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2-local Jordan automorphisms on operator algebras

by

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Abstract. We investigate 2-local Jordan automorphisms on operator algebras. In particular, we show that every 2-local Jordan automorphism of the algebra of all $n \times n$ real or complex matrices is either an automorphism or an anti-automorphism. The same is true for 2-local Jordan automorphisms of any subalgebra of $\mathcal{B}(X)$ which contains the ideal of all compact operators on X, where X is a real or complex separable Banach spaces and $\mathcal{B}(X)$ is the algebra of all bounded linear operators on X.

1. Introduction. A linear mapping ϕ on an algebra $\mathcal{A}$ is called a *local automorphism* if for every $a \in \mathcal{A}$ there exists an automorphism $\phi_a : \mathcal{A} \to \mathcal{A}$ depending on a such that $\phi(a) = \phi_a(a)$. Local mappings were introduced independently by Kadison [5] and Larson and Sourour [7]. Let us point out that they have assumed that these mappings are linear. In [7, Theorem 2.1] Larson and Sourour proved that every surjective local automorphism of the algebra of all bounded linear operators on an infinite-dimensional Banach space is an automorphism. Later, Brešar and Šemrl [1] showed that the surjectivity assumption in this result can be removed in the case of a separable Hilbert space.

In the last few decades a lot of work has been done on local mappings on operator algebras. In many important cases local mappings of some class of transformations on a given algebra are global. But, if we drop the assumption of linearity, then the corresponding statements are no longer true. Without linearity we cannot get nice results even in the case of operator algebras. So, if we drop the linearity assumption, we need some stronger condition on the map. One simple idea is the concept of 2-local mappings which is rather new and relatively few results concerning it have been obtained so far.

Kowalski and Słodkowski [6] proved that if $\mathcal{A}$ is a Banach algebra with identity I and $\phi : \mathcal{A} \to \mathbb{C}$ a mapping with the property that $\phi(I) = 1$ and for every $a, b \in \mathcal{A}$ there exists a multiplicative linear functional $\phi_{a,b}$ on $\mathcal{A}$

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such that $\phi(a) = \phi_{a,b}(a)$ and $\phi(b) = \phi_{a,b}(b)$, then ϕ is linear and multiplicative. Motivated by these considerations, Šemrl [11] introduced the following definition. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-*local automorphism* if for every $a, b \in \mathcal{A}$ there is an automorphism $\phi_{a,b} : \mathcal{A} \to \mathcal{A}$ depending on a and b such that $\phi(a) = \phi_{a,b}(a)$ and $\phi(b) = \phi_{a,b}(b)$. Šemrl proved that every 2-local automorphism of $\mathcal{B}(H)$, the algebra of all bounded linear operators on a separable Hilbert space H, is an automorphism [11, Theorem 1]. Let us point out that in the definition of 2-local mappings no linearity or additivity of ϕ is assumed. Therefore, this notion can be defined on arbitrary algebraic structures and not only on algebras. Moreover, the study of 2-local automorphisms can give important new information about the automorphism groups appearing in different parts of mathematics.

The aim of this paper is to investigate 2-local Jordan automorphisms on some operator algebras. First we will consider 2-local Jordan automorphisms on $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices. Next we obtain analogous results for 2-local Jordan automorphisms on any subalgebra of $\mathcal{B}(X)$ which contains the ideal of all compact operators on X, where X is a real or complex separable Banach space and $\mathcal{B}(X)$ is the algebra of all bounded linear operators on X. For completeness we recall some basic definitions and remarks which we will need later.

Let $\mathcal{A}$ be an algebra over field $\mathbb{F}$. A bijective linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *Jordan automorphism* on $\mathcal{A}$ if $\phi(a^2) = \phi(a)^2$ for every $a \in \mathcal{A}$. Motivated by Šemrl's results we introduce the following definition.

Definition 1.1. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-*local Jordan automorphism* if for every $a, b \in \mathcal{A}$ there exists a Jordan automorphism $\phi_{a,b} : \mathcal{A} \to \mathcal{A}$ depending on a and b such that

$$\phi(a) = \phi_{a,b}(a) \quad \textit{and} \quad \phi(b) = \phi_{a,b}(b). \tag{1}$$

Note that in this definition we do not assume linearity or additivity. But it is easy to show that

$$\phi(\lambda a) = \lambda a$$

for every $a \in \mathcal{A}$ and every scalar $\lambda \in \mathbb{F}$: indeed, there exists a Jordan automorphism $\phi_{\lambda a,a} : \mathcal{A} \to \mathcal{A}$ such that

$$\phi(\lambda a) = \phi_{\lambda a,a}(\lambda a) = \lambda\phi_{\lambda a,a}(a) = \lambda\phi(a).$$

Now, let $a \in \mathcal{A}$. Then there exists a Jordan automorphism $\phi_{a,a^2} : \mathcal{A} \to \mathcal{A}$ such that $\phi(a) = \phi_{a,a^2}(a)$ and $\phi(a^2) = \phi_{a,a^2}(a^2)$. Hence,

$$\phi(a^2) = \phi_{a,a^2}(a^2) = \phi_{a,a^2}(a)^2 = \phi(a)^2.$$

Thus, we proved that

$$\phi(a^2) = \phi(a)^2 \quad \text{for all } a \in \mathcal{A}.$$

Let us also point out that every 2-local automorphism ϕ on $\mathcal{A}$ is injective. Indeed, if $\phi(a) = \phi(b)$ for some $a, b \in \mathcal{A}$, then

$$0 = \phi(a) - \phi(b) = \phi_{a,b}(a) - \phi_{a,b}(b) = \phi_{a,b}(a - b)$$

for an appropriate Jordan automorphism $\phi_{a,b}$ on $\mathcal{A}$. Thus, $a = b$.

2. 2-local Jordan automorphisms on $M_n(\mathbb{C})$. Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices and denote by $E_{ij} \in M_n(\mathbb{C})$, $1 \leq i, j \leq n$, the matrix whose (ij)-entry is 1 and all other entries are 0. As usual, we denote by $\mathrm{tr}(A)$ the trace of a matrix $A \in M_n(\mathbb{C})$. Recall that $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is a similarity invariant map. We denote by δ_{ij}, $1 \leq i, j \leq n$, the Kronecker delta.

The main idea for the proof of the next theorem comes from [9, Section 3.4.3].

THEOREM 2.1. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a 2-local Jordan automorphism. Then ϕ is an automorphism or an anti-automorphism.*

Proof. Let $A, B \in M_n(\mathbb{C})$. Then there exists a Jordan automorphism $\phi_{A,B}$ on $M_n(\mathbb{C})$ with the property (1). According to Herstein's result [3], $\phi_{A,B}$ is an automorphism or an anti-automorphism. We may assume the former (if $\phi_{A,B}$ is an anti-automorphism the proof goes in the same way). Since every automorphism of $M_n(\mathbb{C})$ is inner, there exists an invertible $T \in M_n(\mathbb{C})$ such that

$$\phi(A) = TAT^{-1} \quad \text{and} \quad \phi(B) = TBT^{-1}.$$

Therefore,

$$\mathrm{tr}(\phi(A)\phi(B)) = \mathrm{tr}(AB) \qquad \text{for every } A, B \in M_n(\mathbb{C}). \tag{2}$$

In the next step we will prove that the range of ϕ linearly generates the whole $M_n(\mathbb{C})$. Suppose that

$$\sum_{1 \leq i,j \leq n} \lambda_{ij}\phi(E_{ij}) = 0$$

for some complex numbers λ_{ij}. Fix $1 \leq r, s \leq n$. Then we have

$$\sum_{1 \leq i,j \leq n} \lambda_{ij}\phi(E_{ij})\phi(E_{rs}) = 0,$$

and thus

$$\sum_{1 \leq i,j \leq n} \lambda_{ij}\mathrm{tr}(\phi(E_{ij})\phi(E_{rs})) = 0.$$

By (2), we obtain

$$\sum_{1 \leq i,j \leq n} \lambda_{ij}\mathrm{tr}(E_{ij}E_{rs}) = 0.$$

This yields $\lambda_{ij} = 0$, $1 \leq i, j \leq n$, because $E_{ij}E_{rs} = \delta_{jr}E_{is}$. Therefore, we proved that the matrices $\phi(E_{ij})$, $1 \leq i, j \leq n$, are linearly independent, as desired.

Using linearity of the trace functional and (2) we obtain

$$\begin{aligned}\operatorname{tr}(\phi(A+B)\phi(C)) &= \operatorname{tr}((A+B)C) = \operatorname{tr}(AC) + \operatorname{tr}(BC)\\ &= \operatorname{tr}(\phi(A)\phi(C)) + \operatorname{tr}(\phi(B)\phi(C))\\ &= \operatorname{tr}((\phi(A) + \phi(B))\phi(C))\end{aligned}$$

for every $C \in M_n(\mathbb{C})$ and so, according to the above observations,

$$\phi(A+B) = \phi(A) + \phi(B).$$

Since also

$$\phi(\lambda A) = \lambda\phi(A)$$

for all $A \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$, the map ϕ is linear. So, ϕ is a bijective linear mapping on $M_n(\mathbb{C})$. We already know that $\phi(A^2) = \phi(A)^2$ for every $A \in \mathbb{C}$. Thus, ϕ is a Jordan automorphism of $M_n(\mathbb{C})$. Again using [3], it follows that ϕ is either an automorphism or an anti-automorphism. ∎

Remark 2.2. We actually proved that there exists an invertible matrix $T \in M_n(\mathbb{C})$ such that either

$$\phi(A) = TAT^{-1}, \quad A \in M_n(\mathbb{C}),$$

or

$$\phi(A) = TA^tT^{-1}, \quad A \in M_n(\mathbb{C}).$$

Here A^t denotes the transpose of $A \in M_n(\mathbb{C})$. Let us also mention that analogous results hold true for 2-local Jordan automorphisms of $M_n(\mathbb{F})$, where $\mathbb{F}$ is any algebraically closed field of characteristic 0, and for 2-local Jordan automorphisms of $M_n(\mathbb{R})$. The idea of the proofs is the same.

3. 2-local Jordan automorphisms on $\mathcal{A}$. Let X be a real or complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. We will denote by $\mathcal{F}(X) \subseteq \mathcal{B}(X)$ the subalgebra of bounded finite rank operators. Note that $\mathcal{F}(X)$ is a *prime algebra* (that is, $A, B \in \mathcal{F}(X)$ and $A\mathcal{F}(X)B = \{0\}$ imply $A = 0$ or $B = 0$). We call a subalgebra $\mathcal{A}$ of $\mathcal{B}(X)$ *standard* if it contains $\mathcal{F}(X)$. As usual, the dual space of X will be denoted by X^* and the Banach space adjoint of an operator $A \in \mathcal{B}(X)$ will be denoted by A^*. It is well-known that every automorphism $\phi : \mathcal{A} \to \mathcal{A}$ is of the form $\phi(A) = TAT^{-1}$, where $T : X \to X$ is a fixed bounded invertible linear operator. Analogously, for every anti-automorphism $\phi : \mathcal{A} \to \mathcal{A}$ there exists a bounded invertible linear operator $T : X \to X^*$ such that ϕ is of the form $\phi(A) = TA^*T^{-1}$.

Let $x \in X$ and $f \in X^*$. Then $x \otimes f$ is an operator in $\mathcal{B}(X)$ defined by

$$(x \otimes f)(z) = f(z)x, \quad z \in X.$$

Note that $x \otimes f$ is an operator of rank at most one. Moreover, every operator $A \in \mathcal{F}(X)$ can be written as a finite sum

$$A = \sum_{k=1}^{n} x_k \otimes f_k$$

for some $x_k \in X$ and $f_k \in X^*$, $k = 1, \ldots, n$. Using this representation, the trace of A is defined by

$$\mathrm{tr}(A) = \sum_{k=1}^{n} f_k(x_k).$$

It turns out that tr is a well-defined linear functional on $\mathcal{F}(X)$.

In the proof of Theorem 3.3 we will use the following lemmas.

LEMMA 3.1. *Let $\mathcal{A}$ be an algebra and ϕ a Jordan homomorphism on $\mathcal{A}$. Then*

$$\phi(BAB) = \phi(B)\phi(A)\phi(B) \quad \textit{for all } A, B \in \mathcal{A}.$$

Proof. We know that $\phi(A^2) = \phi(A)^2$ for every $A \in \mathcal{A}$. By linearization, it is easy to see that

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$$

for all $A, B \in A$. Putting $AB + BA$ instead of A in the above equality, we obtain

$$\begin{aligned}\phi((AB + BA)B + B(AB + BA)) &= \phi(AB + BA)\phi(B) + \phi(B)\phi(AB + BA)\\ &= (\phi(A)\phi(B) + \phi(B)\phi(A))\phi(B) + \phi(B)(\phi(A)\phi(B) + \phi(B)\phi(A)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi(AB^2 + B^2A + 2BAB) &= \phi(A)\phi(B^2) + \phi(B^2)\phi(A) + 2\phi(BAB)\\ &= \phi(A)\phi(B)^2 + \phi(B)^2\phi(A) + 2\phi(BAB).\end{aligned}$$

Comparing the last two equalities, we get the desired conclusion. ∎

LEMMA 3.2. *Let $\mathcal{A}$ be a prime algebra and ϕ a linear map on $\mathcal{A}$ such that for all $A, B \in \mathcal{A}$ either $\phi(AB) = \phi(A)\phi(B)$ or $\phi(AB) = \phi(B)\phi(A)$. Then ϕ is either a homomorphism or an anti-homomorphism.*

Proof. If both $\phi(AB) = \phi(A)\phi(B)$ and $\phi(AB) = \phi(B)\phi(A)$ hold for all $A, B \in \mathcal{A}$, then ϕ is a homomorphism and an anti-homomorphism. On the other hand, let $A \in \mathcal{A}$. Then for every $B \in \mathcal{A}$ either $\phi(AB) = \phi(A)\phi(B)$ or $\phi(AB) = \phi(B)\phi(A)$. Denote $\mathcal{A}_1 = \{B \in \mathcal{A} : \phi(AB) = \phi(A)\phi(B)\}$ and $\mathcal{A}_2 = \{B \in \mathcal{A} : \phi(AB) = \phi(A)\phi(B)\}$. Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are additive subsets of the prime algebra $\mathcal{A}$ and $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$. Thus, $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$. Therefore, either $\phi(AB) = \phi(A)\phi(B)$ for every $B \in \mathcal{A}$, or $\phi(AB) = \phi(B)\phi(A)$ for every $B \in \mathcal{A}$.

Suppose that there exist $A, B, C, D \in \mathcal{A}$ such that $\phi(AB) = \phi(A)\phi(B) \neq \phi(B)\phi(A)$ and $\phi(CD) = \phi(D)\phi(C) \neq \phi(C)\phi(D)$. According to the above observations, $\phi(AD) = \phi(A)\phi(D)$ and $\phi(CB) = \phi(B)\phi(C)$. If

$$\begin{aligned}\phi((A+C)(B+D)) &= \phi(A+C)\phi(B+D) \\ &= \phi(A)\phi(B) + \phi(A)\phi(D) + \phi(C)\phi(B) + \phi(C)\phi(D),\end{aligned}$$

then $\phi(CD) = \phi(C)\phi(D)$, a contradiction. Similarly, if

$$\phi((A+C)(B+D)) = \phi(B+D)\phi(A+C),$$

then $\phi(AB) = \phi(B)\phi(A)$, a contradiction. ∎

Recall that a standard operator algebra $\mathcal{A}$ on a complex or real Banach space X is a prime algebra. Indeed, suppose that $C\mathcal{A}D = 0$ for some C, D in $\mathcal{A}$ with $D \neq 0$. Our aim is to prove that $C = 0$. Choose $w \in X$ such that $z = Dw \neq 0$ and a linear functional f on X such that $f(z) \neq 0$. Then $0 = C(x \otimes f)Dw = f(z)Cx$. This implies that $Cx = 0$ for all $x \in X$, as desired. Thus, by Herstein's theorem [3], every Jordan automorphism of $\mathcal{A}$ is either an automorphism or an anti-automorphism.

Now, let $\phi : \mathcal{A} \to \mathcal{A}$ be a 2-local Jordan automorphism of $\mathcal{A}$. Then for all $A, B \in \mathcal{A}$ there exists an automorphism or an anti-automorphism $\phi_{A,B} : \mathcal{A} \to \mathcal{A}$ such that (1) holds. Moreover, for all A, B either there exists a bounded invertible linear operator $T : X \to X$ depending on A and B such that

$$\phi(A) = TAT^{-1} \quad \text{and} \quad \phi(B) = TBT^{-1},$$

or there exists a bounded invertible linear operator $T : X \to X^*$ depending on A and B such that

$$\phi(A) = TA^*T^{-1} \quad \text{and} \quad \phi(B) = TB^*T^{-1}.$$

Theorem 3.3. *Let X be a real or complex Banach space and $\mathcal{A}$ a standard operator algebra on X. Suppose that ϕ is a 2-local Jordan automorphism of $\mathcal{A}$. Then the restriction of ϕ to the subalgebra $\mathcal{F}(X)$ is an endomorphism or an anti-endomorphism.*

Proof. According to Section 2, we may assume that X is infinite-dimensional. Let $P \in \mathcal{A}$ be a nonzero finite rank idempotent, say $\operatorname{rank}(P) = n$. Note that $\phi(P) = \phi_{P,P}(P)$ for an appropriate Jordan automorphism $\phi_{P,P}$ on $\mathcal{A}$. Thus, $\phi(P)$ is also a finite rank idempotent. Moreover, $\operatorname{rank}(P) = \operatorname{rank}(\phi(P)) = n$. Let us denote by $\mathcal{A}_P$ the subalgebra of $\mathcal{A}$ which consists of all operators $A \in \mathcal{A}$ with $PAP = A$. Clearly, $\mathcal{A}_P$ is isomorphic to $M_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. By Lemma 3.1, we have

$$\begin{aligned}\phi(A) &= \phi_{A,P}(A) = \phi_{A,P}(PAP) = \phi_{A,P}(P)\phi_{A,P}(A)\phi_{A,P}(P)\\ &= \phi(P)\phi(A)\phi(P)\end{aligned}$$

for every $A \in \mathcal{A}_P$, where $\phi_{A,P}$ is an appropriate Jordan automorphism of $\mathcal{A}$ with the property (1). It follows that ϕ maps $\mathcal{A}_P$ into $\mathcal{A}_{\phi(P)}$ which is isomorphic to $M_n(\mathbb{F})$ as well. It is also easy to see that

$$\operatorname{tr}(\phi(A)\phi(B)) = \operatorname{tr}(AB)$$

for all $A, B \in \mathcal{A}_P$. Following the arguments given in the previous section, we can show that ϕ is linear on $\mathcal{A}_P$. Thus, ϕ is a homomorphism or an anti-homomorphism on $\mathcal{A}_P$. Since P was an arbitrary finite rank idempotent, for all $A, B \in \mathcal{F}(X)$ either $\phi(AB) = \phi(A)\phi(B)$ or $\phi(AB) = \phi(B)\phi(A)$ and, by Lemma 3.2, the restriction of ϕ to $\mathcal{F}(X)$ is an endomorphism or an anti-endomorphism. ∎

Before giving our next result, we state two theorems which will be used in the proof of Theorem 3.6. The first is a well-known result about locally linearly dependent linear operators.

THEOREM 3.4 ([2]). *Let X be a linear space over an infinite field and $A, B : X \to X$ linear operators. Suppose that for every $x \in X$ the vectors Ax and Bx are linearly dependent. Then either the operators A and B are linearly dependent, or the ranges of A and B are included in the same one-dimensional subspace of X.*

The next theorem can be derived from Hua's arguments in Section I of [4].

THEOREM 3.5 ([4]). *Let X be a real or complex Banach space and $\phi : \mathcal{F}(X) \to \mathcal{F}(X)$ a linear operator which maps rank-one operators to rank-one operators. Suppose that the range of ϕ contains an operator with rank greater than one. Then either there exist linear operators $R : X \to X$ and $S : X^* \to X^*$ such that*

$$\phi(x \otimes f) = Rx \otimes Sf, \quad x \in X,\ f \in X^*, \tag{3}$$

or there exist linear operators $R : X^* \to X$ and $S : X \to X^*$ such that

$$\phi(x \otimes f) = Rf \otimes Sx, \quad x \in X,\ f \in X^*. \tag{4}$$

In the proof of our next theorem we will use some ideas from [9, Theorem 3.4.4] and [8, Proof of Theorem 3.3].

THEOREM 3.6. *Let X be a real or complex separable Banach space and $\mathcal{A}$ a subalgebra of $\mathcal{B}(X)$ which contains the ideal of all compact operators on X. Then every 2-local Jordan automorphism of $\mathcal{A}$ is either an automorphism or an anti-automorphism of $\mathcal{A}$.*

Proof. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a 2-local Jordan automorphism. As X is separable, we can use Ovsepian–Pełczyński's result [10] on the existence of a fundamental total bounded biorthogonal system to find sequences $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that:

- The linear span of the sequence $\{x_n\}_{n=1}^{\infty}$ is dense in X.
- For all $i, j \in \mathbb{N}$, we have $f_i(x_j) = \delta_{ij}$.
- If $x \in X$ and $f_n(x) = 0$ for all $n \in \mathbb{N}$, then $x = 0$.
- We have $\sup\{\|x_n\| \, \|f_n\| : n \in \mathbb{N}\} < \infty$.

Denote

$$P_n = x_n \otimes f_n, \quad n \in \mathbb{N}.$$

Assume that $\{\lambda_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive real numbers converging to zero with $\sum_{n=1}^{\infty} \lambda_n \|P_n\| < 1$. Next, let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \xi_n < \infty$. Denote

$$D = \sum_{n=1}^{\infty} \lambda_n P_n \quad \text{and} \quad C = \sum_{n=1}^{\infty} \xi_n (x_n \otimes f_{n+1}).$$

Note that D and C are compact operators. Composing ϕ with an appropriate automorphism or anti-automorphism of $\mathcal{A}$, if necessary, we may assume that $\phi(D) = D$ and $\phi(C) = C$.

Let $k \in \mathbb{N}$. Since ϕ is a 2-local Jordan automorphism, either there exists a bounded invertible linear operator $U : X \to X$ such that

$$\phi(D) = UDU^{-1} \quad \text{and} \quad \phi(P_k) = UP_kU^{-1},$$

or there exists a bounded invertible linear operator $U : X \to X^*$ such that

$$\phi(D) = UD^*U^{-1} \quad \text{and} \quad \phi(P_k) = UP_k^*U^{-1}.$$

We may assume the former. Set

$$Q_n = UP_nU^{-1}, \quad n \in \mathbb{N}.$$

Recall that $\sum_{n=1}^{\infty} \lambda_n \|Q_n\|$ is an absolutely convergent series. As $\phi(D) = D$, we have

$$\sum_{n=1}^{\infty} \lambda_n Q_n = \sum_{n=1}^{\infty} \lambda_n P_n.$$

Dividing both sides of the above equality by λ_1, we obtain

$$Q_1 + \frac{1}{\lambda_1} \sum_{n=2}^{\infty} \lambda_n Q_n = P_1 + \frac{1}{\lambda_1} \sum_{n=2}^{\infty} \lambda_n P_n.$$

Note that $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ are two sequences of disjoint rank one idempotents. Thus, taking the jth power in the last relation, we get

$$Q_1 + \frac{1}{\lambda_1^j} \sum_{n=2}^{\infty} \lambda_n^j Q_n = P_1 + \frac{1}{\lambda_1^j} \sum_{n=2}^{\infty} \lambda_n^j P_n.$$

By letting j tend to infinity, we conclude that $Q_1 = P_1$ and

$$\frac{1}{\lambda_1}\sum_{n=2}^{\infty}\lambda_n Q_n = \frac{1}{\lambda_1}\sum_{n=2}^{\infty}\lambda_n P_n.$$

Continuing with the same arguments, we can show that $Q_n = UP_nU^{-1} = P_n$ for all $n \in \mathbb{N}$. In particular, $\phi(P_k) = UP_kU^{-1} = P_k$. Since $k \in \mathbb{N}$ was arbitrary, we have

$$\phi(P_n) = P_n, \quad n \in \mathbb{N}. \tag{5}$$

Let ϕ_F be the restriction of ϕ to $\mathcal{F}(X)$. By Theorem 3.3, $\phi_F : \mathcal{F}(X) \to \mathcal{F}(X)$ is an endomorphism or an anti-endomorphism. Without loss of generality we may assume the former. If ϕ_F is an anti-endomorphism, the proof goes in a similar way.

As in the proof of Theorem 3.4.4 in [9], we can find an injective continuous linear operator $T : X \to X$ such that

$$\phi_F(A)T = TA \tag{6}$$

for every $A \in \mathcal{F}(X)$. Note also that ϕ_F preserves rank. Hence, by Theorem 3.5, we have two possibilities: either (3) or (4) holds true. But, since ϕ_F is a homomorphism, the latter cannot occur. Thus, there exist operators $R : X \to X$ and $S : X^* \to X^*$ such that

$$\phi_F(x \otimes f) = Rx \otimes Sf$$

for all $x \in X$ and $f \in X^*$. Using (6), we obtain

$$Tx \otimes f = T(x \otimes f) = (Rx \otimes Sf)T = Rx \otimes T^*Rf.$$

It follows that Tx and Rx are linearly dependent for every $x \in X$ and, according to Theorem 3.4, R is a scalar multiple of T. Thus, we can assume that $R = T$.

In the next step we will show that $S = (T^*)^{-1} = (T^{-1})^*$. Recall that ϕ_F preserves trace as well. This yields

$$(Sf)(Tx) = f(x)$$

for all $x \in X$ and $f \in X^*$. In other words, $(T^*S)f = f$ for every $f \in X^*$. Hence, the range of T^* is closed, and consequently the range of T is closed as well. Moreover, the range of T is dense. Therefore, T and T^* are invertible operators and $S = (T^*)^{-1} = (T^{-1})^*$, as desired. So, we have proved that

$$\phi_F(x \otimes f) = T(x \otimes f)T^{-1}, \qquad x \in X,\ f \in X^*.$$

This yields

$$\phi(A) = TAT^{-1}, \quad A \in \mathcal{F}(X).$$

Let $A \in \mathcal{A}$ and $x \in X$, $f \in X^*$. We already know that $\phi(x \otimes f) = T(x \otimes f)T^{-1}$. On the other hand, since ϕ is a 2-local Jordan automorphism, either there exists a bounded invertible linear operator $U : X \to X$ such that

$$\phi(A) = UAU^{-1} \quad \text{and} \quad \phi(x \otimes f) = U(x \otimes f)U^{-1},$$

or there exists a bounded invertible linear operator $U : X \to X^*$ such that

$$\phi(A) = UA^*U^{-1} \quad \text{and} \quad \phi(x \otimes f) = U(x \otimes f)^*U^{-1}.$$

Since the restriction of ϕ to $\mathcal{F}(X)$ is a homomorphism, we may assume the former. Then

$$\begin{aligned}\phi(x \otimes f)\phi(A)\phi(x \otimes f) &= U(x \otimes f)U^{-1}UAU^{-1}U(x \otimes f)U^{-1} \\ &= U(x \otimes f)A(x \otimes f)U^{-1}.\end{aligned}$$

On the other hand,

$$\phi(x \otimes f)\phi(A)\phi(x \otimes f) = T(x \otimes f)T^{-1}\phi(A)T(x \otimes f)T^{-1}.$$

Consequently,

$$T(x \otimes f)T^{-1}\phi(A)T(x \otimes f)T^{-1} = U(x \otimes f)A(x \otimes f)U^{-1}$$

and so

$$\operatorname{tr}(T(x \otimes f)T^{-1}\phi(A)T(x \otimes f)T^{-1}) = \operatorname{tr}(U(x \otimes f)A(x \otimes f)U^{-1}).$$

This yields

$$f(T^{-1}\phi(A)Tx)f(x) = f(Ax)f(x).$$

If $x \in X$ and $f \in X^*$ are such that $f(x) \neq 0$, then

$$f(T^{-1}\phi(A)Tx) = f(Ax).$$

But if $f(x) = 0$, then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset X$ such that $y_n \to x$ as $n \to \infty$ and $f(y_n) \neq 0$ for all $n \in \mathbb{N}$. This implies that

$$\phi(A) = TAT^{-1}, \quad A \in \mathcal{A}.$$

Finally, we will show that T is the identity map on X and so $\phi(A) = A$ for all $A \in \mathcal{A}$. We know that $\phi(P_n) = TP_nT^{-1}$ and, by (5),

$$Tx_n = \alpha_n x_n \quad \text{and} \quad (T^{-1})^* f_n = \beta_n f_n$$

for some scalars α_n, β_n with $\alpha_n\beta_n = 1$, $n \in \mathbb{N}$. For simplicity, assume that $\alpha_1 = \beta_1 = 1$. We already know that

$$\phi(C) = \phi\Big(\sum_{n=1}^{\infty} \xi_n (x_n \otimes f_{n+1})\Big) = \sum_{n=1}^{\infty} \xi_n (x_n \otimes f_{n+1}) = C.$$

On the other hand,

$$\phi\Big(\sum_{n=1}^{\infty}\xi_n(x_n\otimes f_{n+1})\Big)=\sum_{n=1}^{\infty}\xi_n T(x_n\otimes f_{n+1})T^{-1}$$
$$=\sum_{n=1}^{\infty}\xi_n(Tx_n\otimes (T^{-1})^*f_{n+1})=\sum_{n=1}^{\infty}\xi_n\alpha_n\beta_{n+1}(x_n\otimes f_{n+1}).$$

Thus,

$$\sum_{n=1}^{\infty}\xi_n(x_n\otimes f_{n+1})=\sum_{n=1}^{\infty}\xi_n\alpha_n\beta_{n+1}(x_n\otimes f_{n+1}).$$

Let $k>1$. Then

$$\xi_{k-1}x_{k-1}=\sum_{n=1}^{\infty}\xi_n(x_n\otimes f_{n+1})x_k=\sum_{n=1}^{\infty}\xi_n\alpha_n\beta_{n+1}(x_n\otimes f_{n+1})x_k$$
$$=\xi_{k-1}\alpha_{k-1}\beta_k x_{k-1}.$$

Therefore, $\alpha_n=\beta_n=1$ for all $n\in\mathbb{N}$ and T is the identity map on X. This yields

$$\phi(A)=A,\quad A\in\mathcal{A}. \ \blacksquare$$

REMARK 3.7. Finally, let us point out that in the proof of Theorem 3.6 the assumption of separability of the Banach space X was crucial. However, we do not know whether the conclusion of Theorem 3.6 holds without this assumption as well.

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