}(it_0) \le \frac{\exp(\pi^2+1/4)}{t_0} \cdot \frac{1}{|g(it_0)|} = \frac{\exp(3\pi^2/4+1/4)}{t_0}\tilde{v}_1(it_0).$$

On the other hand, let $f \in Hv(\mathbb{G})$ with $\|f\|_v \le 1$ and $\tilde{v}(t_0) = 1/|f(it_0)|$. Put $k(w) = f(w)/w$. Then

$$|k(x+it)|tv(it) = \frac{|f(x+it)|tv(it)}{\sqrt{x^2+t^2}} \le \|f\|_v \le 1.$$

We obtain

$$t_0\tilde{v}(it_0) = \frac{t_0}{|f(it_0)|} = \frac{1}{|k(it_0)|} \ge \tilde{v}_1(it_0). \ \blacksquare$$

3. Proof of Theorem 1.3. Let v be a standard weight and put

$$b_v = \inf\Big\{b>0 : \sup_{t>0} e^{-bt}v(it) < \infty\Big\}.$$

According to our general assumption on v preceding Theorem 1.3, we have $b_v < \infty$.

Consider the functions $e^{-nt}v(it)$, $t>0$, $n>b_v$. We have $\sup_{t>0} e^{-nt}v(it) = \|\Theta_n\|_v$ where $\Theta_n(w) = e^{inw}$, $w \in \mathbb{G}$. Let $s_n = \inf\{t>0 : e^{-nt}v(it) = \|\Theta_n\|_v\}$ and $t_n = \sup\{t>0 : e^{-nt}v(it) = \|\Theta_n\|_v\}$.

Lemma 3.1.

(a) *Fix* $r_m > 0$ *with* $e^{-mr_m}v(ir_m) = \|\Theta_m\|_v$. *Then*

$$e^{(n-m)r_n}\|\Theta_n\|_v \le \|\Theta_m\|_v \le e^{(n-m)r_m}\|\Theta_n\|_v$$

for any $n > b_v$ *and* $m > b_v$.

(b) *If* $m \le n$ *then* $t_n \le s_m$ *and* $\|\Theta_n\|_v \le \|\Theta_m\|_v$.

(c) $\lim_{n\to\infty} t_n = 0$.

Proof. (a) We have $e^{-mr_n}v(ir_n) \le \|\Theta_m\|_v$. This implies the first inequality. Moreover

$$\|\Theta_m\|_v = e^{(n-m)r_m}e^{-nr_m}v(ir_m) \le e^{(n-m)r_m}\|\Theta_n\|_v.$$

(b) Then (a) implies $(n-m)r_n \le (n-m)r_m$. Hence $r_n \le r_m$ if $n \ge m$. This yields (b).

(c) Let $b > b_v$. Then $v(it) \le ce^{bt}$, $t>0$, for some constant $c>0$. Let $s = \inf_{n>b_v} s_n$ and assume that $s>0$. Fix $0<r<s$. Then we obtain, in

view of (b), since s_n is decreasing,

$$\|\Theta_n\|_v = e^{-ns_n} v(is_n) \le e^{-n(s_n - r)} \frac{v(is_n)}{v(ir)} e^{-nr} v(ir)$$
$$\le e^{-n(s-r)} \frac{v(is_{b_v+1})}{v(ir)} e^{-nr} v(ir) < e^{-nr} v(ir)$$

if $n \ge b_v + 1$ is large enough, a contradiction. Hence $\inf_{n > b_v} s_n = 0$. Combined with (b), this proves (c). ■

Proposition 3.2. *Let v satisfy $(\star)$. Then $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ is bounded.*

Proof. Fix $w_0 = x_0 + it_0 \in \mathbb{G}$. Consider $r = t_0/2$. Then the Cauchy integral formula implies, for any $f \in Hv(\mathbb{G})$,

$$|f'(w_0)| v_1(w_0) = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(w_0 + re^{i\varphi})}{r^2 e^{2i\varphi}} ire^{i\varphi}\, d\varphi \right| t_0 v(it_0)$$
$$\le 2\Big(\sup_{\varphi} |f(w_0 + re^{i\varphi})| v(i(t_0 + r\sin\varphi))\Big) \cdot$$
$$\times \sup_{\varphi} \left(\frac{v(it_0)}{v(i(t_0 + 2^{-1} t_0 \sin\varphi))} \right)$$
$$\le 2c\|f\|_v$$

for some universal constant $c > 0$, in view of $(\star)$. ■

Proposition 3.3. *Let $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ be bounded. Then $\tilde{v}$ satisfies $(\star)$.*

Proof. Fix $t_0 > 0$ such that $\gamma_{\tilde{v}}(t) = \tilde{v}(it)$, $t > 0$, is differentiable at t_0. Find $h \in Hv(\mathbb{G})$ with $\|h\|_v \le 1$ such that $\tilde{v}(it_0) = 1/|h(it_0)|$. We can assume that $h(it_0) = |h(it_0)|$ (otherwise take $h \cdot \overline{h(it_0)}/|h(it_0)|$ instead of h). This implies

$$\sup_{w \in \mathbb{G}} |\operatorname{Re} h(w)| \tilde{v}(w) = h(it_0)\tilde{v}(it_0) = 1 = \|h\|_{\tilde{v}}.$$

Put $\tau(t) = \operatorname{Re} h(it)$. We have $\tau'(t_0)\gamma_{\tilde{v}}(t_0) + \tau(t_0)\gamma'_{\tilde{v}}(t_0) = 0$. Hence

$$\frac{\tau'(t_0)}{\tau(t_0)} = -\frac{\gamma'_{\tilde{v}}(t_0)}{\gamma_{\tilde{v}}(t_0)}.$$

Since $\gamma'_{\tilde{v}}(t_0)$, $\gamma_{\tilde{v}}(t_0)$ and $\tau(t_0)$ are nonnegative, $\tau'(t_0)$ must be nonpositive. Moreover we have $|\tau'(t_0)| \le |h'(it_0)|$ and $\tau(t_0) = h(it_0)$. By assumption and Lemma 2.2,

$$|h'(it_0)| t_0 \tilde{v}(it_0) \le c\|D\| \cdot |h(it_0)| \tilde{v}(it_0)$$

with $c = \exp(3\pi^2/4 + 1/4)$. Hence

$$\frac{\gamma'_{\tilde{v}}(t_0)}{\gamma_{\tilde{v}}(t_0)} = \frac{|\tau'(t_0)|}{|\tau(t_0)|} \le \frac{|h'(it_0)|}{|h(it_0)|} \le \frac{c\|D\|}{t_0} \quad \text{a.e. (with respect to } t_0).$$

This implies that $\log\gamma_{\tilde v}(t) - \|D\|c\log t$ and hence $\tilde v(it)t^{-\|D\|c}$ is decreasing in t. We conclude

$$\frac{\tilde v(it)}{\tilde v(is)} \le \left(\frac{t}{s}\right)^{\|D\|c} \quad \text{for } 0 < s \le t. \ \blacksquare$$

COROLLARY 3.4. *Under the assumptions of Proposition* 3.3 *we have* $b_{\tilde v} = 0$ *and hence* $b_v = 0$.

Proof. $(\star)$ implies $\tilde v(it) \le t^{\|D\|c}\tilde v(i)$ for $t \ge 1$. This implies $b_{\tilde v} = 0$. ∎

PROPOSITION 3.5. *If* $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ *is bounded then* v *is essential. Moreover,*

$$e^{-2\|D\|}\tilde v(it) \le v(it) \le \tilde v(it) \quad \textit{for all } t > 0.$$

Proof. If v is bounded then $1 \in Hv(\mathbb{G})$. Lemma 3.1 implies that $u := \sup_{m>0} t_m = \lim_{m\to 0} t_m = \infty$.

If v is unbounded then by definition and Corollary 3.4 we again obtain $u = \infty$. Indeed, otherwise there is a $t > u$. We have

$$\lim_{m\to 0} e^{-m(t_m - t)}\frac{v(it_m)}{v(it)} = \frac{v(iu)}{v(it)} < 1$$

if t is large enough. Hence

$$\|\Theta_m\|_v = e^{-mt_m}v(it_m) = e^{-m(t_m-t)}\frac{v(it_m)}{v(it)}e^{-mt}v(it) < e^{-mt}v(it)$$

if m is small enough, a contradiction. Hence in any case, for any $t > 0$ there are $m_1, m_2 > 0$ with $t_{m_1} \le t \le t_{m_2}$.

We have $\tilde v(it_m) = v(it_m)$ for all $m > 0$ since

$$v(it_m) \le \tilde v(it_m) \le \frac{\|\Theta_m\|_v}{|e^{-mt_m}|} = v(it_m).$$

Lemma 3.1 implies $s_m = \sup_{k>m} t_k$ and $t_m = \inf_{k<m} s_k$. We have

$$me^{-mt_m}t_m v(it_m) = |ime^{i(it_m m)}|v_1(it_m) \le \|D\|\cdot\|\Theta_m\|_v = \|D\|e^{-mt_m}v(it_m).$$

This yields $t_m m \le \|D\|$ and hence

$$\frac{\tilde v(it_m)}{\tilde v(is_m)} = e^{m(t_m - s_m)} \le e^{2\|D\|}.$$

Now, let $t > 0$. Put $m_1 = \sup\{m > 0 : t_m \le t\}$ and $m_2 = \inf\{m > 0 : s_m \ge t\}$. Then $s_{m_1} \le t \le t_{m_2}$. Lemma 3.1 implies $m_2 \le m_1$. If $m_2 < m < m_1$ then either $s_{m_1} < s_m \le t_m \le t$ or $t \le t_m < t_{m_2}$. In both cases we obtain a contradiction. Hence $m := m_1 = m_2$ and $t \in [s_m, t_m]$, so that

$$\tilde v(it) \le \tilde v(it_m) = v(it_m) \le e^{2\|D\|}v(is_m) \le e^{2\|D\|}v(it) \le e^{2\|D\|}\tilde v(it). \ \blacksquare$$

COROLLARY 3.6. *Let* $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ *be bounded. Then* v *satisfies* $(\star)$.

If $DHv(\mathbb{G}) \subset Hv_1(\mathbb{G})$ then D is a bounded operator $Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ by the closed graph theorem. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4. First we show

PROPOSITION 4.1. *Let v satisfy $(\star)$ and $(\star\star)$. Then $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ is bounded and surjective.*

Proof. We already showed that D is bounded. Since v satisfies $(\star\star)$, it is unbounded. To show the surjectivity take $h \in Hv_1(\mathbb{G})$. Let $w_0 \in \mathbb{G}$ be fixed, say $w_0 = in$ for some integer $n > 0$, and let $w = x + it \in \mathbb{G}$ be arbitrary. Moreover, let Γ be a Jordan curve in $\mathbb{G}$ connecting w_0 and w. Then $(Ih)(w) := \int_\Gamma h(u)\,du$ is holomorphic and $(Ih)' = h$. We now define

$$(I_n h)(x+it) := \int_n^t h(x+is)i\,ds + \int_0^x h(s+in)\,ds$$

(i.e. Γ runs parallel to the axes from in to $x + in$ and then to $x + it$). Then $I_n h$ is holomorphic and $(I_n h)' = h$. Moreover, there are $d, \gamma > 0$ such that, for $t \le n$ and $|x| \le n$,

$$\begin{aligned}|(I_n h)(x+it)|v(it) &\le \sup_{t\le s\le n} |h(x+is)|sv(is)\left|\int_t^n \frac{v(it)}{sv(is)}\,ds\right| \\ &\quad + \sup_{\tilde{x}\in\mathbb{R}} |h(\tilde{x}+in)|nv(in)\left(\int_0^n \frac{1}{n}\,ds\right)\frac{v(it)}{v(in)} \\ &\le d\|h\|_{v_1}\left|\int_t^n \frac{t^\gamma}{s^{\gamma+1}}\,ds\right| + \|h\|_{v_1} \\ &= \frac{d}{\gamma}\|h\|_{v_1}\left|\frac{t^\gamma}{t^\gamma} - \frac{t^\gamma}{n^\gamma}\right| + \|h\|_{v_1} \le \left(\frac{d}{\gamma}+1\right)\|h\|_{v_1}.\end{aligned}$$

In the second inequality we used that v satisfies $(\star\star)$. Hence $(I_n h)_n$ is locally bounded. By Montel's theorem we find a subsequence which converges uniformly on compact subsets to a holomorphic function g. We obtain $\|g\|_v \le (1 + d/\gamma)\|h\|_{v_1} < \infty$. Thus $g \in Hv(\mathbb{G})$ and $g' = h$. ∎

To show the converse we need

LEMMA 4.2. *If $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ is a surjective operator then v is unbounded.*

Proof. Otherwise the function $g(w) = 1/w$, $w \in \mathbb{G}$, is an element of $Hv_1(\mathbb{G})$ since

$$\sup_{t>0,\, x\in\mathbb{R}} \frac{tv(it)}{\sqrt{x^2+t^2}} \le \sup_{t>0} v(it) < \infty.$$

Then there is $h \in Hv(\mathbb{G})$ with $h'(w) = 1/w$, $w \in \mathbb{G}$. Hence there is a constant $c \in \mathbb{C}$ with $h(w) = \log w + c$, $w \in \mathbb{G}$. But this means $\|h\|_v \geq \sup_{t>0} |\log t + c| v(it) = \infty$, a contradiction. ∎

PROPOSITION 4.3. *Let $D : Hv(\mathbb{G}) \to Hv_1(\mathbb{G})$ be bounded and surjective. Then v satisfies $(\star)$ and $(\star\star)$.*

Proof. We already showed that v satisfies $(\star)$. Moreover, by Lemma 4.2 we know that v cannot be bounded. This implies that D is injective because otherwise $Hv(\mathbb{G})$ would contain a constant function different from zero. The open mapping theorem implies that D is an isomorphism between $Hv(\mathbb{G})$ and $Hv_1(\mathbb{G})$. By definition v_1 always satisfies $(\star\star)$. Moreover it also satisfies $(\star)$ since v does. Hence $Hv_1(\mathbb{G})$ is isomorphic to l_∞. It follows that $Hv(\mathbb{G})$ is isomorphic to l_∞. This implies that v satisfies $(\star\star)$ (see [1]). ∎

Finally we note that $DHv(\mathbb{G}) = Hv_1(\mathbb{G})$ implies that D is surjective and bounded by the closed graph theorem. This completes the proof of Theorem 1.4.

Proof of Corollary 1.5. According to [4, Proposition 5], the boundedness of $C_\varphi : Hv_1(\mathbb{G}) \to Hv_2(O)$ is equivalent to

$$\sup_{z \in O} \frac{v_2(z)}{\tilde{v}_1(\varphi(z))} < \infty.$$

(This is a generalization of a corresponding condition for holomorphic functions on the unit disc, see [3, Proposition 2.1].) From Proposition 3.5 we conclude that the boundedness of C_φ is equivalent to

$$\sup_{z \in O} \frac{v_2(z)}{v_1(\varphi(z))} < \infty. \ ∎$$

References

[1] M. A. Ardalani and W. Lusky, *Weighted spaces of holomorphic functions on the upper half-plane*, Math. Scand., to appear.

[2] K. D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), 137–168.

[3] J. Bonet, P. Domański, M. Lindström and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. 64 (1998), 101–118.

[4] J. Bonet, M. Fritz and E. Jorda, *Composition operators between weighted inductive limits of spaces of holomorphic functions*, Publ. Math. Debrecen 67 (2005), 333–348.

[5] A. Harutyunyan and W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. 184 (2008), 233–247.

[6] E. Hewitt and K. R. Stromberg, *Real and Abstract Analysis*, Grad. Texts in Math. 25, Springer, New York, 1975.

[7] W. Lusky, *Growth conditions for harmonic and holomorphic functions*, in: Functional Analysis (Trier, 1994), de Gruyter, 1996, 281–291.

[8] M. A. Stanev, *Weighted Banach spaces of holomorphic functions in the upper half plane*, arXiv:math.FA/9911082 v1 (1999).

Mohammad Ali Ardalani
Department of Mathematics
Faculty of Science
University of Kurdistan
Pasdaran Ave.
Postal code: 66177-15175
Sanandaj, Iran
E-mail: M.Ardalani@uok.ac.ir

Wolfgang Lusky
Institute for Mathematics
University of Paderborn
Warburger Str. 100
D-33098 Paderborn, Germany
E-mail: lusky@math.upb.de

Received October 4, 2011
Revised version May 25, 2012 (7317)