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Smooth renormings of the Lebesgue–Bochner function space $L^1(\mu, X)$

by

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Abstract. We show that, if μ is a probability measure and X is a Banach space, then the space $L^1(\mu, X)$ of Bochner integrable functions admits an equivalent Gâteaux (or uniformly Gâteaux) smooth norm provided that X has such a norm, and that if X admits an equivalent Fréchet (resp. uniformly Fréchet) smooth norm, then $L^1(\mu, X)$ has an equivalent renorming whose restriction to every reflexive subspace is Fréchet (resp. uniformly Fréchet) smooth.

1. Introduction. An important problem in the theory of integration on Banach spaces is whether a given geometrical property of a Banach space X lifts to the corresponding Lebesgue–Bochner space $L^p(\mu, X)$, where μ is a probability measure and $1 \leq p < \infty$. A number of results in this direction appeared during the last three decades (see e.g. [DGJ], [DU], [LS1], [LS2], [S], [ST] and references therein). In particular, it was shown in [LS1] and [LS2] that if $p > 1$ then the canonical norm of $L^p(\mu, X)$ inherits several smoothness properties of the norm of X. The situation becomes different when $p = 1$, since the natural norm of $L^1(\mu)$, which embeds in $L^1(\mu, X)$, is not smooth. However, some of these properties can be transferred to $L^1(\mu, X)$ under a suitable renorming. A first result of this type for the property of locally uniform convexity was obtained in [S] using the Troyanski–Zizler method of renorming in Banach spaces with long sequences of projections (see e.g. [DGZ, Section VII.1]).

In this work we show that several smoothness properties lift from X into $L^1(\mu, X)$ up to renorming. Our technique relies on the construction of an Orlicz–Bochner norm associated to a norm on X with the relevant smoothness properties. In particular, such renormings preserve the lattice structure when X is a Banach lattice.

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All Banach spaces considered in this paper are real. Our notation is standard, and can be found, for instance, in [DGZ], [DU] and [FHHMZ].

Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. We denote by $L^1(\Omega, \Sigma, \mu, X)$, or simply by $L^1(\mu, X)$, the Banach space of all (equivalence classes of) Bochner integrable functions $f : \Omega \to X$, endowed with the norm $\|f\|_{L^1(\mu,X)} = \int_\Omega \|f(\omega)\| \, d\mu(\omega)$. As usual, the symbols S_X and B_X stand for the unit sphere and the closed unit ball of X, i.e., $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \le 1\}$. The topological dual of X is denoted by X^*.

Let U be a non-empty open subset of the Banach space X, let $\varphi : U \to \mathbb{R}$ be a function, and pick $x \in U$. We say that φ is *Gâteaux differentiable* at x if there exists a functional $\varphi'(x) \in X^*$ (the *derivative* of φ at x) such that

$$\lim_{t \to 0} \frac{\varphi(x+th) - \varphi(x)}{t} = \varphi'(x)h \quad \text{for all } h \in X. \tag{1.1}$$

If this limit is uniform with respect to $h \in B_X$, we say that φ is *Fréchet differentiable* at x. Let V be a non-empty subset of U with $\operatorname{dist}(X \setminus U, V) =: \Delta > 0$ (we put $\operatorname{dist}(\emptyset, V) = +\infty$). The function φ is called *uniformly Gâteaux differentiable on* V if it is Gâteaux differentiable at every point of V and the limit (1.1) is uniform in $x \in V$, i.e., if for every $h \in B_X$ and every $\varepsilon > 0$ there is $\delta \in (0, \Delta)$ such that

$$\left| \frac{\varphi(x+th) - \varphi(x)}{t} - \varphi'(x)h \right| < \varepsilon \quad \text{whenever} \quad 0 < |t| < \delta \text{ and } x \in V.$$

Finally, φ is called *uniformly Fréchet differentiable on* V if it is Gâteaux differentiable at every point of V and the limit (1.1) is uniform in both $x \in V$ and $h \in B_X$, i.e., if for every $\varepsilon > 0$ there is $\delta \in (0, \Delta)$ such that

$$\left| \frac{\varphi(x+th) - \varphi(x)}{t} - \varphi'(x)h \right| < \varepsilon \text{ whenever } 0 < |t| < \delta,\ h \in B_X \text{ and } x \in V.$$

The norm $\|\cdot\|$ on the Banach space X is called *Gâteaux smooth* (resp. *Fréchet smooth*) if it is Gâteaux differentiable (resp. Fréchet differentiable) at every point of $X \setminus \{0\}$. The norm $\|\cdot\|$ is called *uniformly Gâteaux smooth* (resp. *uniformly Fréchet smooth*) if it is uniformly Gâteaux differentiable (resp. uniformly Fréchet differentiable) on S_X.

The class of Banach spaces that admit an equivalent Gâteaux smooth norm is quite large; it includes, e.g., all weakly countably determined Banach spaces (see e.g. [DGZ, Section VII.1]). The existence of an equivalent Fréchet smooth norm on a Banach space X implies that it is an *Asplund space*, that is, every separable subspace of X has separable dual (see e.g. [DGZ, Section II.5]). It is also well known (see e.g. [DGZ, Section IV.4]) that X admits an equivalent uniformly Fréchet smooth renorming if, and only if, X is super-reflexive. The structure of Banach spaces with an equivalent uniformly Gâteaux smooth norm was elucidated in [FGZ] and [FGMZ], where it was shown that they are exactly the subspaces of Hilbert-generated spaces (a Banach space X is called *Hilbert-generated* if there exist a Hilbert space H and a bounded linear operator $T : H \to X$ with dense range).

We observe that if μ is a probability measure then the space $L^1(\mu)$, being generated by $L^2(\mu)$, admits an equivalent uniformly Gâteaux smooth norm. In [BF], it was shown that $L^1(\mu)$ admits an equivalent norm whose restriction to every reflexive subspace is Fréchet smooth. A strengthening of this fact was obtained in [GS] (see also [FMZ] and [LPT]), where it was proved that Fréchet smoothness can be replaced by uniform Fréchet smoothness. This result, together with the fact that super-reflexivity is equivalent to the existence of an equivalent uniformly Fréchet smooth renorming, yields a new proof of Rosenthal's theorem in [Ro] that every reflexive subspace of $L^1(\mu)$ is super-reflexive.

The purpose of this paper is to obtain extensions of the results above in the setting of Lebesgue–Bochner spaces. In Section 2 we show that if μ is a probability measure and X is a Gâteaux (resp. uniformly Gâteaux) renormable space, then the space $L^1(\mu, X)$ admits an equivalent Gâteaux (resp. uniformly Gâteaux) smooth norm. In Section 3 we prove that if X admits an equivalent Fréchet (resp. uniformly Fréchet) smooth norm, then the norm on $L^1(\mu, X)$ constructed in Section 2 is such that its restriction to every reflexive subspace is Fréchet (resp. uniformly Fréchet) smooth.

2. Gâteaux smooth and uniformly Gâteaux smooth renormings of $L^1(\mu, X)$. Our first result establishes that the properties of Gâteaux smoothness and uniform Gâteaux smoothness of the space X lift to $L^1(\mu, X)$ up to renorming.

THEOREM 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. If the norm $\|\cdot\|$ is Gâteaux smooth* (*resp. uniformly Gâteaux smooth*), *then the Lebesgue–Bochner space $L^1(\mu, X)$ admits an equivalent Gâteaux smooth* (*resp. uniformly Gâteaux smooth*) *norm.*

As we already mentioned, the proofs of this theorem and the main result of the next section (Theorem 3.1) rely on the construction of a suitable Orlicz–Bochner norm on $L^1(\mu, X)$ (see [RR, p. 213] for details). Let $M : \mathbb{R} \to [0, \infty)$ be an *Orlicz function* (i.e., M is a convex, even function such that M increases on $[0, \infty)$, $M(t) \to \infty$ as $t \to \infty$ and $M(0) = 0$). In addition, we assume that M is *non-degenerate* (that is, M vanishes only at $t = 0$), differentiable, and the functions M, M' and $\mathbb{R} \ni t \mapsto tM'(t)$ are Lipschitzian on $\mathbb{R}$. From now on, the symbol C will denote a common Lipschitz constant for these functions. Examples of such M are $M_1(t) = \int_0^{|t|} \arctan(s)\, ds$, $t \in \mathbb{R}$, or $M_2(t) = t^2$ if $|t| \le 1$ and $M_2(t) = 2|t| - 1$ if $|t| > 1$.

Write, for each $f \in L^1(\mu, X)$,

$$\varphi(f) := \int_\Omega M(\|f(\omega)\|)\, d\mu(\omega). \tag{2.1}$$

Clearly, φ is well defined on all of $L^1(\mu, X)$, is symmetric, convex and satisfies $\varphi(0) = 0$. Moreover, if $f, g \in L^1(\mu, X)$ then

$$\begin{aligned}|\varphi(f) - \varphi(g)| &\le \int_\Omega \big|M(\|f(\omega)\|) - M(\|g(\omega)\|)\big|\, d\mu(\omega)\\ &\le C \int_\Omega \big|\|f(\omega)\| - \|g(\omega)\|\big|\, d\mu(\omega) \le C\|f - g\|_{L^1(\mu,X)}.\end{aligned}$$

So, φ is (globally) C-Lipschitzian (and in particular continuous). Consequently, the set $B = \{f \in L^1(\mu, X) : \varphi(f) \le 1\}$ is symmetric, convex and closed. Let $|\cdot|$ denote the Minkowski functional of B, that is,

$$|f| = \inf\{\varrho > 0 : \varphi(f/\varrho) \le 1\}, \qquad f \in L^1(\mu, X). \tag{2.2}$$

Notice that $|\cdot|$ is the Orlicz–Bochner norm on $L^1(\mu, X)$ associated to M and the norm $\|\cdot\|$. The C-Lipschitz property of φ yields $|f| \le C\|f\|_{L^1(\mu,X)}$ for every $f \in L^1(\mu, X)$. Further, if $f \in L^1(\mu, X)$ and $|f| = 1$, then the convexity of M yields

$$\begin{aligned}1 = \varphi(f) &\ge \int_{\{\|f(\cdot)\|\ge 1\}} M(\|f(\omega)\|)\, d\mu(\omega) \ge M'(1) \int_{\{\|f(\cdot)\|\ge 1\}} (\|f(\omega)\| - 1)\, d\mu(\omega)\\ &\ge M'(1) \int_{\{\|f(\cdot)\|\ge 1\}} \|f(\omega)\|\, d\mu(\omega) - M'(1),\end{aligned}$$

and hence

$$\begin{aligned}\|f\|_{L^1(\mu,X)} &= \int_{\{\|f(\cdot)\|\ge 1\}} \|f(\omega)\|\, d\mu(\omega) + \int_{\{\|f(\cdot)\|< 1\}} \|f(\omega)\|\, d\mu(\omega)\\ &\le \frac{2M'(1) + 1}{M'(1)} =: d.\end{aligned}$$

Therefore, $(1/C)|\cdot| \le \|\cdot\|_{L^1(\mu,X)} \le d|\cdot|$.

We shall prove that $|\cdot|$ satisfies the assertion of Theorem 2.1. First, we shall show that the function φ defined by (2.1) inherits the properties of Gâteaux smoothness and uniform Gâteaux smoothness of the original norm on X. For convenience we put $\|\cdot\|'(0)h = 0$ for every $h \in X$.

Proposition 2.2. *If the norm $\|\cdot\|$ on X is Gâteaux smooth, then the function φ defined by* (2.1) *is Gâteaux differentiable on $L^1(\mu, X)$, and for all $f, h \in L^1(\mu, X)$ we have*

$$\varphi'(f)h = \int_\Omega M'(\|f(\omega)\|)\, \|\cdot\|'(f(\omega))(h(\omega))\, d\mu(\omega). \tag{2.3}$$

Proof. Fix $f \in L^1(\mu, X)$, and pick a direction $h \in L^1(\mu, X)$. The formulas

$$\xi_n(\omega) = n\left[M\left(\left\|f(\omega) + \frac{1}{n}h(\omega)\right\|\right) - M(\|f(\omega)\|)\right], \qquad n \in \mathbb{N},$$

and

$$\xi(\omega) = M'(\|f(\omega)\|)\,\|\cdot\|'(f(\omega))(h(\omega))$$

define measurable functions on Ω. Bearing in mind that the norm $\|\cdot\|$ is Gâteaux smooth, we can deduce easily via the mean value theorem that $\lim_n \xi_n(\omega) = \xi(\omega)$ for all $\omega \in \Omega$. Moreover, since M is C-Lipschitzian, we have $|\xi_n(\omega)| \le C\|h(\omega)\|$ for all $\omega \in \Omega$. As $h \in L^1(\mu)$, Lebesgue's dominated convergence theorem ensures that ξ is integrable and

$$\lim_n \int_\Omega \xi_n(\omega)\,d\mu(\omega) = \int_\Omega \xi(\omega)\,d\mu(\omega),$$

that is,

$$\lim_n n\left(\varphi\left(f + \frac{1}{n}h\right) - \varphi(f)\right) = \int_\Omega M'(\|f(\omega)\|)\,\|\cdot\|'(f(\omega))(h(\omega))\,d\mu(\omega).$$

Now, the convexity of φ guarantees that the one-sided derivative $\varphi'_+(f)h$ exists and that (2.3) holds with $\varphi'(f)h$ replaced by $\varphi'_+(f)h$. Moreover, since the right hand side of (2.3) is linear in h and M' is C-Lipschitz, and $\big|\|\cdot\|'(f(\omega))(h(\omega))\big| \le \|h(\omega)\|$ for all $\omega \in \Omega$, we have $|\varphi'(f)(h)| \le C\|h\|_{L^1(\mu,X)}$. Thus, the function $L^1(\mu, X) \ni h \mapsto \int_\Omega M'(\|f(\omega)\|)\,\|\cdot\|'(f(\omega))(h(\omega))\,d\mu(\omega)$ is linear and bounded, and φ is Gâteaux differentiable at f, as we wanted to show. ■

Now, a "uniform" version of Proposition 2.2 follows.

PROPOSITION 2.3. *If the norm $\|\cdot\|$ on X is uniformly Gâteaux smooth, then the function φ defined by* (2.1) *is uniformly Gâteaux differentiable on all of $L^1(\mu, X)$.*

In the proof, we shall use the following two simple lemmas.

LEMMA 2.4. *Let $(X, \|\cdot\|)$ be a Banach space, let $U \subset X$ be a non-empty open set, and let $\Delta > 0$ be so small that the* (*open*) *set $U_1 := \{x \in U : \operatorname{dist}(x, X \setminus U) > \Delta\}$ is non-empty. Let $\psi : U \to \mathbb{R}$ be a Lipschitzian and Gâteaux differentiable function.*

- (i) *If ψ is uniformly Gâteaux differentiable on U_1, then for every $v \in X$ the function $U_1 \ni x \mapsto \psi'(x)v$ is uniformly continuous.*
- (ii) *If, for every $v \in X$, the function $U \ni x \mapsto \psi'(x)v$ is uniformly continuous, then ψ is uniformly Gâteaux differentiable on U_1.*

Proof. (i) Let $c > 0$ be a Lipschitz constant of ψ. Fix any $0 \ne v \in X$, and pick $\varepsilon > 0$. Find $\tau \in (0, \Delta/\|v\|)$ such that $\left|\frac{1}{\tau}(\psi(x+tv) - \psi(x)) - \psi'(x)v\right| < \varepsilon/3$

whenever $x \in U_1$ (note that $\psi(x+\tau v)$ is defined). Put $\delta = \tau\varepsilon/(6c)$, and consider any $x, y \in U_1$ with $\|x-y\| < \delta$. Then $x+\tau v, y+\tau v \in U$, and

$$\begin{aligned}|\psi'(x)v - \psi'(y)v| &\le \left|\psi'(x)v - \frac{1}{\tau}(\psi(x+\tau v) - \psi(x))\right| \\ &\quad + \left|\frac{1}{\tau}(\psi(y+\tau v) - \psi(y)) - \psi'(y)v\right| \\ &\quad + \frac{1}{\tau}|\psi(x+\tau v) - \psi(y+\tau v)| + \frac{1}{\tau}|\psi(y) - \psi(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{1}{\tau}c\|x-y\| + \frac{1}{\tau}c\|y-x\| < \frac{2\varepsilon}{3} + \frac{2c}{\tau}\delta = \varepsilon.\end{aligned}$$

(ii) Fix any $v \in X$, and take $\varepsilon > 0$. Find $\delta > 0$ so small that $|\psi'(x)v - \psi'(y)v| < \varepsilon$ whenever $x, y \in U$ and $\|x-y\| < \delta$. Now, consider any $x \in U_1$ and any $\tau \in \mathbb{R}$ with $0 < |\tau| < \min\{\delta, \Delta/\|v\|\}$. Then the mean value theorem provides a $\theta \in [0,1]$ such that

$$\left|\frac{1}{\tau}(\psi(x+\tau v) - \psi(x)) - \psi'(x)v\right| = |\psi'(x+\theta\tau v)v - \psi'(x)v|,$$

and bearing in mind that $x+\theta\tau v \in U$ and $\|x+\theta\tau v - x\| < |\tau|\,\|v\| < \delta$ it follows that

$$\left|\frac{1}{\tau}\big(\psi(x+\tau v) - \psi(x)\big) - \psi'(x)v\right| < \varepsilon,$$

as we wanted. ∎

LEMMA 2.5. *Let $(X, \|\cdot\|)$ be a Banach space whose norm is Gâteaux smooth. Then the following statements are mutually equivalent:*

(i) *The norm $\|\cdot\|$ is uniformly Gâteaux smooth.*
(i′) *For every $r > 0$ the norm $\|\cdot\|$ is uniformly Gâteaux differentiable on the set $X \setminus rB_X$.*
(ii) *For every $h \in X$ the function $S_X \ni x \mapsto \|\cdot\|'(x)h$ is uniformly continuous.*
(ii′) *For every $r > 0$ and every $h \in X$ the function $X \setminus rB_X \ni x \mapsto \|\cdot\|'(x)h$ is uniformly continuous.*

Proof. The implications (i′)⇒(i) and (ii′)⇒(ii) are trivial, and the equivalence between (i′) and (ii′) follows from Lemma 2.4. To show (i)⇒(i′) fix $r > 0$, $h \in X$, and $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that $\left|\frac{1}{t}(\|x+th\| - 1) - \|\cdot\|'(x)h\right| < \varepsilon$ whenever $0 < |t| < \delta$ and $x \in S_X$. For any $s \in \mathbb{R}$ with $0 < |s| < r\delta$ and $y \in X \setminus rB_X$ we have $|s/\|y\|| < |s|/r < \delta$, and thus

$$\left|\frac{\|y+sh\| - \|y\|}{s} - \|\cdot\|'(y)h\right| = \left|\frac{1}{s/\|y\|}\left(\left\|\frac{y}{\|y\|} + \frac{s}{\|y\|}h\right\| - 1\right) - \|\cdot\|'\left(\frac{y}{\|y\|}\right)h\right| < \varepsilon.$$

It remains to prove (ii)⇒(ii′). Fix $r > 0$, $h \in X$, and $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that $\big|\|\cdot\|'(x)h - \|\cdot\|'(y)h\big| < \varepsilon$ whenever $x, y \in S_X$ and $\|x - y\| < 2\delta/r$. If $u, v \in X \setminus rB_X$ are such that $\|u - v\| < \delta$, then

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \le \left\| \frac{u}{\|u\|} - \frac{v}{\|u\|} \right\| + \left\| \frac{v}{\|u\|} - \frac{v}{\|v\|} \right\| \le \frac{2\|u - v\|}{\|u\|} < \frac{2\delta}{r},$$

and consequently

$$\big|\|\cdot\|'(u)h - \|\cdot\|'(v)h\big| = \left|\|\cdot\|'\Big(\frac{u}{\|u\|}\Big)h - \|\cdot\|'\Big(\frac{v}{\|v\|}\Big)h\right| < \varepsilon.$$

The assertion (ii′) is thus proved. ∎

Proof of Proposition 2.3. We have already proved that φ is C-Lipschitzian and Gâteaux differentiable on all of $L^1(\mu, X)$. So, according to Lemma 2.4 it is enough to show that for every $h \in L^1(\mu, X)$, the function $L^1(\mu, X) \ni f \mapsto \varphi'(f)h$ is uniformly continuous. Assume first that $h = x\chi_E$ for some $x \in X \setminus \{0\}$ and some measurable set $E \subset \Omega$. (The symbol χ_E stands for the characteristic function of E.) Fix $\varepsilon > 0$ and pick $0 < r < \varepsilon/(8\|x\|)$ such that

$$M'(2r) < \frac{\varepsilon}{8\|x\|}. \tag{2.4}$$

Since the norm $\|\cdot\|$ is uniformly Gâteaux smooth, thanks to Lemma 2.5 there is $0 < s < \min\{r, 1\}$ such that

$$\big|\|\cdot\|'(u)x - \|\cdot\|'(v)x\big| < \frac{\varepsilon}{4C} \quad \text{whenever} \quad u, v \in X \setminus rB_X \text{ and } \|u - v\| < s. \tag{2.5}$$

Put $\delta = s\varepsilon/(8C\|x\|)$, and let $f, g \in L^1(\mu, X)$ with $\|f - g\|_{L^1(\mu,X)} < \delta$. From Proposition 2.2 we get

$$\begin{aligned} |\varphi'(f)h - \varphi'(g)h| &= \Big| \int_E M'(\|f(\omega)\|)\, \|\cdot\|'(f(\omega))(x)\, d\mu(\omega) \\ &\qquad - \int_E M'(\|g(\omega)\|)\, \|\cdot\|'(g(\omega))(x)\, d\mu(\omega) \Big| \\ &\le \int_E |M'(\|f(\omega)\|) - M'(\|g(\omega)\|)|\, \big|\|\cdot\|'(f(\omega))x\big|\, d\mu(\omega) \\ &\qquad + \int_E M'(\|g(\omega)\|)\, \big|\|\cdot\|'(f(\omega))x - \|\cdot\|'(g(\omega))x\big|\, d\mu(\omega). \end{aligned} \tag{2.6}$$

Let us estimate the integrals on the right hand side. For brevity we write, for each $\omega \in \Omega$,

$$a(\omega) = \big(M'(\|f(\omega)\|) - M'(\|g(\omega)\|)\big)\|\cdot\|'(f(\omega))x$$
$$b(\omega) = M'(\|g(\omega)\|)\big(\|\cdot\|'(f(\omega))x - \|\cdot\|'(g(\omega))x\big).$$

Bearing in mind that M' is C-Lipschitzian and $\big|\|\cdot\|'(f(\omega))x\big| \le \|x\|$ for all $\omega \in E$, we have

$$\int_E |a(\omega)|\,d\mu(\omega) \le C\int_E \big|\|f(\omega)\| - \|g(\omega)\|\big|\,\|x\|\,d\mu(\omega) \le C\|x\|\,\|f-g\|_{L^1(\mu,X)} < \varepsilon/8. \tag{2.7}$$

Let us now estimate $\int_E |b(\omega)|\,d\mu(\omega)$. Put $E_1 = \{\omega \in E : \|g(\omega)\| < 2r\}$ and $E_2 = E \setminus E_1$. Using (2.4) we get

$$\int_{E_1} |b(\omega)|\,d\mu(\omega) \le 2\|x\| \int_{E_1} M'(\|g(\omega)\|)\,d\mu(\omega) \le \varepsilon\mu(E_1)/4 \le \varepsilon/4. \tag{2.8}$$

Further, we put $E_{21} = \{\omega \in E_2 : \|f(\omega) - g(\omega)\| < s\}$ and $E_{22} = E_2 \setminus E_{21}$. For each $\omega \in E_{21}$ we have $\|g(\omega)\| \ge 2r$, and thus $\|f(\omega)\| \ge \|g(\omega)\| - \|f(\omega) - g(\omega)\| > 2r - s > r$. Using (2.5) with $u := f(\omega)$ and $v := g(\omega)$, and taking into account that M' is bounded by C, we deduce that $|b(\omega)| \le C\frac{\varepsilon}{4C} = \varepsilon/4$ whenever $\omega \in E_{21}$. Therefore,

$$\int_{E_{21}} |b(\omega)|\,d\mu(\omega) \le \frac{\varepsilon}{4}\mu(E_{21}) \le \frac{\varepsilon}{4}. \tag{2.9}$$

It remains to estimate $\int_{E_{22}} |b(\omega)|\,d\mu(\omega)$. According to Chebyshev's inequality it follows that

$$\mu(E_{22}) \le \frac{1}{s}\int_{E_{22}} \|f(\omega) - g(\omega)\|\,d\mu(\omega) \le \frac{1}{s}\|f-g\|_{L^1(\mu,X)} < \frac{\delta}{s} < \frac{\varepsilon}{8C\|x\|},$$

and consequently

$$\int_{E_{22}} |b(\omega)|\,d\mu(\omega) \le 2C\|x\|\mu(E_{22}) < \frac{\varepsilon}{4}.$$

Adding this inequality to (2.8) and (2.9) we get

$$\int_E |b(\omega)|\,d\mu(\omega) < 3\varepsilon/4,$$

and bearing in mind (2.6) and (2.7) we obtain

$$|\varphi'(f)(x\chi_E) - \varphi'(g)(x\chi_E)| < \varepsilon.$$

Thus, for each $x \in X$ and each measurable set $E \subset \Omega$, the function $L^1(\mu, X) \ni f \mapsto \varphi'(f)(x\chi_E)$ is uniformly continuous. It follows that if σ is any simple function in $L^1(\mu, X)$, then $L^1(\mu, X) \ni f \mapsto \varphi'(f)\sigma$ is uniformly continuous as well.

Now, fix $h \in L^1(\mu, X)$. For any $f, g \in L^1(\mu, X)$ and each simple function $\sigma \in L^1(\mu, X)$ we have

$$\begin{aligned}|\varphi'(f)h - \varphi'(g)h| &\le |\varphi'(f)(h-\sigma)| + |\varphi'(f)(\sigma) - \varphi'(g)(\sigma)| + |\varphi'(g)(\sigma - h)| \\ &\le 2C\|h-\sigma\|_{L^1(\mu,X)} + |\varphi'(f)(\sigma) - \varphi'(g)(\sigma)|.\end{aligned}$$

Since the set of simple functions is $\|\cdot\|_{L^1(\mu,X)}$-dense in $L^1(\mu, X)$, we deduce that $L^1(\mu, X) \ni f \mapsto \varphi'(f)h$ is uniformly continuous. Now, it remains to apply Lemma 2.4. ∎

Proof of Theorem 2.1. Let $|\cdot|$ be the equivalent norm on $L^1(\mu, X)$ defined by (2.2). Assume that $\|\cdot\|$ is Gâteaux smooth. An argument as in [FZ, p. 664] shows that $|\cdot|$ is also Gâteaux smooth, and

$$|\cdot|'(f)h = \frac{\varphi'(f)h}{\varphi'(f)f} \quad \text{whenever} \quad f, h \in L^1(\mu, X) \text{ and } |f| = 1. \tag{2.10}$$

For completeness we shall prove this. Fix $f \in L^1(\mu, X)$ with $|f| = 1$. Pick any ζ in the subdifferential $\partial|\cdot|(f)$ and $h \in \ker\zeta$. Put $\gamma(t) = \varphi(f + th)$ for $t \in \mathbb{R}$. Since, by Proposition 2.2, the function φ is Gâteaux differentiable on $L^1(\mu, X)$, we see that γ is differentiable on $\mathbb{R}$, with $\gamma'(t) = \varphi'(f + th)h$ for every $t \in \mathbb{R}$. On the other hand, for each $t \in \mathbb{R}$ we have $|f + th| \ge \zeta(f) = 1$, and bearing in mind that φ is convex we obtain

$$1 = \varphi\left(\frac{f+th}{|f+th|}\right) \le \frac{\varphi(f+th)}{|f+th|} \le \gamma(t).$$

Therefore, $\gamma(t) \ge \gamma(0)$ for all $t \in \mathbb{R}$, and so $0 = \gamma'(0) = \varphi'(f)h$, that is, $h \in \ker\varphi'(f)$. We have proved that $\ker\zeta \subseteq \ker\varphi'(f)$. Hence, there is $a \in \mathbb{R}$ such that $\zeta = a\varphi'(f)$. So, $\partial|\cdot|(f)$ is a singleton, and $|\cdot|$ is Gâteaux differentiable at f. Moreover, since $1 = \zeta(f) = a\varphi'(f)f$ we have $a = (\varphi'(f)f)^{-1}$, and (2.10) is proved.

Now, suppose that $\|\cdot\|$ is uniformly Gâteaux smooth. According to Proposition 2.3 and Lemma 2.4, for every $h \in L^1(\mu, X)$ the function $L^1(\mu, X) \ni h \mapsto \varphi'(f)h$ is uniformly continuous. We shall show that so is $S_{L^1(\mu,X)} \ni f \mapsto |\cdot|'(f)h$. For $f \in L^1(\mu, X)$, write

$$\lambda(f) = \varphi'(f)f.$$

We claim that λ is C-Lipschitzian. Indeed, (2.3) yields

$$\lambda(f) = \int_\Omega M'(\|f(\omega)\|)\|f(\omega)\|\,d\mu(\omega),$$

and since $\mathbb{R} \ni t \mapsto tM'(t)$ is C-Lipschitzian we deduce that

$$\begin{aligned}|\lambda(f) - \lambda(g)| &\le \int_\Omega \big|M'(\|f(\omega)\|)\|f(\omega)\| - M'(\|g(\omega)\|)\|g(\omega)\|\big|\,d\mu(\omega) \\ &\le C\|f-g\|_{L^1(\mu,X)}\end{aligned} \tag{2.11}$$

for all $f, g \in L^1(\mu, X)$. Moreover, from the convexity of φ we get

$$\lambda(f) \geq \varphi(f) - \varphi(0) = 1 \quad \text{whenever} \quad |f| = 1. \tag{2.12}$$

Now, fix $h \in L^1(\mu, X)$, and consider any $f, g \in L^1(\mu, X)$ with $|f| = |g| = 1$. Thanks to (2.10)–(2.12) we have

$$\begin{aligned}
\big| |\cdot|'(f)h - |\cdot|'(g)h \big| &= \left| \frac{\varphi'(f)h}{\lambda(f)} - \frac{\varphi'(g)h}{\lambda(g)} \right| \\
&\leq \frac{1}{\lambda(f)} \big|\varphi'(f)h - \varphi'(g)h\big| + \frac{1}{\lambda(f)\lambda(g)} |\lambda(f) - \lambda(g)| \, \big|\varphi'(g)h\big| \\
&\leq \big|\varphi'(f)h - \varphi'(g)h\big| + |\lambda(f) - \lambda(g)| C \|h\|_{L^1(\mu,X)} \\
&\leq \big|\varphi'(f)h - \varphi'(g)h\big| + C^2 \|f - g\|_{L^1(\mu,X)} \|h\|_{L^1(\mu,X)}.
\end{aligned}$$

Since $L^1(\mu, X) \ni f \mapsto \varphi'(f)h$ is uniformly continuous, it follows that so is $S_{L^1(\mu,X)} \ni f \mapsto \|\cdot\|'(f)h$, and applying Lemma 2.5 we deduce that the norm $|\cdot|$ is uniformly Gâteaux smooth. ∎

Remark 2.6. The equivalent norm $|\cdot|$ we constructed on $L^1(\mu, X)$ is a lattice norm whenever X is a Banach lattice (see e.g. [LT, p. 1] for details). Indeed, assume that $(X, \|\cdot\|, \leq)$ is a Banach lattice. For $f, g \in L^1(\mu, X)$ we write

$$f \preceq g \quad \text{if and only if} \quad f(\omega) \leq g(\omega) \text{ for almost all } \omega \in \Omega.$$

Since the function M is increasing on $[0, \infty)$ we deduce easily that $|f| \leq |g|$ whenever $f, g \in L^1(\mu, X)$ and $f \vee (-f) \leq g \vee (-g)$.

Remark 2.7. Notice that another uniformly Gâteaux smooth renorming on $L^1(\mu, X)$ can be achieved indirectly using the characterization of uniformly Gâteaux smooth renormable Banach spaces mentioned at the beginning. Indeed, if X admits an equivalent uniformly Gâteaux smooth norm, then there exists a Hilbert-generated space Y such that $X \subset Y$ ([FGZ]). On the other hand, imitating an argument in [D] (see also [DU, p. 252, Corollary 11]) we deduce that, if a Banach space Z is generated by a Hilbert space H, then $L^1(\mu, Z)$ is generated by the Hilbert space $L^2(\mu, H)$. (Given a Banach space $(E, \|\cdot\|)$, we denote by $L^2(\mu, E)$ the Banach space of all (equivalence classes of) strongly measurable functions $f : \Omega \to E$ such that $\|f\|^2_{L^2(\mu,E)} = \int_\Omega \|f(\omega)\|^2 \, d\mu(\omega) < \infty$.) Consequently, $L^1(\mu, Y)$ is Hilbert-generated, and hence it admits an equivalent uniformly Gâteaux smooth norm. Since $L^1(\mu, X)$ embeds into $L^1(\mu, Y)$ it follows that $L^1(\mu, X)$ has an equivalent uniformly Gâteaux smooth norm as well.

3. Fréchet smooth and uniformly Fréchet smooth renormings on reflexive subspaces of $L^1(\mu, X)$. In this section, we consider the problem of lifting the properties of Fréchet smoothness and uniform Fréchet smoothness of a Banach space X to $L^1(\mu, X)$. It is well known that if λ is the Lebesgue measure on $[0,1]$, then the space $L^1(\lambda)$ contains an isomorphic copy of ℓ^1. (Indeed, if $\{A_n\}_{n\in\mathbb{N}}$ is a partition of $[0,1]$ into Lebesgue measurable non-negligible sets, then the closed linear span of the characteristic functions of A_n's is a subspace of $L^1(\lambda)$, isometric to ℓ^1.) Since ℓ^∞ is non-separable it follows that $L^1(\lambda)$ is not an Asplund space, and $L^1(\lambda)$ admits no equivalent Fréchet smooth renorming. Thus, an analogue of Theorem 2.1 in the cases of Fréchet or uniformly Fréchet smoothness does not make sense. We have, however, the following result.

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. If the norm $\|\cdot\|$ is Fréchet smooth (resp. uniformly Fréchet smooth), then $L^1(\mu, X)$ admits an equivalent norm whose restriction to every reflexive subspace is Fréchet smooth (resp. uniformly Fréchet smooth).*

As an immediate consequence of the parenthetic version of this theorem we obtain the following extension of the aforementioned result by Rosenthal in the setting of Lebesgue–Bochner spaces.

COROLLARY 3.2. *Let (Ω, Σ, μ) be a probability space. If X is a super-reflexive Banach space, then every reflexive subspace of $L^1(\mu, X)$ is super-reflexive.*

In the proof of Theorem 3.1 we shall use the following analogues of Propositions 2.2 and 2.3.

PROPOSITION 3.3. *Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. Let $\varphi : L^1(\mu, X) \to [0, \infty)$ be defined by (2.1) and Y be a reflexive subspace of $L^1(\mu, X)$. If the norm $\|\cdot\|$ is Fréchet smooth, then φ is "Y-Fréchet differentiable" at every $f \in L^1(\mu, X)$, that is,*

$$\lim_{t\to 0} \sup\left\{\left|\frac{\varphi(f+th)-\varphi(f)}{t} - \varphi'(f)h\right| : h \in B_Y\right\} = 0.$$

In particular, φ restricted to Y is then Fréchet differentiable on Y.

Proof. Fix $f \in L^1(\mu, X)$. In order to guarantee the μ-measurability of some functions defined below, we shall first perform a separable reduction for the Fréchet differentiability of φ at f. For $h \in B_Y$ and $n \in \mathbb{N}$, write

$$R_n(f,h) = n\left[\varphi\left(f + \frac{1}{n}h\right) - \varphi(f)\right] - \varphi'(f)h,$$

and let S be a countable subset of B_Y such that

$$\sup\{R_n(f,h) : h \in B_Y\} = \sup\{R_n(f,h) : h \in S\} \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

We may and do assume that $h(\Omega)$ is a separable subset of X for every $h \in S$

and also that $f(\Omega)$ is separable. Define

$$Y_0 = \overline{\operatorname{span}(S)} \quad \text{and} \quad X_0 = \overline{\operatorname{span}\Big(\bigcup_{h\in S} h(\Omega) \cup f(\Omega)\Big)}.$$

It is clear that X_0 is a separable subspace of X, and a simple argument reveals that

$$h(\Omega) \subset X_0 \quad \text{for every } h \in Y_0. \tag{3.2}$$

Moreover, φ is Y-Fréchet differentiable at f if (and only if) it is Y_0-Fréchet differentiable at f. Indeed, from (3.1) we get

$$\sup\{R_n(f,h) : h \in B_Y\} = \sup\{R_n(f,h) : h \in S\} \le \sup\{R_n(f,h) : h \in B_{Y_0}\}.$$

Thus, if φ is Y_0-Fréchet differentiable at f then

$$\lim_n \sup\{R_n(f,h) : h \in B_Y\} = 0,$$

and the convexity of φ ensures that φ is Fréchet differentiable at f. Therefore, it suffices to prove that φ is Y_0-Fréchet differentiable at f.

Fix $\varepsilon > 0$. Since Y_0 is reflexive, the ball B_{Y_0} is uniformly integrable (see e.g. [DU, p. 104]), i.e., there is $N = N_{Y_0}(\varepsilon) > 1$ such that

$$\int_{\{\|h(\cdot)\|>N\}} \|h(\omega)\|\, d\mu(\omega) < \varepsilon \quad \text{for } h \in B_{Y_0}. \tag{3.3}$$

Now, fix any $h \in B_{Y_0}$ and, for each $n \in \mathbb{N}$, put $g_n = f + \frac{1}{n}h$. The convexity of φ yields

$$\varphi(g_n) - \varphi(f) \ge \varphi'(f)(g_n - f) = \frac{1}{n}\varphi'(f)h,$$
$$\varphi(f) - \varphi(g_n) \ge \varphi'(g_n)(f - g_n) = -\frac{1}{n}\varphi'(g_n)h.$$

Thus,

$$0 \le R_n(f,h) \le \varphi'(g_n)h - \varphi'(f)h,$$

and using Proposition 2.2 we get

$$\begin{aligned} 0 \le R_n(f,h) \le &\int_\Omega M'(\|g_n(\omega)\|)\, \|\cdot\|'(g_n(\omega))(h(\omega))\, d\mu(\omega) \\ &- \int_\Omega M'(\|f(\omega)\|)\, \|\cdot\|'(\|f(\omega)\|)(h(\omega))\, d\mu(\omega). \end{aligned}$$

(Recall that we put $\|\cdot\|'(0) = 0$.) Define $E = \{\omega \in \Omega : \|h(\omega)\| \le N\}$. As M' is bounded by C and $\big|\|\cdot\|'(u)v\big| \le \|v\|$, we obtain

$$M'(\|f(\omega)\|)\, \big|\|\cdot\|'(f(\omega))(h(\omega))\big| \le C\|h(\omega)\|,$$
$$M'(\|g_n(\omega)\|)\, \big|\|\cdot\|'(g_n(\omega))(h(\omega))\big| \le C\|h(\omega)\|,$$

for all $\omega \in \Omega$ and any $n \in \mathbb{N}$. From inequality (3.3) it then follows that

$$\int_{\Omega\setminus E} M'(\|f(\omega)\|)\,\big|\|\cdot\|'(f_n(\omega))(h(\omega))\big|\,d\mu(\omega) \le C\varepsilon,$$

$$\int_{\Omega\setminus E} M'(\|g_n(\omega)\|)\,\big|\|\cdot\|'(g_n(\omega))(h(\omega))\big|\,d\mu(\omega) \le C\varepsilon.$$

Consequently,

$$\begin{aligned}(3.4)\qquad 0 &\le R_n(f,h)\\ &\le 2C\varepsilon + \int_E \big|[M'(\|g_n(\omega)\|) - M'(\|f(\omega)\|)]\|\cdot\|'(g_n(\omega))(h(\omega))\big|\,d\mu(\omega)\\ &\quad + \int_E M'(\|f(\omega)\|)\big|\|\cdot\|'(g_n(\omega))(h(\omega)) - \|\cdot\|'(f(\omega))(h(\omega))\big|\,d\mu(\omega).\end{aligned}$$

For $\omega \in \Omega$, write

$$\begin{aligned} a_n(\omega) &= [M'(\|g_n(\omega)\|) - M'(\|f(\omega)\|)]\,\|\cdot\|'(g_n(\omega))(h(\omega)),\\ (3.5)\qquad b_n(\omega) &= M'(\|f(\omega)\|)[\|\cdot\|'(g_n(\omega))(h(\omega)) - \|\cdot\|'(f(\omega))(h(\omega))].\end{aligned}$$

Since M' is C-Lipschitzian we have

$$\begin{aligned}\int_E |a_n(\omega)|\,d\mu(\omega) &\le C\int_E \|g_n(\omega) - f(\omega)\|\,\|h(\omega)\|\,d\mu(\omega) = \frac{C}{n}\int_E \|h_n(\omega)\|^2\,d\mu(\omega)\\ &\le \frac{CN^2}{n}\mu(E) \le \frac{CN^2}{n},\end{aligned}$$

and using (3.4) we get

$$(3.6)\qquad 0 \le R_n(f,h) \le 2C\varepsilon + \frac{CN^2}{n} + \int_E |b_n(\omega)|\,d\mu(\omega).$$

It remains to estimate the integral $\int_E |b_n(\omega)|\,d\mu(\omega)$.

For each $\omega \in \Omega$ we write

$$\xi_n(\omega) = \sup\bigg\{\bigg\|\|\cdot\|'\Big(f(\omega) + \frac{N}{n}u\Big) - \|\cdot\|'(f(\omega))\bigg\| : u \in B_{X_0}\bigg\}$$

if $\|f(\omega)\| > N/n$, and $\xi_n(\omega) = 2$ otherwise. (Notice that ξ_n does not depend upon $h \in B_{Y_0}$.) We claim that ξ_n is a μ-measurable function. Indeed, let T be a countable dense subset of B_{X_0}. Then

$$\xi_n(\omega) = \sup\bigg\{\bigg\|\|\cdot\|'\Big(f(\omega) + \frac{N}{n}u\Big) - \|\cdot\|'(f(\omega))\bigg\| : u \in T\bigg\}$$

whenever $\|f(\omega)\| > N/n$, because the Fréchet smoothness of $\|\cdot\|$ guarantees that the mapping $X_0 \setminus \{0\} \ni x \mapsto \|\cdot\|'(x) \in X^*$ is norm-to-norm continuous (see [FHHMZ, Corollary 7.24]). The latter supremum is over the countable family of functions $\Omega \ni \omega \mapsto \big\| \|\cdot\|'(f(\omega)+\frac{N}{n}u) - \|\cdot\|'(f(\omega))\big\|$, $u \in T$, each being μ-measurable. It follows that ξ_n is also μ-measurable.

Now, from (3.2) we get

$$|b_n(\omega)| \le CN\xi_n(\omega) \quad \text{for all } \omega \in E.$$

Taking into account that $M'(0) = 0$, we have

$$\begin{aligned}\int_E |b_n(\omega)|\,d\mu(\omega) &= \int_{E\setminus f^{-1}(0)} |b_n(\omega)|\,d\mu(\omega) \le CN \int_{E\setminus f^{-1}(0)} \xi_n(\omega)\,d\mu(\omega)\\ &\le CN \int_{\Omega\setminus f^{-1}(0)} \xi_n(\omega)\,d\mu(\omega),\end{aligned}$$

and using (3.6) we obtain

$$0 \le R_n(f,h) \le 2C\varepsilon + \frac{CN^2}{n} + CN \int_{\Omega\setminus f^{-1}(0)} \xi_n(\omega)\,d\mu(\omega).$$

Since this is true for every $h \in B_{Y_0}$, we have

(3.7)
$$0 \le \sup\{R_n(f,h) : h \in B_{Y_0}\} \le 2C\varepsilon + \frac{CN^2}{n} + CN \int_{\Omega\setminus f^{-1}(0)} \xi_n(\omega)\,d\mu(\omega).$$

Now, we shall prove that

(3.8)
$$\int_{\Omega\setminus f^{-1}(0)} \xi_n(\omega)\,d\mu(\omega) \to 0 \quad \text{as } n \to \infty.$$

Since the mapping $X \setminus \{0\} \ni x \mapsto \|\cdot\|'(x) \in X^*$ is norm-to-norm continuous it follows that $\xi_n(\omega) \to 0$ as $n \to \infty$ for every $\omega \in \Omega \setminus f^{-1}(0)$. On the other hand, $0 \le \xi_n(\omega) \le 2$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$. Therefore, (3.8) follows from Lebesgue's dominated convergence theorem.

From (3.7) and (3.8) we get

$$0 \le \limsup_n \sup\{R_n(f,h) : h \in B_{Y_0}\} \le 2C\varepsilon.$$

Letting $\varepsilon > 0$ go to zero in this inequality we obtain

$$\sup\bigg\{n\bigg[\varphi\bigg(f+\frac{1}{n}h\bigg) - \varphi(f)\bigg] - \varphi'(f)h : h \in B_{Y_0}\bigg\} \to 0 \quad \text{as } n \to \infty,$$

and since φ is convex and $Y_0 \ni h \mapsto \varphi'(f)h$ is linear, it follows that

$$\lim_{t\to 0} \sup\bigg\{\bigg|\frac{\varphi(f+th)-\varphi(f)}{t} - \varphi'(f)h\bigg| : h \in B_{Y_0}\bigg\} = 0,$$

as we wanted to show. ∎

Proposition 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. Let $\varphi : L^1(\mu, X) \to [0,\infty)$ be defined by (2.1) and Y be a reflexive subspace of $L^1(\mu, X)$. If the norm $\|\cdot\|$ is uniformly Fréchet smooth,* *then the derivative φ' is "Y-uniformly continuous", that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|\varphi'(f)h - \varphi'(g)h| < \varepsilon \quad \textit{whenever} \quad f, g \in L^1(\mu, X),\ h \in B_Y \ \textit{and}\ \|f - g\|_{L^1(\mu,X)} < \delta.$$

In particular, φ restricted to Y is then uniformly Fréchet differentiable on Y.

In the proof of this proposition we shall use the following Fréchet counterparts of Lemmas 2.4 and 2.5. Their proofs are left to the reader.

LEMMA 3.5. *Let $(X, \|\cdot\|)$, U, U_1, Δ and ψ be as in Lemma* 2.4.

(i) *If ψ is uniformly Fréchet differentiable on U_1, then the mapping $U_1 \ni x \mapsto \psi'(x) \in X^*$ is norm-to-norm uniformly continuous.*
(ii) *If $U \ni x \mapsto \psi'(x) \in X^*$ is norm-to-norm uniformly continuous, then ψ is uniformly Fréchet differentiable on U_1.*

LEMMA 3.6. *Let $(X, \|\cdot\|)$ be a Banach space whose norm is Gâteaux smooth. Then the following statements are mutually equivalent:*

(i) *The norm $\|\cdot\|$ is uniformly Fréchet smooth.*
(i′) *For every $r > 0$ the norm $\|\cdot\|$ is uniformly Fréchet differentiable on the set $X \setminus rB_X$.*
(ii) *For every $h \in X$ the mapping $S_X \ni x \mapsto \|\cdot\|'(x) \in S_{X^*}$ is norm-to-norm uniformly continuous.*
(ii′) *For every $r > 0$ the mapping $X \setminus rB_X \ni x \mapsto \|\cdot\|'(x) \in S_{X^*}$ is norm-to-norm uniformly continuous.*

Proof of Proposition 3.4. Fix $\varepsilon > 0$. Since Y is reflexive we can find $N = N_Y(\varepsilon) > 1$ such that

$$\int_{\{\|h(\cdot)\| > N\}} \|h(\omega)\|\, d\mu(\omega) < \varepsilon \quad \text{for } h \in B_Y. \tag{3.9}$$

On the other hand, thanks to Lemma 3.6 there exists $0 < \gamma < \min\{1, \varepsilon/N\}$ such that

(3.10)
$$\big\|\|\cdot\|'(x) - \|\cdot\|'(y)\big\| < \frac{\varepsilon}{N} \quad \text{whenever} \quad x, y \in X \setminus \frac{\varepsilon}{N} B_X \text{ and } \|x - y\| < \gamma.$$

Let $\delta = \gamma\varepsilon/N$. Consider any $f, g \in L^1(\mu, X)$ with $\|f - g\|_{L^1(\mu,X)} < \delta$, and take $h \in B_Y$. For $\omega \in \Omega$, write

$$a(\omega) = \big(M'(\|f(\omega)\|) - M'(\|g(\omega)\|)\big)\|\cdot\|'(f(\omega))(h(\omega)),$$
$$b(\omega) = M'(\|g(\omega)\|)\big(\|\cdot\|'(f(\omega))(h(\omega)) - \|\cdot\|'(g(\omega))(h(\omega))\big),$$

and put $E = \{\omega \in \Omega : \|h(\omega)\| \le N\}$. Using (3.9) and proceeding as in the proof of (3.4) we deduce that

$$|\varphi'(f)h - \varphi'(g)h| \le 2C\varepsilon + \int_E |a(\omega)|\,d\mu(\omega) + \int_E |b(\omega)|\,d\mu(\omega). \tag{3.11}$$

Further, bearing in mind that M' is C-Lipschitz we get

$$\begin{aligned}\int_E |a(\omega)|\,d\mu(\omega) &\le C\int_E \big|\|f(\omega)\| - \|g(\omega)\|\big|\|h(\omega)\|\,d\mu(\omega) \\ &\le CN\|f-g\|_{L^1(\mu,X)} < C\varepsilon.\end{aligned} \tag{3.12}$$

Now, we shall estimate the integral $\int_E |b(\omega)|\,d\mu(\omega)$. Define

$$\begin{aligned} E_1 &= \{\omega \in E : \|f(\omega) - g(\omega)\| \ge \gamma\}, \qquad E_2 = E \setminus E_1,\\ E_{21} &= \{\omega \in E_2 : \|g(\omega)\| \le 2\varepsilon/N\}, \qquad E_{22} = E_2 \setminus E_{21}.\end{aligned}$$

Then $\mu(E_1) < 1/\gamma\|f-g\|_{L^1(\mu,X)} < \delta/\gamma = \varepsilon/N$, and consequently

$$\int_{E_1} |b(\omega)|\,d\mu(\omega) \le 2C\int_{E_1} \|h(\omega)\|\,d\mu(\omega) \le 2CN\mu(E_1) < 2C\varepsilon. \tag{3.13}$$

On the other hand, as $M'(\|g(\omega)\|) \le C\|g(\omega)\| \le 2C\varepsilon/N$ for each $\omega \in E_{21}$, it follows that

$$\int_{E_{21}} |b(\omega)|\,d\mu(\omega) \le \frac{2C\varepsilon}{N}\int_{E_{21}} 2\|h(\omega)\|\,d\mu(\omega) \le 4C\epsilon. \tag{3.14}$$

It remains to estimate $\int_{E_{22}} |b(\omega)|\,d\mu(\omega)$. For each $\omega \in E_{22}$ we have $\|g(\omega)\| > 2\varepsilon/N$, and so $\|f(\omega)\| > \varepsilon/N$. Applying now (3.10) with $x := f(\omega)$ and $y := g(\omega)$ we get

$$\big|\|\cdot\|'(f(\omega))(h(\omega)) - \|\cdot\|'(g(\omega))(h(\omega))\big| < \frac{\varepsilon}{N}N = \varepsilon \quad \text{for every } \omega \in E_{22},$$

and consequently

$$\int_{E_{22}} |b(\omega)|\,d\mu(\omega) \le C\varepsilon.$$

Adding this to (3.13) and (3.14) we obtain $\int_E |b(\omega)|\,d\mu(\omega) < 7C\varepsilon$, and taking into account (3.11) and (3.12) yields

$$|\varphi'(f)h - \varphi'(g)h| < 10C\varepsilon.$$

Therefore, φ' is Y-uniformly continuous. Finally, an appeal to Lemma 3.5 shows that the restriction of φ to Y is uniformly Fréchet differentiable. ∎

Proof of Theorem 3.1. Let $|\cdot|$ be the equivalent norm on $L^1(\mu, X)$ given by (2.2). Assume that the norm $\|\cdot\|$ on X is Fréchet smooth. From the end of the proof of Theorem 2.1 we know that

(3.15)

$$\big||\cdot|'(f_1)h - |\cdot|'(f_2)h\big| \le |\varphi'(f_1)h - \varphi'(f_2)h| + C^2\|f_1 - f_2\|_{L^1(\mu,X)}\|h\|_{L^1(\mu,X)}$$

for all $f_1, f_2, h \in L^1(\mu, X)$ such that $|f_1| = |f_2| = 1$. Fix any $f \in L^1(\mu, X)$ with $|f| = 1$. We shall show that $|\cdot|$ is Y-Fréchet differentiable at f. Consider any $0 < t < 1$ and $h \in Y$ with $|h| \le 1$. The convexity of $|\cdot|$ yields

$$t|\cdot|'(f)h \le |f + th| - |f| \le t|\cdot|'(f + th)h = t|\cdot|'\Big(\frac{f + th}{|f + th|}\Big)h.$$

Combining this with (3.15) and bearing in mind that $\|\cdot\|_{L^1(\mu,X)} \le d|\cdot|$ for some constant $d > 0$, we get

$$\begin{aligned}
0 &\le \frac{|f + th| - |f|}{t} - |\cdot|'(f)h \\
&\le \Big|\varphi'\Big(\frac{f + th}{|f + th|}\Big)h - \varphi'(f)h\Big| + C^2\Big\|\frac{f + th}{|f + th|} - f\Big\|_{L^1(\mu,X)}\|h\|_{L^1(\mu,X)} \\
&\le \Big|\varphi'\Big(\frac{f + th}{|f + th|}\Big)h - \varphi'(f)h\Big| + \frac{2d^2t}{1 - t}.
\end{aligned}$$

Therefore,

$$\sup_{h\in B_{(Y,|\cdot|)}}\Big|\frac{|f + th| - |f|}{t} - |\cdot|'(f)h\Big| \le \sup_{h\in B_{(Y,|\cdot|)}}\Big|\varphi'\Big(\frac{f + th}{|f + th|}\Big)h - \varphi'(f)h\Big| + \frac{2d^2t}{1 - t}$$

for all $t \in (0, 1)$. By Proposition 3.3, the function φ is Y-Fréchet differentiable at f. So, φ' is Y-norm-to-norm continuous at f, and thus

$$\lim_{t\to 0^+}\sup_{h\in B_{(Y,|\cdot|)}}\Big|\varphi'\Big(\frac{f + th}{|f + th|}\Big)h - \varphi'(f)h\Big| = 0.$$

Consequently,

$$\lim_{t\to 0^+}\sup_{h\in B_{(Y,|\cdot|)}}\Big|\frac{|f + th| - |f|}{t} - |\cdot|'(f)h\Big| = 0,$$

and the norm $|\cdot|$ is Y-Fréchet differentiable at f.

Assume further that the norm $\|\cdot\|$ on X is uniformly Fréchet smooth. Fix $f, g \in L^1(\mu, X)$ with $|f| = |g| = 1$. From (3.15) we get

$$\sup_{h\in B_{(Y,|\cdot|)}}\big||\cdot|'(f)h - |\cdot|'(g)h\big| \le \sup_{h\in B_{(Y,|\cdot|)}}|\varphi'(f)h - \varphi'(g)h| + C^2d\|f - g\|_{L^1(\mu,X)}.$$

Since, by the former proposition, the mapping $L^1(\mu, X) \ni f \mapsto \varphi'(f)$ is Y-uniformly continuous, so is $S_{(L^1(\mu,X),|\cdot|)} \ni f \mapsto |\cdot|'(f)$, and using Lemma

3.6 we conclude that the norm $|\cdot|$ restricted to Y is uniformly Fréchet smooth. ∎

REMARK 3.7. The parenthetic part of Theorem 3.1 can be proved indirectly, using the well known theorem by Figiel and Pisier [FP] (cf. [LT, II, Theorem 1.e.9]) that $L^2(\mu, X)$ is super-reflexive if (and only if) X is super-reflexive, and some renorming results of [FMZ] on strongly super-reflexive generated Banach spaces. According to [FMZ], we say that a Banach space E is *strongly generated* by a Banach space Y if there exists a bounded linear operator $T : Y \to E$ such that, for every $\varepsilon > 0$ and every weakly compact set $K \subset E$, there is $c > 0$ such that $K \subset cT(B_Y) + \varepsilon B_E$. In [FMZ, Corollary 8], it was shown that if E is strongly generated by a super-reflexive space, then E admits an equivalent norm whose restriction to every reflexive subspace of E is uniformly Fréchet smooth. Further, an argument as in the proof of [FMZ, Proposition 12] shows that if μ is a probability measure and X is a Banach space, then $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$. Thus, if X has an equivalent uniformly Fréchet smooth norm, then $L^1(\mu, X)$ is strongly generated by the super-reflexive space $L^2(\mu, X)$, and $L^1(\mu, X)$ admits an equivalent norm whose restriction to every reflexive subspace is uniformly Fréchet smooth.

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