STUDIA MATHEMATICA 184 (3) (2008)

A stronger Dunford–Pettis property

by

H. Carrión (São Paulo), P. Galindo (Valencia) and
M. L. Lourenço (São Paulo)

Abstract. We discuss a strong version of the Dunford–Pettis property, earlier named (DP^*) property, which is shared by both ℓ_1 and ℓ_∞. It is equivalent to the Dunford–Pettis property plus the fact that every quotient map onto c_0 is completely continuous. Other weak sequential continuity results on polynomials and analytic mappings related to the (DP^*) property are shown.

There are several classes of subsets of a Banach space which play a significant role in describing its topological properties, and hence in the study of operators and polynomials. One can think of the classes of compact, weakly compact, limited sets, etc. Often in the literature when dealing with the complete continuity of either operators or polynomials one faces Banach spaces which are either Schur spaces or Grothendieck spaces with the Dunford–Pettis property; see the work of F. Bombal and G. Emmanuele [1], M. González and J. Gutiérrez [12] and J. Jaramillo, A. Prieto and I. Zalduendo [14]. What do these classes of spaces have in common? We found that, among other things, they share the property that weakly compact sets are limited and also that such property had been introduced by J. Borwein, M. Fabian and J. Vanderwerff in [3] under the name of (DP^*) property.

In this note we obtain characterizations of this property in terms of c_0-valued linear operators. For instance, it is equivalent to the Dunford–Pettis property plus the fact that every quotient map onto c_0 is completely continuous. It is shown that any non-limited operator between Banach spaces with the (DP^*) property fixes a copy of ℓ_1. We also relate it to weak sequential continuity properties of polynomials and analytic mappings defined on Ba-

2000 *Mathematics Subject Classification*: Primary 46B20; Secondary 46G20.

Key words and phrases: Dunford–Pettis property, complete continuity, polynomial.

Research of P. Galindo supported partially by Ccint (Universidade de São Paulo) and BFM-FEDER 2003-07540 (DGI Spain).

Research of M. L. Lourenço supported partially by CNPq (472416/2004-9) and BFM-FEDER 2003-07540 (DGI Spain).

nach spaces enjoying this property; in particular, polynomials are completely continuous when the range space is Gelfand–Phillips.

1. Generalities. A subset L of a Banach space X is said to be *limited* if weak* null sequences in the dual space X^* converge uniformly on L. If all limited sets in a Banach space X are relatively compact, then X is said to be a *Gelfand–Phillips space*. All separable spaces and all weakly compactly generated spaces are Gelfand–Phillips spaces. Recall that $L \subset X$ is said to be *conditionally weakly compact* if any sequence in L has a weakly Cauchy subsequence. A Banach space X has the *Grothendieck property* if weak* convergent sequences in X^* are weakly convergent.

Given Banach spaces X, Y and a positive integer n we denote by $P(^nX, Y)$ the space of all continuous n-homogeneous polynomials from X into Y. To each $P \in P(^nX, Y)$ we can associate a unique symmetric n-linear mapping $\check{P}$ from X^n to Y such that $P(x) = \check{P}(x, \ldots, x)$ for all $x \in X$. The space of all holomorphic functions from X into Y will be denoted by $H(X, Y)$. For any $f \in H(X, Y)$ and $A \subset X$, we define $\|f\|_A = \sup_{x \in A} \|f(x)\|$. As is customary, we say that f is *completely continuous* if it maps weakly convergent sequences into convergent sequences.

For unexplained notation on Banach spaces we refer to [11] and on polynomials and holomorphic mappings to [10].

Definition 1.1. A Banach space X is said to have the (DP^*) *property* whenever all weakly compact sets in X are limited.

In other words, this is equivalent to the fact that for any weakly null sequence (x_n) in X and any weak* convergent sequence (φ_n) in X^*, $\lim_n \varphi_n(x_n) = 0$.

Recall that X has the *Dunford–Pettis property* if for any weakly null sequence (x_n) in X and any weakly convergent sequence (φ_n) in X^*, $\lim_n \varphi_n(x_n) = 0$. That is, weakly null sequences in X are uniformly convergent on weakly compact sets in X^*, or equivalently, on conditionally weakly compact sets. But observe that Definition 1.1 does not mean that weakly null sequences in X are uniformly convergent on weak* compact sets in X^*, as this is just the Schur property of X; and not even on weak* conditionally compact sets, since the unit ball of the dual of a separable space is weak* conditionally compact.

Proposition 1.2. *The Banach space X has the (DP^*) property if, and only if, every conditionally weakly compact set L in X is a limited set. In particular, if X has the (DP^*) property, then it is finite-dimensional or contains a copy of ℓ_1, and $P(^nX)$ contains a copy of ℓ_∞ for $n \geq 2$.*

Proof. Assume X has the (DP^*) property. If L is not limited, there is a sequence (φ_n) weak* null in X^* such that $(\|\varphi_n\|_L)$ is not null. Thus, we can pick $\delta > 0$ and $x_n \in L$ so that $|\varphi_n(x_n)| > \delta$. There is no loss of generality in assuming that (x_n) itself is a weakly Cauchy sequence. Since the sequence $(\varphi_n(x_m))_n$ is null, there is n_m such that $|\varphi_{n_m}(x_m)| < \delta/2$. Thus $|\varphi_{n_m}(x_{n_m} - x_m)| \geq |\varphi_{n_m}(x_{n_m})| - |\varphi_{n_m}(x_m)| \geq \delta/2$. However, the sequence $(x_{n_m} - x_m)_m$ is weakly null in X, so $\lim_m \varphi_{n_m}(x_{n_m} - x_m) = 0$, contrary to the former inequality.

The converse statement is obvious since every weakly compact set is conditionally weakly compact.

To see that X contains a copy of ℓ_1, observe that otherwise by Rosenthal's ℓ_1 theorem, the unit ball of X would be conditionally weakly compact, hence limited. According to the Josefson–Nissenzweig theorem this is only possible if X is finite-dimensional. In case X contains a copy of ℓ_1, it has ℓ_2 as a quotient, $q : X \to \ell_2$. Then $q^t : P(^n\ell_2) \to P(^nX)$ is an embedding and it is known that $P(^n\ell_2)$ contains a copy of ℓ_∞. ■

REMARK 1.3 (cf. [1], [3], [14]). If X is a Schur space, then X has the (DP^*) property. Also every Grothendieck Banach space with the Dunford–Pettis property enjoys the (DP^*) property. Also every Banach space with the Pełczyński (P) property, the Dunford–Pettis property and without complemented copies of c_0 enjoys the (DP^*) property.

As a consequence, H^∞ has the (DP^*) property since Bourgain showed that it has the Dunford–Pettis property [4] and it is a Grothendieck space [5].

REMARK 1.4. If X is a Gelfand–Phillips space, then either X is a Schur space or X lacks the (DP^*) property. In particular, a separable space with the (DP^*) property must be a Schur space. Indeed, if X is not a Schur space, then there is a weakly null sequence which is not norm convergent, hence it cannot be limited.

The above result slightly improves Proposition 5 of [14] (see also [3]) as it points out that if X has the (DP^*) property and is a Gelfand–Phillips space (instead of X^* having weak* sequentially compact unit ball), then it is a Schur space. Let us recall the existence of a Gelfand–Phillips space whose dual unit ball is not weak* sequentially compact [18].

PROPOSITION 1.5. *If X and Y are Banach spaces with the (DP^*) property, then so is $X \times Y$ as well.*

Proof. It suffices to check that the product of two limited sets $A \subset X$ and $B \subset Y$ is limited. Indeed, let $(\varphi_n) \subset (X \times Y)^* \approx X^* \oplus Y^*$ be weak* null. Then $\varphi_n = \phi_n + \psi_n$ for $\phi_n \in X^*$ and $\psi_n \in Y^*$ with (ϕ_n) and (ψ_n) weak* null. Since $\|\psi_n\|_B \to 0$ and $\|\phi_n\|_A \to 0$, we have $\|\varphi_n\|_{A \times B} \leq \|\phi_n\|_A + \|\psi_n\|_B \to 0$. ■

EXAMPLES. (a) In general, a quotient space of a space with the (DP^*) property does not share this property: think of ℓ_2 as a quotient of ℓ_1. If X

has the (DP^*) property and $Y \subset X$ does not contain ℓ_1, then X/Y has the (DP^*) property: Assume that for some weakly null sequence $(\chi_n) \subset X/Y$, and for some weak* convergent sequence $(\Phi_n) \subset (X/Y)^*$, we have $|\Phi_n(\chi_n)| > \delta > 0$. Then we may find a weak* convergent sequence $(\varphi_n) \subset X^*$ and, by Lohman's lifting result (see [8, p. 212]), a weakly Cauchy sequence $(x_k) \subset X$ such that $x_k \in \chi_{n_k}$ and $|\varphi_{n_k}(x_k)| = |\Phi_{n_k}(\chi_{n_k})| > \delta > 0$, which contradicts the (DP^*) property of X since (x_k) is a conditionally weakly compact set.

However, complemented subspaces do inherit this property, although general closed subspaces do not: just recall $c_0 \subset \ell_\infty$. Actually, according to Remark 1.4, every closed subspace of a Banach space X has the (DP^*) property if, and only if, X is a Schur space.

(b) Also the (DP^*) property is neither inherited from the dual, as in the case of c_0, nor inherited by the dual, as in the case of $\ell_1(\ell_2^n)$ which is a Schur space whose dual $\ell_\infty(\ell_2^n)$ lacks the Dunford–Pettis property [22].

(c) The above proposition enlarges the class of spaces enjoying the (DP^*) property and provides examples, like $\ell_1 \times \ell_\infty$, of spaces with the (DP^*) property which are neither Schur nor Grothendieck. Observe also that the proposition implies that the bidual of the disc algebra A has the (DP^*) property. Indeed, it is known ([21, p. 11]) that $A^{**} = H^\infty \oplus V_{\text{sing}}^*$ where V_{sing} denotes the space of measures on the unit sphere of $\mathbb{C}$ singular with respect to the Lebesgue measure. As pointed out there, V_{sing}^* is a $C(K)$-space for an extremely disconnected compact Hausdorff space K, hence a Grothendieck space.

(d) Furthermore, the tensor product $\ell_\infty \widehat{\otimes}_\pi \ell_\infty$ has been shown in [2] to lack the Dunford–Pettis property, hence also the (DP^*).

(e) The space L_1 does not have the (DP^*) property, since otherwise, as a separable space, it must be, according to 1.4, a Schur space, and this is precluded by the fact that it contains a copy of ℓ_2. Furthermore, since L_1 is a weakly sequentially complete Banach lattice, it is a complemented subspace of its bidual (see, for instance, Theorem 1.c.4 in [19, II]), which therefore cannot have the (DP^*) property. As a consequence, ℓ_∞^* does not have (DP^*) either because, thanks to an old result of Pełczyński, $\ell_\infty \approx L_\infty$ $(\simeq L_1^*)$, so ℓ_∞^* is isomorphic to L_1^{**}.

(f) Clearly in a weakly sequentially complete Banach space the notions of conditionally weakly compact and relatively weakly compact set coincide. Therefore if X has the (DP^*) property, then limited sets in X are relatively weakly compact if, and only if, X is weakly sequentially complete.

Proposition 1.6. *If X^* has the (DP^*) property and no sequence in X^* equivalent to the unit basis of ℓ_1 converges in the weak* topology, then X has the (DP^*) property.*

Proof. Let $(\varphi_n) \subset X^*$ and $(x_n) \subset X$ be sequences weak* convergent and weakly null respectively, such that for some $\varepsilon > 0$, $|\varphi_n(x_n)| > \varepsilon$. According to the assumption, (φ_n) does not have subsequences equivalent to the unit basis of ℓ_1. By passing to subsequences, we may suppose that (φ_n) is a weakly Cauchy sequence. Further, (x_n) is a weak* null sequence in X^*, so $\lim_n |\varphi_n(x_n)| = 0$ because E^* has the (DP^*) property. ∎

2. Operators. In this section we obtain several characterizations of the (DP^*) property in terms of linear operators into c_0.

PROPOSITION 2.1. *X has the (DP^*) property if, and only if, every operator $T : X \to c_0$ is completely continuous. In particular, if X has the (DP^*) property, then it does not contain complemented copies of c_0.*

Proof. Let $(x_n) \subset X$ be a weakly null sequence and let (φ_n) be a weak* null sequence in X^*. Define $T : X \to c_0$ by $T(x) = (\varphi_n(x))$. Then $\|T(x_m)\| = \sup_n |\varphi_n(x_m)|$ tends to 0 as $m \to \infty$, so (φ_n) converges uniformly to 0 on $\{x_n\}$.

Conversely, let $T : X \to c_0$ be an operator, and let $(x_n) \subset X$ be a weakly null sequence. If $\|T(x_m)\|$ does not converge to 0, there is no loss of generality in assuming that $\|T(x_m)\| > \delta$ for all m and some $\delta > 0$. For each m, there is a canonical projection from c_0, say π_{k_m}, such that $\|T(x_m)\| = |\pi_{k_m}(T(x_m))|$. The sequence (k_m) cannot be bounded, since otherwise we may take $N > k_m$ for all m, and then considering only the first N coordinates we would obtain a mapping $x \in X \mapsto (T_i(x))_{i=1}^N \in \mathbb{C}^N$ for which $(T_i(x_n))_{i=1}^N$ would be a non-null sequence. Therefore $(\pi_{k_m} \circ T)$ is a weak* null sequence in X^*, and by assumption $(\pi_{k_m} \circ T)$ must converge uniformly to 0 on (x_n). A contradiction. ∎

For a $C(K)$-space, being a Grothendieck space is equivalent to the (DP^*) property. This is so because whenever a $C(K)$-space does not contain a complemented copy of c_0, it must be a Grothendieck space [20, p. 230]. Recall that $\ell_\infty/c_0 \approx C(\beta\mathbb{N} \setminus \mathbb{N})$ is a Grothendieck space, hence it has the (DP^*) property.

THEOREM 2.2. *X has the (DP^*) property if, and only if, X has the Dunford–Pettis property and every quotient mapping $q : X \to c_0$ is completely continuous.*

Proof. We begin with the sufficiency. Let (φ_n) be a weak* null sequence in X^* and (x_n) a weakly null sequence in X. Assume that $|\varphi_n(x_n)| > \delta > 0$ for all $n \in \mathbb{N}$. By Rosenthal's ℓ_1 theorem, we may suppose that either (φ_n) is a weakly Cauchy sequence or it is equivalent to the ℓ_1 basis. In the first case, given x_m, there is n_m such that $|\varphi_n(x_m)| < \delta/2$ for $n \geq n_m$. Further,

$$|(\varphi_{n_m} - \varphi_m)(x_m)| \geq |\varphi_m(x_m)| - |\varphi_{n_m}(x_m)| \geq \delta/2.$$

Now the sequence $(\varphi_{n_m} - \varphi_m)_m$ is weakly null in X^* so by the Dunford–Pettis property, $\lim_m(\varphi_{n_m} - \varphi_m)(x_m) = 0$. This contradiction leaves us with the case where (φ_n) is equivalent to the ℓ_1 basis. Then the mapping $x \in X \mapsto q(x) = (\varphi_n(x)) \in c_0$ is surjective because its transpose q^* is an isomorphic embedding. Hence, by assumption, q must be a completely continuous operator, so $(q(x_n))$ is a null sequence in c_0. This is a contradiction since $\|q(x_n)\| \geq |\varphi_n(x_n)| > \delta$.

The necessity is obvious from Proposition 2.1. ∎

Note that the above proof also shows that for spaces X such that X^* does not contain a copy of ℓ_1, the (DP^*) property and the Dunford–Pettis property are equivalent.

Recall that an operator $T : X \to Y$ is called *limited* (respectively, *conditionally weakly compact*) if T takes the unit ball of X into a limited (respectively, conditionally weakly compact) subset of Y.

THEOREM 2.3. *Assume X and Y have the (DP^*) property. If $T : X \to Y$ is a non-limited operator, then T fixes a copy of ℓ_1.*

Proof. If T is not limited, then there is a sequence (φ_n) weak* null in Y^* such that $\|T^*(\varphi_n)\|$ is not null. Thus, we can pick $\delta > 0$ and $x_n \in X$ with $\|x_n\| \leq 1$ so that $|(\varphi_n \circ T)(x_n)| > \delta$. We claim that (x_n) has no weakly Cauchy subsequence. If the claim is false, then there is no loss of generality in assuming that (x_n) is itself weakly Cauchy. Since the sequence $(\varphi_n(T(x_m)))_n$ is null, there is n_m such that $|(\varphi_{n_m} \circ T)(x_m)| < \delta/2$. Further,

$$|\varphi_{n_m}(T(x_{n_m}) - T(x_m))| \geq |\varphi_{n_m}(T(x_{n_m}))| - |\varphi_{n_m}(T(x_m))| \geq \delta/2.$$

However, the sequence $(x_{n_m} - x_m)_m$ is weakly null in X, so by the (DP^*) property of X, $\lim_m(\varphi_{n_m} \circ T)(x_{n_m} - x_m) = 0$, which contradicts the previous statement. Thus the claim holds. Therefore by Rosenthal's ℓ_1 theorem there is a subsequence of (x_n) which is equivalent to the ℓ_1 basis. For simplicity, we assume again that such a subsequence is the whole sequence.

Now we deal with $(T(x_n))$. Bearing in mind the (DP^*) property of Y and the weak* convergence of (φ_n), the above calculations also show that $(T(x_n))$ has no weakly Cauchy subsequence. Finally, Rosenthal's ℓ_1 theorem gives us a subsequence $(T(x_{n_k}))$ of $(T(x_n))$ equivalent to the ℓ_1 basis. Therefore we have found a copy of ℓ_1 fixed by T. ∎

We may deduce from the above theorem that the quotient mapping $\ell_\infty \to \ell_\infty/c_0$ fixes a copy of ℓ_1 since, as an open mapping, it is not limited.

The following is an extension of Corollary in [6] that follows straight from Proposition 1.2.

REMARK 2.4. If Y enjoys the (DP^*) property and the operator $T : X \to Y$ is a conditionally weakly compact, then T is limited.

3. Polynomials and analytic mappings. The object of this section is to relate the (DP^*) property to weak sequential continuity properties of polynomials and analytic mappings defined on Banach spaces enjoying it. In particular, polynomials are completely continuous when the range space is Gelfand–Phillips.

PROPOSITION 3.1. *Let X and Y be Banach spaces with $c_0 \subseteq Y$. If every operator $T : X \to Y$ is completely continuous, then X has the (DP^*) property and every polynomial $P \in P(^nX, Y)$ is completely continuous.*

Proof. The assumption implies that every $T : X \to c_0$ is completely continuous since $c_0 \subseteq Y$. Thus X has the (DP^*) property by Proposition 2.1.

The second statement is proved by induction on the degree of P. It is obvious for $n = 1$. So, assume it is true for n. For $P \in P(^{n+1}X, Y)$ we show first that P maps weakly null sequences into null sequences. Let $(x_m) \subset X$ be a weakly null sequence. The inductive hypothesis shows that for each $x \in X$, the n-homogeneous polynomial $z \in X \mapsto \check{P}(x, z^n) \in Y$ is completely continuous, hence $\lim_m \check{P}(x, x_m^n) = 0$. Choose $\varphi_m \in Y^*$ such that $\|\varphi_m\| = 1$ and $\varphi_m(P(x_m)) = \|P(x_m)\|$. Then $T : X \to c_0 \subseteq Y$ given by

$$T(x) = (\varphi_m(\check{P}(x, x_m^n)))_m$$

is a well defined operator which is completely continuous since X has the (DP^*) property. Therefore,

$$0 = \lim_m \|T(x_m)\| = \lim_m \|(\varphi_m(\check{P}(x_m, x_m^n)))_m| = \lim_m \|P(x_m)\|,$$

as we wanted. Now for a sequence $(x_n) \subset E$ weakly convergent to a it suffices to observe the identity

$$P(x_m) - P(a) = P(x_m - a) + \sum_{j=1}^{n} \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j).$$

By the above, $\lim P(x_m - a) = 0$ and moreover every mapping $z \in X \mapsto \check{P}(z^{n+1-j}, a^j) \in Y$, $j = 1, \ldots, n$, is a polynomial of degree not greater than n, so by induction $\lim_m \check{P}((x_m - a)^{n+1-j}, a^j) = 0$. Thus $\lim_m P(x_m) = P(a)$. ∎

The above theorem does not necessarily hold if the assumption $c_0 \subseteq Y$ is removed. Just consider $X = \ell_2$ and $Y = \ell_1$, for which all operators $L : \ell_2 \to \ell_1$ are compact (Pitt's theorem). Obviously ℓ_2 lacks the (DP^*) property and the 2-homogeneous polynomial $(x_n) \in \ell_2 \mapsto (x_n^2) \in \ell_1$ is not completely continuous.

We observe that the (DP^*) property yields a polynomial version of itself.

REMARK 3.2. *If X has the (DP^*) property, then pointwise convergent sequences in $P(^kX)$ converge uniformly to 0 on weakly null sequences in X.* Indeed, as a consequence of [13, Theorem 5], for any sequence (P_n) pointwise convergent in $P(^kX)$, to say P, we know that $P_n - P$ converges to 0 uniformly on limited sets in E, and, in particular, on any weakly null sequence (x_n) in X. Further, since E also has the Dunford–Pettis property, P is completely continuous. Therefore, $\lim_n P_n(x_n) = \lim_n (P_n - P)(x_n) = 0$.

As a consequence of Proposition 3.1 we recover Theorem 17 in [12], that is, if X has the (DP^*) property, then every polynomial $P : X \to c_0$ is completely continuous. Our next result extends this in the same way Theorem 6 in [1] did. We obtain a slight extension valid for some holomorphic mappings; we include in the proof the polynomial case just for the reader's convenience.

PROPOSITION 3.3. *If X has the (DP^*) property and Y is a Gelfand–Phillips space, then every polynomial $P : X \to Y$ is completely continuous. Further, any $f \in H(X, Y)$ which is bounded on weakly compact (resp. limited) sets is weakly continuous on them.*

Proof. Since polynomials map limited sets into limited sets [13], P maps weakly compact subsets of X into limited sets in Y, hence into relatively compact sets, and further the weak and norm topologies coincide on those images. Now if $(x_n) \subset X$ converges weakly to a, then for each $\varphi \in Y^*$, $(\varphi \circ P)(x_n) \to (\varphi \circ P)(a)$ because of [13, Theorem 3], that is, $P(x_n)$ converges weakly to $P(a)$. Further, by [13, Proposition 7], P is weakly continuous on limited sets in X.

Since f may be uniformly approximated on weakly compact (resp. limited) sets by its Taylor series at 0, the weak continuity of f on weakly compact (resp. limited) sets follows from that of the polynomials in the Taylor series. ∎

In [7, Fact 1], relating to a question of Pełczyński, it is shown that a separable Banach space X is Schur if, and only if, every symmetric bilinear separately compact map $X \times X \to c_0$ is completely continuous. It is also remarked that this may fail for nonseparable spaces. Our next remark points out that the (DP^*) property is the due property of X for the stated condition to hold.

COROLLARY 3.4. *A Banach space X has the (DP^*) property if, and only if, every symmetric bilinear separately compact map $X \times X \to c_0$ is completely continuous.*

Proof. The necessity follows from Proposition 3.3 and the polarization formula. Conversely, let $T \in L(X, c_0)$. We claim that the symmetric bilinear map $T \otimes T : X \times X \to c_0$ given by $T \otimes T(x, y) = T(x)T(y)$ (pointwise product) is separately compact. Without loss of generality, assume $\|T\| = 1$. Now, fix $x \in X$. For given $\varepsilon > 0$ there is n such that $|T(x)_m| \le \varepsilon$ for all $m \ge n$; so

$$T(x)T(B_X) \subset (\|T(x)\|\mathbb{D})^n + \varepsilon B_{c_0}.$$

Hence $T(x)T$ is a compact mapping, as claimed. Therefore, by assumption, $T \otimes T$ is completely continuous. Thus if (x_n) is a weakly null sequence in X, then $\|T \otimes T(x_n, x_n)\| = \|(T(x_n))^2\| = \|(T(x_n))\|^2 \to 0$, showing that T is completely continuous. ∎

It is clear that no Gelfand–Phillips space can contain a (complemented) copy of ℓ_∞. However in the above proposition we cannot replace the condition on the space Y by the non-containment of copies of ℓ_∞ without some extra assumption on X. This is pointed out by Haydon's example of a compact space K such that $C(K)$ is a Grothendieck space without copies of ℓ_∞ ([15]), for which the identity mapping is not completely continuous.

Let us recall that a subset L of X is *bounding* if every $f \in H(X)$ is bounded on L.

Corollary 3.5. *All holomorphic functions on X are completely continuous if, and only if, all holomorphic functions on X are bounded on weakly compact sets. Under this condition, X has the (DP^*) property, and if Y is a Gelfand–Phillips space, then any $f \in H(X, Y)$ is completely continuous.*

Proof. $\Rightarrow$. If $f \in H(X)$ and there is a weakly compact set $L \subset X$ such that f is unbounded on L, then we can find a weakly convergent sequence $(x_n) \subset L$ such that $|f(x_n)| \geq n$, contradicting the assumption.

$\Leftarrow$. Now the weakly compact sets are bounding, hence limited (see for instance [17]). Therefore X has the (DP^*) property and so we may apply the above proposition.

The final statement follows from noticing that, under the assumption, any holomorphic mapping on X is bounded on each weakly compact set L, so we may approximate it by some polynomial which is completely continuous on L. ∎

This corollary actually requires dealing with *all* holomorphic functions: Recall that for the reflexive Tsirelson space, T^*, a holomorphic function on T^* is completely continuous if, and only if, it is bounded on weakly compact sets. Nevertheless this class does not exhaust the space of all holomorphic functions on T^*, and clearly T^* does not have the (DP^*) property.

Let $\mathcal{B}$ denote the class of Banach spaces which satisfy the equivalent conditions in Corollary 3.5, that is, *Banach spaces whose weakly compact sets are bounding.* The above corollary shows that any $X \in \mathcal{B}$ satisfies the (DP^*) property, a fact that also follows from the proof of Theorem 3 in [14]. The spaces in the class $\mathcal{B}$ may be characterized in an analogous way to Proposition 2.1: *$X \in \mathcal{B}$ if, and only if, all $f \in H(X, c_0)$ are completely continuous.*

The space ℓ_∞ belongs to $\mathcal{B}$. This follows from Theorem 1 in [16].

We claim that also the space $\ell_\infty/c_0 \approx C(\beta\mathbb{N}\setminus\mathbb{N})$ is in $\mathcal{B}$. First, we prove that for any weakly null sequence $(f_n) \subset C(\beta\mathbb{N}\setminus\mathbb{N})$ there is a weakly null sequence of extensions $(\widetilde{f_n}) \subset C(\beta\mathbb{N})$. In order to do that, define in a preliminary step $\widetilde{f_n}(m) = 1/m$ for $m \leq n$; the functions $\widetilde{f_n}$ are continuous in the compact subset $\{1,\ldots,n\} \cup \beta\mathbb{N}\setminus\mathbb{N}$ which have continuous extensions to $\beta\mathbb{N}$ which we also denote by $\widetilde{f_n}$ and which satisfy $\|\widetilde{f_n}\| \leq \max\{\|f_n\|,1\}$. The resulting sequence $(\widetilde{f_n})$ clearly converges pointwise to 0 in $\beta\mathbb{N}$ and is uniformly bounded. Thus the Lebesgue dominated convergence theorem guarantees that it is weakly null. To prove the claim, assume to the contrary that there is a $g \in H(C(\beta\mathbb{N}\setminus\mathbb{N}))$ unbounded on some weakly compact set $L \subset C(\beta\mathbb{N}\setminus\mathbb{N})$. Thus there is a sequence $(x_n) \subset L$ weakly convergent to x_0 such that $(g(x_n))$ is an unbounded sequence. Put $f_n = x_n - x_0$. If $G(x) = g(x+x_0)$, then $G \in H(C(\beta\mathbb{N}\setminus\mathbb{N}))$, and the sequence $(G(f_n))$ is unbounded. If $q : C(\beta\mathbb{N}) \to C(\beta\mathbb{N}\setminus\mathbb{N})$ is the restriction map, then $G \circ q \in H(C(\beta\mathbb{N}))$ and $(G \circ q(\widetilde{f_n}))$ is unbounded. This is a contradiction, since $(\widetilde{f_n})$ is a weakly null sequence.

Next we show that $\mathcal{B}$ is stable under cartesian product and that the class of Banach spaces with the (DP^*) property is wider than $\mathcal{B}$.

Proposition 3.6. *If $X, Y \in \mathcal{B}$, then $X \times Y$ also belongs to $\mathcal{B}$.*

Proof. Let $f \in H(X \times Y)$ and consider a weakly compact set in $X \times Y$ which we may suppose to be $A \times B$ with $A \subset X$ and $B \subset Y$ both weakly compact. We check that the collection $\{f(x,\cdot)\}_{x\in A} \subset H(Y)$ is τ_0-bounded. Indeed, for any compact subset K of Y, the collection $\{f(\cdot,y)\}_{y\in K} \subset (H(X),\tau_0)$ is bounded, hence τ_δ-bounded ([9, 2.44, 2.46]). In addition, since A is bounding in X, the sup norm on A, $\|\cdot\|_A$, is a τ_δ-continuous seminorm in $H(X)$ by [9, 4.18], so

$$\sup_{x\in A}\sup_{y\in K}|f(x,y)| = \sup_{y\in K}\sup_{x\in A}|f(x,y)| = \sup_{y\in K}\|f(\cdot,y)\|_A < \infty,$$

as we wanted. Now, since B is bounding in Y, $\{f(x,\cdot)\}_{x\in A}$ is bounded for the $\|\cdot\|_B$ seminorm, hence $\{|f(x,y)|\}_{x\in A, y\in B}$ is bounded, and so $A \times B$ is bounding in $X \times Y$. ∎

Example 3.7. *There is a Banach space with the (DP^*) property which does not belong to $\mathcal{B}$.* Let E be the Banach space constructed by Josefson in [17]. It contains a copy of c_0, its unit basis (e_k) is a limited set in E (ibid., Lemma 1) and E/c_0 is a Schur space. Actually, E has the (DP^*) property. Indeed, let (x_n) be a weakly null sequence in E. If $q : E \to E/c_0$ is the quotient mapping, then $(q(x_n))$ is a null sequence, so we can choose $a_n \in c_0$ such that $\|x_n + a_n\| \leq \|q(x_n)\| + 1/n$. Then $(x_n + a_n)$ is a null sequence, and therefore (a_n) is a weakly null sequence in c_0. Since

$$\{x_n : n \in \mathbb{N}\} \subset \{x_n + a_n : n \in \mathbb{N}\} - \{a_n : n \in \mathbb{N}\},$$

to show that $\{x_n : n \in \mathbb{N}\}$ is limited it is sufficient to prove the limitedness in E of $\{a_n : n \in \mathbb{N}\}$. Suppose the latter set is not limited in E. Then (a_n) cannot be norm null and we may apply the Bessaga–Pełczyński selection principle (see [11, 6.21]) to obtain an also non-limited subsequence (a_{n_k}) of (a_n) that is equivalent to a block basic sequence of the unit basis (e_k) which in turn is equivalent (up to normalization) to (e_k) ([11, 6.22]). Since (e_k) is a limited set in E, so also is (a_{n_k}), contrary to assumption.

On the other hand, all bounding sets in E are relatively compact. If all the weakly compact sets were bounding, they would also be relatively compact or, in other words, E would be a Schur space. This is not possible since E contains a copy of c_0.

Acknowledgements. This note was prepared while the second author visited USP and the third visited UV. Both thank each institution for the support received.

References

[1] F. Bombal and G. Emmanuele, *Remarks on completely continuous polynomials*, Quest. Math. 20 (1997), 85–93.

[2] F. Bombal and I. Villanueva, *On the Dunford–Pettis property of the tensor product of $C(K)$ spaces*, Proc. Amer. Math. Soc. 129 (2001), 1359–1363.

[3] J. Borwein, M. Fabian and J. Vanderwerff, *Characterizations of Banach spaces via convex and other locally Lipschitz functions*, Acta Math. Vietnam. 22 (1997), 53–69.

[4] J. Bourgain, *New Banach space properties of the disc algebra and H^∞*, Acta Math. 152 (1984), 1–48.

[5] —, *H^∞ is a Grothendieck space*, Studia Math. 75 (1982), 193–226.

[6] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. 119 (1984), 55–58.

[7] F. Cabello, R. García and I. Villanueva, *On a question of Pełczyński about multilinear operators*, Bull. Polish Acad. Sci. Math. 48 (2000), 341–345.

[8] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, 1984.

[9] S. Dineen, *Complex Analysis on Locally Convex Spaces*, North-Holland Math. Stud. 57, North-Holland, 1981.

[10] —, *Complex Analysis on Infinite Dimensional Spaces*, Springer, 1999.

[11] M. Fabian, P. Habala, P. Hajek, V. Montesinos, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books in Math., Springer, 2001.

[12] M. González and J. Gutiérrez, *Polynomial Grothendieck properties*, Glasgow Math. J. 37 (1995), 211–219.

[13] P. Galindo, *Polynomials and limited sets*, Proc. Amer. Math. Soc. 124 (1996), 1481–1488.

[14] J. Jaramillo, A. Prieto and I. Zalduendo, *Sequential convergences and Dunford–Pettis properties*, Ann. Acad. Sci. Fenn. 25 (2000), 467–475.

[15] R. Haydon, *A non reflexive Grothendieck space that does not contain ℓ_∞*, Israel J. Math. 40 (1981), 65–73.
[16] B. Josefson, *Bounding subsets of $\ell_\infty(A)$*, J. Math. Pures Appl. 57 (1978), 397–421.
[17] —, *A Banach space containing non-trivial limited sets but no non-trivial bounding set*, Israel J. Math. 71 (1990), 321–327.
[18] —, *A Gelfand–Phillips space not containing ℓ_1 whose dual ball is not weak* sequentially compact*, Glasgow Math. J. 43 (2001), 125–128.
[19] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer, 1979.
[20] C. Niculescu, *Weak compactness in Banach lattices*, J. Operator Theory 6 (1981), 217–231.
[21] A. Pełczyński, *Banach Spaces of Analytic Functions and Absolutely Summing Operators*, CBMS Reg. Conf. Ser. Math. 30, Amer. Math. Soc., Providence, RI, 1977.
[22] C. Stegall, *Duals of certain spaces with the Dunford–Pettis property*, Notices Amer. Math. Soc. 19 (1972), 799.

Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281
CEP 05315-970, São Paulo, Brazil
E-mail: leinad@ime.usp.br
mllouren@ime.usp.br

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Valencia
46100 Burjasot (Valencia), Spain
E-mail: galindo@uv.es

Received October 17, 2005
Revised version November 12, 2007 (5778)