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Spectral gap lower bound for the one-dimensional fractional Schrödinger operator in the interval

by

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Abstract. We prove a uniform lower bound for the difference $\lambda_2 - \lambda_1$ between the first two eigenvalues of the fractional Schrödinger operator $(-\Delta)^{\alpha/2} + V$, $\alpha \in (1,2)$, with a symmetric single-well potential V in a bounded interval (a,b), which is related to the Feynman–Kac semigroup of the symmetric α-stable process killed upon leaving (a,b). "Uniform" means that the positive constant C_α appearing in our estimate $\lambda_2 - \lambda_1 \geq C_\alpha(b-a)^{-\alpha}$ is independent of the potential V. In the general case of $\alpha \in (0,2)$, we also find a uniform lower bound for the difference $\lambda_* - \lambda_1$, where λ_* denotes the smallest eigenvalue corresponding to an antisymmetric eigenfunction. One of our key arguments used in proving the spectral gap lower bound is a certain integral inequality which is known to be a consequence of the Garsia–Rodemich–Rumsey lemma. We also study some basic properties of the corresponding eigenfunctions.

1. Introduction and statement of results. The main purpose of this paper is to prove a uniform lower bound for the spectral gap of the fractional Schrödinger operator with symmetric single-well potential on a bounded interval of the real line. Such an operator is related to the Feynman–Kac semigroup of the killed symmetric α-stable process. To obtain this bound we study some basic properties of the first and the second eigenfunctions. Mainly we use the fact that the first eigenfunction is unimodal and symmetric, which is a consequence of the rearrangement inequality due to F. J. Almgren and E. H. Lieb [AL]. We also prove differentiability of all eigenfunctions. Another main argument used in proving our spectral gap lower bound is a certain integral inequality which has important consequences in the embedding theory of Sobolev spaces of fractional order. This inequality is known to be a consequence of the Garsia–Rodemich–Rumsey lemma (abbreviated as GRR lemma) [GRR] (see also [Ka]).

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Our work is motivated by the classical results of M. Ashbaugh and R. Benguria [AB1, AB2], where a similar spectral problem was studied for the classical Schrödinger operator with a symmetric single-well potential on a bounded interval.

Before we describe our results in detail let us recall some basic definitions and facts. Let $(X_t)_{t\geq 0}$ be a symmetric α-stable process in $\mathbb{R}$ of order $\alpha \in (0,2)$ with the characteristic function $\mathbb{E}^0[\exp(i\xi X_t)] = \exp(-t|\xi|^\alpha)$, $\xi \in \mathbb{R}$, $t > 0$. As usual, $\mathbb{E}^x$ denotes the expected value of the process starting at $x \in \mathbb{R}$. Let $(a,b) \subset \mathbb{R}$ be a bounded interval and let $\tau_{(a,b)} = \inf\{t \geq 0 : X_t \notin (a,b)\}$ be the first exit time of X_t from (a,b).

The Feynman–Kac semigroup $(T_t)_{t\geq 0}$ for the symmetric α-stable process X_t killed upon leaving (a,b) and for a potential $V \in \mathcal{V}^\alpha((a,b))$ is defined as

$$T_t f(x) = \mathbb{E}^x\Big[\exp\Big(-\int_0^t V(X_s)\,ds\Big) f(X_t);\ \tau_{(a,b)} > t\Big], \tag{1.1}$$

for $f \in L^2((a,b))$, $t > 0$, $x \in (a,b)$, where $\mathcal{V}^\alpha((a,b))$ is the class of functions $V\colon (a,b) \to \mathbb{R}$ satisfying the following three conditions:

(i) Integrability: V extended to $\mathbb{R}$ by putting 0 outside (a,b) is in the Kato class $\mathcal{K}^\alpha$ for the symmetric α-stable process X_t. (The formal definition of $\mathcal{K}^\alpha$ is given in Section 2.)
(ii) Symmetry: $V(x) = V(b+a-x)$ for $x \in (a,b)$.
(iii) Monotonicity: V is nonincreasing in $(a,(a+b)/2]$.

In the above definition we assume that potentials V are defined on the interval (a,b). However, very often, it is useful to view V as a function extended to the whole real line $\mathbb{R}$ by putting $V = 0$ outside (a,b). Notice also that under assumption (i) the above Feynman–Kac semigroup is well defined (see [BB1, BB2]). Moreover, it immediately follows from assumptions (ii) and (iii) that V is a symmetric function, which is bounded from below, nonincreasing in $(a,(a+b)/2]$ and nondecreasing in $[(a+b)/2, b)$. Following [AB1], we refer to potentials from the class $\mathcal{V}^\alpha((a,b))$ as *symmetric single-well potentials* on (a,b).

The operators T_t are symmetric and form a strongly continuous semigroup on $L^2((a,b))$. The infinitesimal generator of the semigroup $(T_t)_{t\geq 0}$ is the fractional Schrödinger operator $(-\Delta)^{\alpha/2} + V$ on (a,b) with homogeneous Dirichlet exterior conditions (that is, outside (a,b)).

In recent years Schrödinger operators based on nonlocal pseudodifferential operators have been intensively studied. One of the most well known result is the so-called Hardy–Lieb–Thirring inequality obtained in 2008 by R. Frank, E. Lieb and R. Seiringer [FLS], which is connected with the problem of the stability of matter [LS]. In the last 30 years many results concerning fractional Schrödinger operators and relativistic Schrödinger operators have been obtained [CMS, HIL, Z, CS1–CS3, BB1, BB2, KS, KK, KL, LMa]. Those results concern functional integration, structure of spectrum, conditional gauge theorem, estimates of eigenfunctions and intrinsic ultracontractivity, and are mostly obtained by using probabilistic and potential-theoretic methods.

The boundedness of the interval (a,b) implies that for any $t>0$ the operator T_t is compact. It follows from the theory of semigroups that there exists an orthonormal basis of eigenfunctions $\{\varphi_n\}$ in $L^2((a,b))$ and the corresponding sequence of eigenvalues

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

satisfying

$$T_t \varphi_n = e^{-\lambda_n t}\varphi_n.$$

We may and do choose the basis $\{\varphi_n\}$ so that each φ_n is either *symmetric* (i.e., $\varphi_n(x) = \varphi_n(a+b-x)$) or *antisymmetric* (i.e., $\varphi_n(x) = -\varphi_n(a+b-x)$) for $x \in (a,b)$. Moreover, each eigenfunction φ_n is continuous and bounded and all λ_n have finite multiplicities. Additionally, λ_1 is simple and the corresponding eigenfunction, called the *ground state* eigenfunction, can be assumed to be strictly positive on (a,b). It is not very difficult to verify that due to symmetry of V also φ_1 is symmetric in (a,b). The function φ_n is an eigenfunction of the related fractional Schrödinger operator corresponding to the eigenvalue λ_n.

Our main concern in this paper is the difference $\lambda_2 - \lambda_1 > 0$, which is called the *spectral gap*. All the above defined objects depend on the stability parameter $\alpha \in (0,2)$, the interval (a,b) and the potential $V \in \mathcal{V}^\alpha((a,b))$. However, for simplicity, we prefer to omit this dependence in our notation.

The analogous spectral problem has been widely studied for classical Schrödinger operators $-\Delta + V$ acting on $L^2(D)$ with Dirichlet boundary conditions, where D is a bounded domain in $\mathbb{R}^d$, $d \ge 1$. Motivated by problems in mathematical physics concerning the behaviour of free boson gases, M. van den Berg [Be] made the following conjecture. If $D \subset \mathbb{R}^d$ is convex with $\operatorname{diam}(D) < \infty$ and V is a nonnegative convex potential in D, then

$$\lambda_{2,D}^V - \lambda_{1,D}^V > \frac{3\pi^2}{\operatorname{diam}(D)^2}, \tag{1.2}$$

where $\lambda_{1,D}^V$ and $\lambda_{2,D}^V$ are the first and the second eigenvalues of $-\Delta + V$ acting on $L^2(D)$ with Dirichlet boundary conditions. This problem has been widely studied by many authors [S-Y, YZ, Da2, BM, Sm, Li, AC]. In particular, the strict inequality (1.2) was obtained in 2010 by B. Andrews and J. Clutterbuck [AC]. Let us point out that this conjecture for intervals on the real line and for arbitrary nonnegative convex potentials was proved earlier by R. Lavine [Lav].

The classical result which is most closely related to ours was obtained by M. Ashbaugh and R. Benguria [AB1, AB2]. They studied this problem in one dimension when D is just a bounded interval and proved the inequality (1.2) for the different class of *symmetric single-well potentials* V that are integrable in D. This class includes the symmetric convex potentials, as well as a variety of nonconvex (but symmetric) potentials.

The problem of eigenvalue estimates and the spectral gap lower bound has also been studied for the fractional Laplacian $-(-\Delta)^{\alpha/2}$ (i.e. $V \equiv 0$) on bounded domains of $\mathbb{R}^d$ with Dirichlet exterior conditions [CS5, DB, BK2–BK4, DK, Kw1, DBM]. In one dimension (when D is an interval) eigenvalue gaps estimates follow from results in [BK1] ($\alpha = 1$) and [CS4] ($\alpha > 1$). Moreover, the recent papers [K-S] ($\alpha = 1$), [Kw2] ($\alpha \in (0, 2)$) contain new asymptotic formulas for eigenvalues, which can be used to find numerical bounds for eigenvalue gaps.

Now we formulate the main results of this paper. The variational formula below for the eigenvalue gaps will be the starting point of our proofs. In fact, it is a fractional extension of classical variational formula for the eigenvalue gaps of the classical Schrödinger operators which can be found for example in [Sm]. For the version of this formula for the fractional Laplacian (i.e. $V \equiv 0$) we refer to [DK]. Denote by $L^2((a,b),\varphi_1^2)$ the space of square-integrable functions on (a,b) with measure $\varphi_1^2(x)\,dx$.

Proposition 1.1. *Assume that $\alpha \in (0,2)$. Let $V \in \mathcal{V}^\alpha((a,b))$, $-\infty < a < b < \infty$. Then for every $n \ge 2$ we have*

$$\lambda_n - \lambda_1 = \inf_{f \in \mathcal{F}_n} \frac{\mathcal{A}_{-\alpha}}{2} \int_a^b\int_a^b \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\,dx\,dy, \tag{1.3}$$

where

$$\mathcal{F}_n = \Big\{ f \in L^2((a,b),\varphi_1^2) : \int_a^b f^2(x)\varphi_1^2(x)\,dx = 1, \\ \int_a^b f(x)\varphi_1(x)\varphi_i(x)\,dx = 0,\ 1 \le i \le n-1 \Big\}$$

and

$$\mathcal{A}_\gamma = \frac{\Gamma((1-\gamma)/2)}{2^\gamma \sqrt{\pi}\,|\Gamma(\gamma/2)|}. \tag{1.4}$$

Moreover, the infimum in (1.3) *is achieved for $f = \varphi_n/\varphi_1$.*

Proposition 1.1 is a consequence of the standard variational formula for eigenvalues and a special case of [C-Z, Theorems 2.6 and 2.8]. We include a short proof at the end of Section 2 for the reader's convenience.

Let us recall that our orthonormal basis $\{\varphi_n\}$ is so chosen that each φ_n is either symmetric or antisymmetric in (a,b). It follows from the fact that $\{\varphi_n\}$ is an orthonormal basis that among φ_n there are infinitely many antisymmetric functions in (a,b). Denote by λ_* the smallest eigenvalue corresponding to an antisymmetric eigenfunction φ_*. It is a natural hypothesis that $\lambda_* = \lambda_2$. For the classical Schrödinger operator on the interval this fact is well known. It is a consequence of the Courant–Hilbert theorem which states that φ_2 has exactly two nodal domains (the interval consists of exactly two subintervals on which the sign of φ_2 is fixed). In our case this problem is more complicated. This is due to the fact that no version of the Courant–Hilbert theorem is known for operators which are nonlocal. Despite this fact, the hypothesis was proved by R. Bañuelos and T. Kulczycki [BK2] for $\alpha = 1$ and $V \equiv 0$. Moreover, recently, M. Kwaśnicki [Kw2] proved it for $\alpha \in (1,2)$ and $V \equiv 0$.

Our first result is the following lower bound for $\lambda_* - \lambda_1$.

THEOREM 1.2. *Assume that $\alpha \in (0,2)$. Let $V \in \mathcal{V}^\alpha((a,b))$, $-\infty < a < b < \infty$. Let λ_* be the smallest eigenvalue corresponding to an antisymmetric eigenfunction φ_*. Then*

$$\lambda_* - \lambda_1 \ge \frac{\mathcal{A}_{-\alpha}}{(b-a)^\alpha}. \tag{1.5}$$

Our proof of the above theorem is based on monotonicity and symmetry properties of the ground state eigenfunction (Lemma 3.1). These properties of φ_1 are a direct consequence of the rearrangement inequality of Almgren and Lieb [AL].

The next theorem is the main result of this paper.

THEOREM 1.3. *Assume that $\alpha \in (1,2)$. Let $V \in \mathcal{V}^\alpha((a,b))$, $-\infty < a < b < \infty$. Then*

$$\lambda_2 - \lambda_1 \ge \frac{C_\alpha^{(3)}}{(b-a)^\alpha} \quad \text{with} \quad C_\alpha^{(3)} = \frac{\mathcal{A}_{-\alpha}}{4}\left(\frac{\alpha-1}{16(\alpha+1)}\right)^2. \tag{1.6}$$

Since we do not know whether the second eigenfunction is antisymmetric, in the proof of Theorem 1.3 we also have to consider the case that it is symmetric. The crucial tool in this case is the following integral inequality due to Garsia, Rodemich and Rumsey [GRR].

LEMMA 1.4. *Let $\alpha \in (1,2)$. Then for any continuous function f on $[a,b]$,*

$$\int_a^b\int_a^b \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy \ge C_\alpha^{(4)} \frac{(f(b)-f(a))^2}{(b-a)^{\alpha-1}} \tag{1.7}$$

with $C_\alpha^{(4)} = \left(\frac{\alpha-1}{16(\alpha+1)}\right)^2$.

The following counterexample shows that for $\alpha \in (0,1)$ the inequality (1.7) does not hold with any positive constant. This range of α requires different arguments. The case $\alpha = 1$ also remains open.

EXAMPLE 1.5. Let $\alpha \in (0,1)$. Let f be a C^∞ function such that

$$f(x) = \begin{cases} 0, & x < 1/4, \\ \in [0,1), & 1/4 \le x < 1/2, \\ 1, & x \ge 1/2. \end{cases} \tag{1.8}$$

Set $f_n(x) = f(nx)$, $n \ge 1$. Clearly, each f_n is a C^∞ function such that $f_n(0) = 0$, $f_n(1) = 1$. However,

$$\int_0^1\int_0^1 \frac{(f_n(x) - f_n(y))^2}{|x-y|^{1+\alpha}}\, dx\, dy \to 0 \quad \text{as } n \to \infty. \tag{1.9}$$

The justification of the above example is a very special version of a similar reasoning in [D, Section 2] and is given in Section 5.

Note that the constants $C_\alpha^{(3)}$ and $C_\alpha^{(4)}$ in Theorem 1.3 and in the inequality (1.7) are not optimal. As we will see below, the inequality (1.7) is an important argument used in proving the bound (1.6). Indeed, we have $C_\alpha^{(3)} = (\mathcal{A}_{-\alpha}/4)\ C_\alpha^{(4)}$. It follows that by improving the constant in (1.7), one improves the constant in (1.6). Notice also that in view of Theorem 1.2 another way to improve the constant in Theorem 1.3 is to show that $\lambda_* = \lambda_2$.

A consequence of (1.7) is the following fractional version of the weighted Poincaré inequality, which may be of independent interest.

COROLLARY 1.6. *Let $\alpha \in (1,2)$ and let $g : [a,b] \to \mathbb{R}$ be continuous, nonincreasing and strictly positive in $[a,b)$. Then for any continuous function f on $[a,b]$ such that $f(a) = 0$,*

$$\int_a^b\int_a^b \frac{(f(x) - f(y))^2}{|x-y|^{1+\alpha}} g(x)g(y)\, dx\, dy \ge \frac{C_\alpha^{(4)}}{(b-a)^\alpha} \int_a^b f^2(x) g^2(x)\, dx. \tag{1.10}$$

Our last theorem, on differentiability of eigenfunctions, is completely independent of the above eigenvalue gaps estimates.

THEOREM 1.7. *Let $\alpha \in (1,2)$ and $V \in \mathcal{V}^\alpha((a,b))$, $-\infty < a < b < \infty$. Then all eigenfunctions φ_n are differentiable in (a,b). Moreover, if $[c,d] \subset (a,b)$, then there exists a constant $C_{V,n,\alpha,a,b,c,d}$ such that for all $x \in [c,d]$ we have*

$$\left|\frac{d}{dx}\varphi_n(x)\right| \le C_{V,n,\alpha,a,b,c,d} \|\varphi_n\|_\infty.$$

The paper is organized as follows. In Section 2 we introduce additional notation and collect various facts which are used later. In particular, we justify the variational formulas for eigenvalue gaps. In Section 3 we discuss properties of eigenfunctions. Section 4 contains the proof of Theorem 1.2. In Section 5 we prove Lemma 1.4, Example 1.5, Corollary 1.6 and our main theorem concerning the spectral gap lower bound.

2. Preliminaries. Let $\alpha \in (0,2)$. By $C_{\alpha,\kappa}$ we always mean a strictly positive and finite constant depending on α and the parameters κ. We adopt the convention that constants in proofs may change their value from one occurence to another. However, very often, especially in the statements of our results, we write $C_\kappa^{(1)}, C_\kappa^{(2)}$ etc. to distinguish between constants.

We now summarize the properties of the symmetric α-stable process and some facts from its potential theory. For further information on the potential theory of stable processes we refer to [CS2, B-V].

Let $X = (X_t)_{t\geq 0}$ be the standard one-dimensional symmetric α-stable process with Lévy measure $\nu(dx) = \mathcal{A}_{-\alpha}|x|^{-1-\alpha}\,dx$, where the constant $\mathcal{A}_{-\alpha}$ is given by (1.4). By $\mathbb{P}^x$ we denote the distribution of X starting at $x \in \mathbb{R}$. For each fixed $t > 0$ the transition density $p(t, y-x)$, $t > 0$, $x, y \in \mathbb{R}$, of X is a continuous and bounded function on $\mathbb{R}\times\mathbb{R}$ satisfying

$$C_\alpha^{-1}\left(\frac{t}{|y-x|^{1+\alpha}} \wedge t^{-1/\alpha}\right) \leq p(t,y-x) \leq C_\alpha\left(\frac{t}{|y-x|^{1+\alpha}} \wedge t^{-1/\alpha}\right). \tag{2.1}$$

It is known that when $\alpha < 1$, the process X is transient with potential kernel [BG]

$$K^{(\alpha)}(y-x) = \int_0^\infty p(t,y-x)\,dt = \mathcal{A}_\alpha|y-x|^{\alpha-1}, \qquad x,y\in\mathbb{R}.$$

Whenever $\alpha \geq 1$ the process is recurrent (pointwise recurrent when $\alpha > 1$). In this case we can consider the compensated kernel [BGR], that is, for $\alpha \geq 1$ we put

$$K^{(\alpha)}(y-x) = \int_0^\infty (p(t,y-x) - p(t,x_0))\,dt,$$

where $x_0 = 0$ for $\alpha > 1$, and $x_0 = 1$ for $\alpha = 1$. In this case

$$K^{(1)}(x) = \frac{1}{\pi}\log\frac{1}{|x|}$$

and

$$K^{(\alpha)}(x) = (2\Gamma(\alpha)\cos(\pi\alpha/2))^{-1}|x|^{\alpha-1}, \qquad x\in\mathbb{R},$$

for $\alpha > 1$. Note that $K^{(\alpha)}(x) \leq 0$ if $\alpha > 1$.

We say that a Borel function $V : \mathbb{R} \to \mathbb{R}$ belongs to the *Kato class* $\mathcal{K}^\alpha$ corresponding to the symmetric α-stable process X if V satisfies either of the two equivalent conditions (see [Z] and [BB2, (2.5)])

$$\lim_{\epsilon\to 0}\sup_{x\in\mathbb{R}}\int_{|y-x|<\epsilon}|V(y)|\,|K^{(\alpha)}(y-x)|\,dy=0,$$

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}\mathbb{E}^x\Big[\int_0^t|V(X_s)|\,ds\Big]=0.$$

For instance, if $V(x)=(1-x^2)^{-\beta}$, $\beta>0$, then $V\in\mathcal{K}^\alpha$ and $V\in\mathcal{V}^\alpha((-1,1))$ provided that $\beta<\alpha\wedge 1$. It can be verified directly that for every $\alpha\in(0,2)$, $\mathcal{K}^\alpha\subset L^1_{\mathrm{loc}}(\mathbb{R})$.

We denote by $p_D(t,x,y)$ the transition density of the process killed upon exiting an open bounded set $D\subset\mathbb{R}$. It satisfies the relation

$$p_D(t,x,y)=p(t,y-x)-\mathbb{E}^x[p(t-\tau_D,y-X_{\tau_D});\,\tau_D\le t],\quad x,y\in D,\ t>0,$$

where $\tau_D=\inf\{t\ge 0: X_t\notin D\}$ is the first exit time from D. For every $t>0$, $x,y\in D$, we have

$$0<p_D(t,x,y)\le p(t,y-x). \tag{2.2}$$

The Green operator of an open bounded set D, denoted by G_D, is defined by

$$G_Df(x)=\mathbb{E}^x\Big[\int_0^{\tau_D}f(X_t)\,dt\Big]=\int_D G_D(x,y)f(y)\,dy$$

for nonnegative Borel functions f on $\mathbb{R}$, where

$$G_D(x,y)=\int_0^\infty p_D(t,x,y)\,dt$$

is called the *Green function* for D.

We now discuss some properties of the Feynman–Kac semigroup for the fractional Schrödinger operator with a potential V on a bounded interval of $\mathbb{R}$. For the rest of this section we assume that $D=(a,b)\subset\mathbb{R}$, $a<b$, and $V\in\mathcal{V}^\alpha((a,b))$. We refer the reader to [BB1, BB2, CS1, CS2, CZ] for a more systematic treatment of fractional Schrödinger operators.

The *V-Green operator* for (a,b) is defined by

$$G^V_{(a,b)}f(x)=\int_0^\infty T_tf(x)\,dt=\mathbb{E}^x\Big[\int_0^{\tau_{(a,b)}}e^{-\int_0^t V(X_s)\,ds}f(X_t)\,dt\Big],$$

for nonnegative Borel functions f on (a,b). The corresponding gauge function is given by (see e.g. [BB1, p. 58], [CS2, CZ])

$$u_{(a,b)}(x)=\mathbb{E}^x[e^{-\int_0^{\tau_{(a,b)}}V(X_s)\,ds}],\quad x\in(a,b).$$

When it is bounded in (a,b), then $((a,b),V)$ is said to be *gaugeable*. It is easy to check that if $V\ge 0$ on (a,b), then gaugeability holds.

The following perturbation type formula, for a potential V such that $((a,b),V)$ is gaugeable and for any bounded function f, will be an important argument in the proof of differentiability of eigenfunctions (see [BB1, (9)]):

$$G^V_{(a,b)}f(x) = G_{(a,b)}f(x) - G_{(a,b)}(VG^V_{(a,b)}f)(x), \qquad x \in (a,b). \tag{2.3}$$

Recall that the class $\mathcal{V}^\alpha((a,b))$ contains signed potentials V. So we do not exclude the case that the operators T_t are not sub-Markovian. However, each operator T_t can be made sub-Markovian by adding a constant to the potential V. Clearly, the eigenvalue gaps $\lambda_n - \lambda_1$, $n \ge 1$, are invariant under the potential translations.

We now justify the variational formula in Proposition 1.1.

Proof of Proposition 1.1. By using the translation invariance of the eigenvalue gaps, we may and do assume that $V \ge 0$ and that the corresponding Feynman–Kac semigroup $(T_t)_{t\ge 0}$ is sub-Markovian. For every $t > 0$ define an operator $\widetilde{T}_t$ by

$$\widetilde{T}_t f = e^{\lambda_1 t}\varphi_1^{-1}T_t(\varphi_1 f), \qquad f \in L^2((a,b),\varphi_1^2).$$

It is easy to see that the operators $\widetilde{T}_t$, $t > 0$, form a semigroup of symmetric Markov operators on $L^2((a,b),\varphi_1^2)$ such that

$$\widetilde{T}_t\left(\frac{\varphi_n}{\varphi_1}\right) = e^{-(\lambda_n-\lambda_1)t}\,\frac{\varphi_n}{\varphi_1}, \qquad n \ge 1.$$

Let

$$\widetilde{\mathcal{E}}(f,f) = \lim_{t\to 0^+}\frac{1}{t}(f - \widetilde{T}_t f, f)_{L^2((a,b),\varphi_1^2)} \tag{2.4}$$

for $f \in L^2((a,b),\varphi_1^2)$. It is known that the form $\widetilde{\mathcal{E}}$ with its natural domain

$$\mathcal{D}(\widetilde{\mathcal{E}}) = \{f \in L^2((a,b),\varphi_1^2) : \widetilde{\mathcal{E}}(f,f) < \infty\}$$

is the Dirichlet form corresponding to the semigroup $(\widetilde{T}_t)_{t>0}$ [FOT, p. 23]. By the standard variational formula for eigenvalues we have

$$\lambda_n - \lambda_1 = \inf_{f\in\mathcal{F}_n}\widetilde{\mathcal{E}}(f,f), \qquad n \ge 2,$$

and the infimum is achieved for $f = \varphi_n/\varphi_1$. Thus to complete the proof of Proposition 1.1 it is enough to see that

$$\widetilde{\mathcal{E}}(f,f) = \frac{\mathcal{A}_{-\alpha}}{2}\int_a^b\!\!\int_a^b \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy. \tag{2.5}$$

However, the equality (2.5) is a special case of more general results in [C-Z, Theorems 2.6 and 2.8]. ∎

3. Properties of eigenfunctions. The following lemma is a direct consequence of the standard variational formula for the ground state eigenvalue and the rearrangement inequality of Almgren and Lieb [AL].

LEMMA 3.1. *Let $\alpha \in (0,2)$ and $V \in \mathcal{V}^{\alpha}((a,b))$, $-\infty < a < b < \infty$. Then φ_1 is symmetric and unimodal in (a,b), i.e., φ_1 is nondecreasing in $(a,(a+b)/2)$ and nonincreasing in $((a+b)/2,b)$.*

Proof. With no loss of generality we may and do assume that (a,b) is a symmetric interval (i.e. $a=-b$). By the standard variational formula for the smallest eigenvalue, we have

$$\lambda_1 = \inf_{f \in \mathcal{G}} \mathcal{E}(f,f) \tag{3.1}$$

with

$$\mathcal{E}(f,f) = \frac{\mathcal{A}_{-\alpha}}{2} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy + \int_{\mathbb{R}} V(x) f^2(x)\,dx,$$

and

$$\mathcal{G} = \{f \in L^2(\mathbb{R}) : \|f\|_2 = 1 \text{ and } f = 0 \text{ on } (a,b)^c\}.$$

It is known that the infimum in (3.1) is achieved for $f=\varphi_1$. Moreover, by taking $\Psi(x)=x^2$, $f=g=\varphi_1$, $W_n(x)=|x|^{-1-\alpha}\wedge n$ ($n\in\mathbb{N}$) in [AL, Corollary 2.3] with $n\to\infty$, and by monotonicity properties of V, we directly deduce that $\mathcal{E}(\breve{\varphi}_1,\breve{\varphi}_1) \le \mathcal{E}(\varphi_1,\varphi_1)$, where $\breve{\varphi}_1$ is the *symmetric decreasing rearrangement* of φ_1. It follows from the definition (see e.g. [AL, Definition 1.3]) that also $\breve{\varphi}_1 \in \mathcal{G}$. However, since the eigenvalue λ_1 is simple, this clearly means that $\varphi_1 = \breve{\varphi}_1$ and thus it is symmetric and unimodal on (a,b). ∎

We need the following auxiliary lemma, which is a version of [BNK, Lemma 5.2] and [BJ, Lemma 10].

LEMMA 3.2. *Let $\alpha \in (1,2)$, $-\infty < a < b < \infty$ and $f \in L^1((a,b))$ be such that for every interval $[c,d] \subset (a,b)$ we have $\sup_{x\in[c,d]} |f(x)| < \infty$. Then*

$$\frac{d}{dx} G_{(a,b)} f(x) = \int_a^b \frac{\partial}{\partial x} G_{(a,b)}(x,y) f(y)\,dy, \qquad x \in (a,b), \tag{3.2}$$

and for every interval $[c,d] \subset (a,b)$ there is a constant $C_{\alpha,f,a,b,c,d} < \infty$ such that

$$\left| \frac{d}{dx} G_{(a,b)} f(x) \right| \le C_{\alpha,f,a,b,c,d}, \qquad x \in [c,d]. \tag{3.3}$$

Proof. Recall that $\alpha \in (1,2)$. Then by [BNK, (5)] we have

$$G_{(a,b)}(x,y) = K^{(\alpha)}(x-y) - H(x,y), \qquad x,y \in (a,b),\ x \ne y,$$

where

$$K^{(\alpha)}(x-y) = (2\Gamma(\alpha)\cos(\pi\alpha/2))^{-1}|x-y|^{\alpha-1},$$
$$H(x,y) = \mathbb{E}^x[K^{(\alpha)}(X_{\tau_{(a,b)}} - y)].$$

Hence

$$\frac{d}{dx}G_{(a,b)}f(x) = \lim_{h\to 0}\int_a^b \frac{K^{(\alpha)}(x+h-y) - K^{(\alpha)}(x-y)}{h} f(y)\,dy$$
$$- \lim_{h\to 0}\int_a^b \frac{H(x+h,y) - H(x,y)}{h} f(y)\,dy, \qquad x \in (a,b).$$

Notice that both partial derivatives $\frac{\partial}{\partial x}K^{(\alpha)}(x-y)$, $x \neq y$, and $\frac{\partial}{\partial x}H(x,y)$ exist (see [BNK, (10)]). For $x \in (a,b)$ denote $\delta(x) = (b-x)\wedge(x-a)$. From [BNK, Lemma 3.2] we have $\left|\frac{\partial}{\partial x}H(x,y)\right| \le C_{\alpha,a,b}\delta(x)^{-1}$. It follows that

$$\left|\frac{\partial}{\partial x}G_{(a,b)}(x,y)\right| \le \left|\frac{\partial}{\partial x}K^{(\alpha)}(x-y)\right| + \left|\frac{\partial}{\partial x}H(x,y)\right| \le C_\alpha|x-y|^{\alpha-2} + C_{\alpha,a,b}\delta(x)^{-1} \tag{3.4}$$

for $x, y \in (a,b)$, $x \neq y$. This inequality and properties of f imply that for each fixed $x \in (a,b)$ the integral on the right hand side of (3.2) is absolutely convergent. Thus, to obtain (3.2) it is enough to show that for any $x \in (a,b)$,

$$\lim_{h\to 0}\int_a^b |F_h^{(1)}(x,y)|\,|f(y)|\,dy + \lim_{h\to 0}\int_a^b |F_h^{(2)}(x,y)|\,|f(y)|\,dy = 0, \tag{3.5}$$

where

$$F_h^{(1)}(x,y) = \frac{K^{(\alpha)}(x+h-y) - K^{(\alpha)}(x-y)}{h} - \frac{\partial}{\partial x}K^{(\alpha)}(x-y),$$
$$F_h^{(2)}(x,y) = \frac{H(x+h,y) - H(x,y)}{h} - \frac{\partial}{\partial x}H(x,y).$$

Fix now $x \in (a,b)$. Let $|h| < \delta(x)/4$. From [BNK, Lemma 3.2] and Lagrange's theorem we obtain

$$|F_h^{(2)}(x,y)| \le C_{\alpha,a,b}\delta(x)^{-1}. \tag{3.6}$$

This estimate and the fact that $f \in L^1((a,b))$ give that

$$\lim_{h\to 0}\int_a^b |F_h^{(2)}(x,y)|\,|f(y)|\,dy = 0$$

by the dominated convergence theorem.

It suffices to show that

$$\lim_{h\to 0}\int_a^b |F_h^{(1)}(x,y)|\,|f(y)|\,dy = 0 \tag{3.7}$$

for each fixed $x \in (a,b)$. Let $\beta \in (0,1/2)$ and

$$\int_a^b |F_h^{(1)}(x,y)|\,|f(y)|\,dy = \int_{(x-\beta\delta(x),x+\beta\delta(x))} |F_h^{(1)}(x,y)|\,|f(y)|\,dy + \int_{(a,b)\cap(x-\beta\delta(x),x+\beta\delta(x))^c} |F_h^{(1)}(x,y)|\,|f(y)|\,dy. \tag{3.8}$$

Fix $x \in (a,b)$ and $\varepsilon > 0$. We will show that for sufficiently small $|h|$ the left hand side of (3.8) is smaller than ε. Let $[c,d] \subset (a,b)$ be such that $x \in (c,d)$. Denote $M = \sup_{x\in[c,d]} |f(x)|$. Let β be small enough so that $(x-\beta\delta(x), x+\beta\delta(x)) \subset [c,d]$. It is known (see [BNK, proof of Lemma 5.2]) that

$$|F_h^{(1)}(x,y)| \le C_\alpha(|x+h-y|^{\alpha-2} \vee |x-y|^{\alpha-2}), \tag{3.9}$$

for $y \in (a,b)$, $y \ne x$, $y \ne x+h$. Hence for any $h \in \mathbb{R}$,

$$\int_{x-\beta\delta(x)}^{x+\beta\delta(x)} |F_h^{(1)}(x,y)|\,|f(y)|\,dy \le MC_\alpha \int_{x-\beta\delta(x)}^{x+\beta\delta(x)} (|x+h-y|^{\alpha-2} + |x-y|^{\alpha-2})\,dy$$
$$\le 2MC_\alpha \int_{-\beta\delta(x)}^{\beta\delta(x)} |y|^{\alpha-2}\,dy.$$

Let β be so small that the above integral is smaller than $\varepsilon/2$. Clearly, $F_h^{(1)}(x,y) \to 0$ as $h \to 0$ for any $x \ne y$. By (3.9),

$$|F_h^{(1)}(x,y)| \le C_\alpha(2\beta^{-1}\delta(x)^{-1} \vee 1)$$

for $y \in (a,b) \cap (x-\beta\delta(x), x+\beta\delta(x))^c$ and $h \in (-\beta\delta(x)/2, \beta\delta(x)/2)$. Since $f \in L^1((a,b))$, the second integral on the right hand side of (3.8) tends to 0 as h tends to 0 by the bounded convergence theorem. Hence for $|h|$ sufficiently small that integral is smaller than $\varepsilon/2$. This finishes the proof of (3.7). Thus (3.2) is proved. The boundedness property (3.3) is a simple consequence of (3.4) and the properties of f. ∎

Proof of Theorem 1.7. The starting point is the system of eigenequations

$$T_t\varphi_n = e^{-\lambda_n t}\varphi_n, \qquad n \ge 1. \tag{3.10}$$

Since we do not exclude that V is a signed potential, it may happen that $\lambda_n < 0$ for finitely many n. Put

$$\eta = \begin{cases} 0 & \text{if } \inf V > 0, \\ 1 & \text{if } \inf V = 0, \\ -2\inf V & \text{if } \inf V < 0. \end{cases}$$

Denote $V_\eta = V + \eta$. Then $V_\eta > 0$ and $((a,b), V_\eta)$ is gaugeable (see p. 274). By (3.10) we clearly have

$$e^{-(\lambda_n+\eta)t}\varphi_n(x) = \mathbb{E}^x[e^{-\int_0^t V_\eta(X_s)ds}\varphi_n(X_t);\, \tau_{(a,b)} > t], \qquad x \in (a,b).$$

Since $\lambda_n + \eta > 0$ for all $n \geq 1$, integrating over t we obtain

$$\varphi_n(x) = (\lambda_n + \eta) G^{V_\eta}_{(a,b)} \varphi_n(x), \qquad x \in (a,b),\, n \geq 1.$$

Applying now the perturbation formula (2.3) we get

$$\varphi_n(x) = (\lambda_n + \eta) G_{(a,b)} \varphi_n(x) - G_{(a,b)}(V_\eta \varphi_n)(x), \qquad x \in (a,b),\, n \geq 1,$$

which can be rewritten as

$$\varphi_n(x) = (\lambda_n + \eta)\int_a^b G_{(a,b)}(x,y)\varphi_n(y)\,dy - \int_a^b G_{(a,b)}(x,y)V_\eta(y)\varphi_n(y)\,dy.$$

Since $\|\varphi_n\|_\infty < \infty$, and $V_\eta \in L^1((a,b))$ and is bounded in any interval $[c,d] \subset (a,b)$, the assumptions of Lemma 3.2 are satisfied. Thus for $x \in (a,b)$ we have

$$\frac{d}{dx}\varphi_n(x) = (\lambda_n+\eta)\int_a^b \frac{\partial}{\partial x} G_{(a,b)}(x,y)\varphi_n(y)\,dy - \int_a^b \frac{\partial}{\partial x} G_{(a,b)}(x,y)V_\eta(y)\varphi_n(y)\,dy.$$

A direct consequence of Lemma 3.2 is that also for any interval $[c,d] \subset (a,b)$ there is a constant $C_{V,\alpha,n,a,b,c,d}$ such that for all $x \in [c,d]$ we have

$$\left|\frac{d}{dx}\varphi_n(x)\right| \leq C_{V,\alpha,n,a,b,c,d}. \ \blacksquare$$

4. Lower bound for $\lambda_* - \lambda_1$

Proof of Theorem 1.2. Let $0 < a < \infty$. With no loss of generality we provide the argument for the symmetric interval $(-a,a)$ only. Let $V \in \mathcal{V}^\alpha((-a,a))$. Recall that our orthonormal basis $\{\varphi_n\}$ is so chosen that each φ_n is either symmetric or antisymmetric. Let n_0 be the smallest natural number such that φ_{n_0} is antisymmetric in $(-a,a)$. Thus $\varphi_* = \varphi_{n_0}$. Let $f = \varphi_*/\varphi_1 = \varphi_{n_0}/\varphi_1$. Then for every $\varepsilon \in (0,a)$ we have

$$\begin{aligned}
&\int_{-a}^{a}\int_{-a}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad \geq \int_{\varepsilon}^{a}\int_{\varepsilon}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad\quad + \int_{\varepsilon}^{a}\int_{-a}^{-\varepsilon} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad\quad + \int_{-a}^{-\varepsilon}\int_{-a}^{-\varepsilon} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad\quad + \int_{-a}^{-\varepsilon}\int_{\varepsilon}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy.
\end{aligned}$$

Simple changes of variables in the last three integrals and the fact that f is antisymmetric give that the last sum can be transformed to

$$2\int_\varepsilon^a\int_\varepsilon^a\left(\frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}+\frac{(f(x)+f(y))^2}{(x+y)^{1+\alpha}}\right)\varphi_1(x)\varphi_1(y)\,dx\,dy.$$

Clearly, this is greater than or equal to

$$2\int_\varepsilon^a\int_\varepsilon^a\frac{(f(x)-f(y))^2+(f(x)+f(y))^2}{(x+y)^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy$$
$$=4\int_\varepsilon^a\int_\varepsilon^a\frac{f^2(x)+f^2(y)}{(x+y)^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy,$$

which, by symmetry, is equal to

$$8\int_\varepsilon^a\int_\varepsilon^a\frac{f^2(x)}{(x+y)^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy.$$

Thus, by Lemma 3.1, we have

$$\int_{-a}^a\int_{-a}^a\frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy$$
$$\ge 8\int_\varepsilon^a\int_\varepsilon^a\frac{f^2(x)}{(x+y)^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dy\,dx\ge 8\int_\varepsilon^a\int_\varepsilon^x\frac{f^2(x)}{(x+y)^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dy\,dx$$
$$\ge 8\int_\varepsilon^a\int_\varepsilon^x dy\,\frac{f^2(x)}{(2x)^{1+\alpha}}\varphi_1^2(x)\,dx\ge\frac{4}{(2a)^\alpha}\int_\varepsilon^a\frac{x-\varepsilon}{x}f^2(x)\varphi_1^2(x)\,dx.$$

Now, letting $\varepsilon\to 0$, we obtain

$$\int_{-a}^a\int_{-a}^a\frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy\ge\frac{4}{(2a)^\alpha}\int_0^a f^2(x)\varphi_1^2(x)\,dx$$
$$=\frac{2}{(2a)^\alpha}\int_{-a}^a f^2(x)\varphi_1^2(x)\,dx=\frac{2}{(2a)^\alpha}.$$

Since $f=\varphi_*/\varphi_1$ is antisymmetric, the assertion of Theorem 1.2 follows easily from Proposition 1.1. ∎

5. Spectral gap estimate

Proof of Lemma 1.4. The lemma follows by taking $\psi(x)=x^2$ and $p(x)=|x|^{(\alpha+1)/2}$ in [GRR, Lemma 1.1]. ∎

Proof of Theorem 1.3. With no loss of generality we provide the argument for the symmetric interval $(-a,a)$, $0<a<\infty$, only. Let $V\in\mathcal{V}^\alpha((-a,a))$. Recall that the orthonormal basis $\{\varphi_n\}$ is so chosen that each φ_n is either symmetric or antisymmetric. If φ_2 is antisymmetric, then Theorem 1.3 follows from Theorem 1.2. Assume now that φ_2 is symmetric. Clearly, the function φ_2/φ_1 is continuous in each interval $[a_0, b_0]$ such that $-a < a_0 < b_0 < a$. By Proposition 1.1, it is enough to estimate from below the double integral

$$\int_{-a}^{a}\int_{-a}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \quad \text{with} \quad f = \varphi_2/\varphi_1.$$

Note that f is symmetric on $(-a, a)$, it changes sign in $(-a, a)$ and $\int_{-a}^{a} f^2(x)\varphi_1^2(x)\,dx = 1$.

Let $a_0 = \min\{x \in [0, a) : f(x) = 0\}$. Consider the following two cases.

Case 1. Assume that

$$\int_{a_0}^{a} f^2(x)\varphi_1^2(x)\,dx \geq 1/4.$$

We have

$$\begin{aligned}
&\int_{-a}^{a}\int_{-a}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad \geq \int_{a_0}^{a}\int_{a_0}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad\quad + \int_{-a}^{-a_0}\int_{-a}^{-a_0} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \\
&\qquad = 2\int_{a_0}^{a}\int_{a_0}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy.
\end{aligned}$$

Let now $b_0 \in [a_0, a)$ be such that $f^2(b_0)\varphi_1^2(b_0) = \max_{x\in(a_0,a)} f^2(x)\varphi_1^2(x)$. We have

$$\int_{a_0}^{a}\int_{a_0}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\varphi_1(x)\varphi_1(y)\,dx\,dy \geq \varphi_1^2(b_0)\int_{a_0}^{b_0}\int_{a_0}^{b_0} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy,$$

which, by Lemma 1.4, is larger than

$$\begin{aligned}
\frac{C_\alpha^{(4)}}{(b_0-a_0)^{\alpha-1}} f^2(b_0)\varphi_1^2(b_0) &\geq \frac{C_\alpha^{(4)}}{(b_0-a_0)^{\alpha-1}}\,\frac{1}{(a-a_0)}\int_{a_0}^{a} f^2(x)\varphi_1^2(x)\,dx \\
&\geq \frac{1}{4}\frac{C_\alpha^{(4)}}{(a-a_0)^{\alpha}}.
\end{aligned}$$

It follows that

$$\int_{-a}^{a}\int_{-a}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\,dx\,dy \geq \frac{1}{2}\,\frac{C_\alpha^{(4)}}{(2a)^\alpha},$$

which ends the proof in the first case.

Case 2. Suppose now that

$$\int_0^{a_0} f^2(x)\varphi_1^2(x)\,dx \geq 1/4.$$

Notice that

$$\Big(\int_{a_0}^{a} f(x)\varphi_1^2(x)\,dx\Big)^2 \leq \int_{a_0}^{a} f^2(x)\varphi_1^2(x)\,dx \int_{a_0}^{a} \varphi_1^2(x)\,dx \leq \varphi_1^2(a_0)(a-a_0)\int_{a_0}^{a} f^2(x)\varphi_1^2(x)\,dx \tag{5.1}$$

by the Schwarz inequality and Lemma 3.1, and

$$-\int_{a_0}^{a} f(x)\varphi_1^2(x)\,dx = \int_0^{a_0} f(x)\varphi_1^2(x)\,dx \tag{5.2}$$

since f is symmetric and $\int_{-a}^{a} f(x)\varphi_1^2(x)\,dx = 0$. Without loosing generality we may and do assume that $f \geq 0$ on $[0,a_0]$. Let $a^* \in [0,a_0)$ be such that $f(a^*) = \max_{x\in[0,a_0)} f(x)$. Note that $\int_0^a f^2(x)\varphi_1^2(x)\,dx = 1/2$. By (5.1) and (5.2), we have

$$\begin{aligned} 1/4 &\geq \int_{a_0}^{a} f^2(x)\varphi_1^2(x)\,dx \geq \frac{(\int_{a_0}^{a} f(x)\varphi_1^2(x)\,dx)^2}{\varphi_1^2(a_0)(a-a_0)} = \frac{(\int_0^{a_0} f(x)\varphi_1^2(x)\,dx)^2}{\varphi_1^2(a_0)(a-a_0)} \\ &= \frac{f^2(a^*)(\int_0^{a_0} f(x)\varphi_1^2(x)\,dx)^2}{f^2(a^*)\varphi_1^2(a_0)(a-a_0)} \geq \frac{(\int_0^{a_0} f^2(x)\varphi_1^2(x)\,dx)^2}{f^2(a^*)\varphi_1^2(a_0)(a-a_0)}, \end{aligned}$$

which implies that

$$f^2(a^*)\varphi_1^2(a_0) \geq 1/(4(a-a_0)). \tag{5.3}$$

We have

$$\begin{aligned} &\int_{-a}^{a}\int_{-a}^{a} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\,dx\,dy \\ &\qquad \geq \int_0^{a_0}\int_0^{a_0} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\,dx\,dy \\ &\qquad\quad + \int_{-a_0}^{0}\int_{-a_0}^{0} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\,dx\,dy \end{aligned}$$

$$= 2 \int_0^{a_0} \int_0^{a_0} \frac{(f(x) - f(y))^2}{|x - y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\, dx\, dy$$
$$\geq 2\varphi_1^2(a_0) \int_{a^*}^{a_0} \int_{a^*}^{a_0} \frac{(f(x) - f(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy.$$

Now, using Lemma 1.4 and (5.3), we obtain

$$\int_{-a}^{a} \int_{-a}^{a} \frac{(f(x) - f(y))^2}{|x - y|^{1+\alpha}} \varphi_1(x)\varphi_1(y)\, dx\, dy \geq 2\varphi_1^2(a_0) \frac{C_\alpha^{(4)}}{(a_0 - a^*)^{\alpha-1}} f^2(a^*)$$
$$\geq \frac{1}{2} \frac{C_\alpha^{(4)}}{(2a)^\alpha}. \ \blacksquare$$

Justification of Example 1.5. We clearly have

$$\int_0^1 \int_0^1 \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy \leq \int_0^1 \int_0^{1/(2n)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy$$
$$+ \int_0^{1/(2n)} \int_0^1 \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy.$$

By symmetry, the right hand side above is equal to

$$2 \int_0^1 \int_0^{1/(2n)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy.$$

Denote the last double integral by J_n. We have

$$J_n \leq \int_0^{2/n} \int_0^{1/(2n)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy + \int_{2/n}^{1} \int_0^{1/(2n)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy$$
$$= I_{n,1} + I_{n,2}.$$

Recall that $f_n(x) = f(nx)$, where f is a C^∞ function. Observe that for $x, y \in [0, 2]$ we have

$$|f_n(x) - f_n(y)| \leq \sup_{z\in[0,2]} |f_n'(z)|\, |x - y| = n \sup_{z\in[0,2]} |f'(z)|\, |x - y| \leq Cn|x - y|.$$

Hence

$$I_{n,1} = \int_0^{2/n} \int_0^{1/(2n)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{1+\alpha}}\, dx\, dy \leq C^2 n^2 \int_0^{2/n} \int_{y-2/n}^{y+2/n} |x - y|^{1-\alpha}\, dx\, dy$$
$$\leq C_\alpha n^{\alpha-1}.$$

Similarly,

$$I_{n,2} \leq \int_{2/n}^{1} \int_{0}^{1/(2n)} \frac{1}{|x-y|^{1+\alpha}}\, dx\, dy \leq \frac{1}{n} \int_{2/n}^{\infty} \left| \frac{1}{n} - y \right|^{-1-\alpha} dy = C_\alpha n^{\alpha-1}. \ \blacksquare$$

Proof of Corollary 1.6. Let $b_0 \in (a,b]$ be such that

$$f^2(b_0)g^2(b_0) = \max_{x\in(a,b]} f^2(x)g^2(x).$$

We have

$$\int_a^b\int_a^b \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} g(x)g(y)\, dxdy \geq g^2(b_0) \int_a^{b_0}\int_a^{b_0} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\, dx\, dy,$$

which, by Lemma 1.4, is larger than

$$\begin{aligned} \frac{C_\alpha^{(4)}}{(b_0-a)^{\alpha-1}} f^2(b_0)g^2(b_0) &\geq \frac{C_\alpha^{(4)}}{(b_0-a)^{\alpha-1}} \frac{1}{(b-a)} \int_a^b f^2(x)g^2(x)\, dx \\ &= \frac{C_\alpha^{(4)}}{(b-a)^{\alpha}} \int_a^b f^2(x)g^2(x)\, dx. \ \blacksquare \end{aligned}$$

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