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On the boundedness of the differentiation operator between weighted spaces of holomorphic functions

by

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Abstract. We give necessary and sufficient conditions on the weights v and w such that the differentiation operator $D : Hv(\Omega) \to Hw(\Omega)$ between two weighted spaces of holomorphic functions is bounded and onto. Here $\Omega = \mathbb{C}$ or $\Omega = \mathbb{D}$. In particular we characterize all weights v such that $D : Hv(\Omega) \to Hw(\Omega)$ is bounded and onto where $w(r) = v(r)(1 - r)$ if $\Omega = \mathbb{D}$ and $w = v$ if $\Omega = \mathbb{C}$. This leads to a new description of normal weights.

1. Introduction. Let Ω be the complex plane $\mathbb{C}$ or the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A *weight* v on Ω is a continuous non-increasing function $v : [0, a[\to]0, \infty[$ where $a = 1$ if $\Omega = \mathbb{D}$ and $a = \infty$ if $\Omega = \mathbb{C}$. We assume that $\lim_{r\to a} v(r) = 0$ if $a = 1$ and $\lim_{r\to a} r^m v(r) = 0$ for all $m \geq 0$ if $a = \infty$. For a function $h : \Omega \to \mathbb{C}$ and $r \in [0, a[$ put

$$M_\infty(h, r) = \sup_{|z|=r} |h(z)| \quad \text{and} \quad \|h\|_v = \sup_{0\leq r<a} M_\infty(h, r)v(r).$$

We consider the Banach space

$$Hv(\Omega) = \{h : \Omega \to \mathbb{C} \text{ holomorphic: } \|h\|_v < \infty\}$$

endowed with the norm $\| \cdot \|_v$. Hence, a holomorphic function h satisfies $h \in Hv(\Omega)$ if and only if $M_\infty(h, r) = O(1/v(r))$ as $r \to a$.

There is an extensive literature about the Banach spaces $Hv(\Omega)$ and their generalisations to other domains $\Omega \subset \mathbb{C}^n$ or to corresponding spaces of harmonic functions (see e.g. [19, 20, 21, 2, 11, 8, 12, 15, 16, 14]). Moreover, many authors study special classes of operators between such spaces. For example, the authors of [3, 6, 9] discuss multiplication operators $M_\varphi f = \varphi \cdot f$, $f \in Hv(\Omega)$, where φ is a fixed holomorphic function. Other papers ([7, 5, 22]

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and many more) deal with composition operators $C_\varphi f = f \circ \varphi$, $f \in Hv(\Omega)$, where $\varphi : \Omega \to \Omega$ is again a fixed holomorphic function.

Also, there is a vast literature on interpolation and sampling in these weighted spaces of holomorphic functions (e.g. [17, 18, 1, 10]). Here the operators

$$T : Hv(\mathbb{D}) \to l_\infty, \qquad f \mapsto (f(z_n)v(z_n))_n,$$

are studied where $(z_n)_n \subset \mathbb{D}$ is a given sequence, which is called a set of interpolation if T is surjective and a sampling set if T is a monomorphism. A nice survey of all these results is given in [4].

In our paper we discuss the question of what kind of growth condition the derivative $Dh = h'$ satisfies. In Section 2 we introduce necessary and sufficient conditions on weights v and w such that $D : Hv(\Omega) \to Hw(\Omega)$ is bounded and sometimes onto. In Section 3 we investigate the case $\Omega = \mathbb{D}$ and $w(r) = (1-r)v(r)$ while in Section 4 we focus on $\Omega = \mathbb{C}$ and $w = v$.

To this end we make some further assumptions on v which do not restrict generality. We can always fix radii $r_1 < r_2 < \cdots < a$ such that $v(r_n) = 2v(r_{n+1})$ for all n and change $v(r)$ keeping monotonicity for $r_n < r < r_{n+1}$ without changing $Hv(\Omega)$. Therefore we can always assume that v is continuously differentiable. Moreover in the following, for any $n > 0$, the function $\gamma_n(r) = r^n v(r)$ plays an important role. Put

$$r_n = \min\{r : r \text{ is a global maximum point of } \gamma_n\},$$
$$s_n = \max\{r : r \text{ is a global maximum point of } \gamma_n\}.$$

1.1. Lemma. *If $m < n$ then $s_m \le r_n$.*

Proof. We have

$$s_m^n v(s_m) \le r_n^n v(r_n) \le r_n^{n-m} s_m^m v(s_m).$$

Hence $s_m^{n-m} \le r_n^{n-m}$ and thus $s_m \le r_n$. ■

So, if $r_m < r < s_m$ then r cannot be a global maximum point for γ_n for any $n \ne m$. For those m with $r_m < s_m$ we change v on the interval $[r_m, s_m]$. Define $\widetilde{v}(r) = (r_m/r)^m v(r_m)$ if $r_m < r < s_m$. Then all $r \in [r_m, s_m]$ are global maximum points of $r^m \widetilde{v}(r)$ and we obtain $r^m \widetilde{v}(r) = r_m^m v(r_m)$. Moreover, $\widetilde{v}(r_m) = v(r_m)$ and $\widetilde{v}(s_m) = v(s_m)$. According to [16, Corollary 5.4], $\|\cdot\|_v$ is equivalent to a norm which depends exclusively on the global maximum points of the functions γ_m. So in the following we assume that any $r \in [r_m, s_m]$, for any $m > 0$, is a global maximum point of γ_m. This is no loss of generality, otherwise we go over to $\widetilde{v}$ where $\|\cdot\|_{\widetilde{v}}$ is equivalent to $\|\cdot\|_v$.

1.2. Lemma. *We have $\lim_{n\to\infty} r_n = a$.*

Proof. According to Lemma 1.1, r_n is increasing. Put $r = \lim_{n\to\infty} r_n$ and assume $r < a$.

CASE $a = 1$. Here we obtain
$$1 \geq \left(\frac{2^{-1}(1+r)}{r_n}\right)^n \frac{v(2^{-1}(1+r))}{v(r_n)}.$$
Since
$$\lim_{n\to\infty}\left(\frac{1}{2}\left(\frac{1}{r_n}+\frac{r}{r_n}\right)\right)^n = \infty \quad \text{and} \quad \lim_{n\to\infty} v(r_n) = v(r)$$
we arrive at a contradiction.

CASE $a = \infty$. Here we have
$$1 \geq \left(\frac{1+r}{r_n}\right)^n \frac{v(r+1)}{v(r_n)} \geq \left(\frac{1}{r}+1\right)^n \frac{v(r+1)}{v(r_n)}.$$
Again we get a contradiction for large n. ∎

According to our assumptions, any r in $[0, a[$ is a global maximum point for some γ_n. We have $\gamma_n'(r) = 0$ if and only if $-rv'(r)/v(r) = n$. Hence if $\gamma_n'(r) = 0$ then $\gamma_m'(r) \neq 0$ for any $m \neq n$. This means that r is a global maximum point of γ_n and all local maximum points of γ_n are also global.

2. The differentiation and integration operators $Hv(\Omega)\to Hw(\Omega)$ **for general** w**.** Let v and w be two weights. Assume that $Hw(\Omega)$ is isomorphic to l_∞. For each n fix a maximum point r_n of $r^n w(r)$. According to [16] there are numbers $0 < m_1 < m_2 < \cdots$, $t_{n,k} \in \mathbb{R}$ and operators
$$(T_n h)(z) = \sum_{m_{n-1}\leq k < m_{n+1}} t_{n,k}\alpha_k z^k \quad \text{for } h(z) = \sum_k \alpha_k z^k$$
such that

(1)
$$c_1 \sup_n \sup_{r_{m_{n-1}}\leq r\leq r_{m_{n+1}}} M_\infty(T_n h, r)w(r) \leq \|h\|_w \leq c_2 \sup_n \sup_{r_{m_{n-1}}\leq r\leq r_{m_{n+1}}} M_\infty(T_n h, r)w(r)$$

for all $h \in Hw(\Omega)$ and some $c_1, c_2 > 0$. Moreover there is a universal constant $\gamma > 0$ such that

(2)
$$M_\infty(T_n h, r) \leq \gamma M_\infty(h, r) \quad \text{for all } n, h \text{ and } r.$$

Finally, either

(3)
$$\sup_n \max\left(\left(\frac{r_{m_n}}{r_{m_{n-1}}}\right)^{m_n}\frac{w(r_{m_n})}{w(r_{m_{n-1}})}, \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_{n-1}}\frac{w(r_{m_{n-1}})}{w(r_{m_n})}\right) < \infty$$

and

(3′)
$$0 < \inf_n \frac{m_{n+1}-m_n}{m_n - m_{n-1}} \leq \sup_n \frac{m_{n+1}-m_n}{m_n - m_{n-1}} < \infty,$$

or

(4)
$$\sup(m_{n+1} - m_{n-1}) < \infty.$$

In the latter case we can split T_n further, i.e. we can assume

$$d_1 \sup_{n\in\mathbb{Z}_+} |\alpha_n| r_n^n w(r_n) \le \|h\|_w \le d_2 \sup_{n\in\mathbb{Z}_+} |\alpha_n| r_n^n w(r_n) \tag{5}$$

for some $d_1, d_2 > 0$ and all $h = \sum_k \alpha_k z^k \in Hw(\Omega)$. ((4) is not possible for $\Omega = \mathbb{D}$, see [15].)

If (3) holds then we have

$$\sup_n \left(\frac{r_{m_n}}{r_{m_{n-1}}}\right)^{m_n - m_{n-1}} < \infty. \tag{6}$$

Now [16, Lemma 3.1] implies, for any $r \in [r_{m_{n-1}}, r_{m_{n+1}}]$,

$$\begin{aligned} M_\infty(T_n h, r) w(r) &\le 2\left(\frac{r}{r_{m_{n+1}}}\right)^{m_{n-1}} \frac{w(r)}{w(r_{m_{n+1}})} M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) \\ &\le 2\left(\frac{r_{m_{n-1}}}{r_{m_{n+1}}}\right)^{m_{n-1}} \frac{w(r_{m_{n-1}})}{w(r_{m_{n+1}})} M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) \\ &= 2\left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_{n-1}} \frac{w(r_{m_{n-1}})}{w(r_{m_n})} \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{w(r_{m_n})}{w(r_{m_{n+1}})} \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_n - m_{n-1}} \\ &\quad \times M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) \\ &\le d M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) \end{aligned}$$

for some universal constant $d > 0$.

For the last inequality we used (3), (3′) and (6). (According to (3′) there is a universal constant c with $m_n - m_{n-1} \le c(m_{n+1} - m_n)$ for all n. Hence

$$\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_n - m_{n-1}} \le \left(\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1} - m_n}\right)^c,$$

and this is uniformly bounded by (6).)

Therefore, in this case, (1) implies

$$\begin{aligned} \widetilde{c}_1 \sup_n M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) &\le \|h\|_w \\ &\le \widetilde{c}_2 \sup_n M_\infty(T_n h, r_{m_{n+1}}) w(r_{m_{n+1}}) \end{aligned} \tag{7}$$

for some $\widetilde{c}_1, \widetilde{c}_2 > 0$.

It is known that $Hw(\Omega)$ is isomorphic to l_∞ if and only if

$\forall b_1 > 1\ \exists b_2 > 1\ \exists c > 0\ \forall m, n \ge c,$

$$|m - n| \ge c \quad \text{and} \quad \left(\frac{r_m}{r_n}\right)^m \frac{w(r_m)}{w(r_n)} \le b_1$$

$$\Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{w(r_n)}{w(r_m)} \le b_2.$$

Examples include $(1-r)^\alpha$, $\alpha > 0$, $\exp(-(1-r)^{-1})$ on $\mathbb{D}$, $\exp(-\alpha r)$, $\exp(-\log^2 r)$ on $\mathbb{C}$.

If $Hw(\Omega)$ is not isomorphic to l_∞ then it is isomorphic to the space $H_\infty = \{h : \mathbb{D} \to \mathbb{C} : h \text{ holomorphic and bounded}\}$ (see [16]). Here we still obtain estimates similar to (1) but (2)–(4) will fail to hold.

Now we investigate the differentiation operator $D : Hv(\Omega) \to Hw(\Omega)$. Let $h(z) = \sum_k \alpha_k z^k$. For $n > 0$ define the Cesàro mean σ_n by

$$(\sigma_n h)(z) = \sum_{k \le n} \frac{[n]-k}{[n]}\, \alpha_k z^k,$$

where $[n]$ is the largest integer $\le n$. Moreover, for $j \in \mathbb{Z}$, define the shift U_j by

$$(U_j h)(re^{i\varphi}) = \sum_k \alpha_k r^k e^{i(k+j)\varphi}.$$

We formally extend the definition of T_n to $T_n U_j h$ by putting

$$(T_n U_j h)(re^{i\varphi}) = \sum_{m_{n-1} \le k+j < m_n} t_{n,k} \alpha_k r^k e^{i(k+j)\varphi}.$$

Define $g(\varrho e^{i\varphi}) = (U_j h)(\varrho r e^{i\varphi})$. Then (2) applied to g with $\varrho = 1$ implies

$$M_\infty(T_n U_j h, r) \le \gamma M_\infty(U_j h, r) \le \gamma M_\infty(h, r) \quad \text{for all } r \text{ and } n.$$

2.1. Theorem.

(a) *Let $Hw(\Omega)$ be isomorphic to l_∞. If*

$$\limsup_{r \to a} \left(-\frac{w'(r)}{v(r)} \right) < \infty$$

then $D : Hv(\Omega) \to Hw(\Omega)$ is bounded.

(b) *Let s_n be a global maximum point of $r^n v(r)$. If $D : Hv(\Omega) \to Hw(\Omega)$ is bounded then*

$$\limsup_{n \in \mathbb{Z}_+,\, n \to \infty} \left(-\frac{v'(s_n)}{v^2(s_n)}\, w(s_n) \right) < \infty.$$

If, *in addition*, $\limsup_{n \in \mathbb{Z}_+,\, n \to \infty} s_{n+1}/s_n < \infty$, *then also*

$$\limsup_{r \to a} \left(-\frac{v'(r)}{v^2(r)}\, w(r) \right) < \infty.$$

(*In* (b), *$Hw(\Omega)$ need not be isomorphic to l_∞.*)

Proof. (a) Fix n. Assume that (7) holds. Then it suffices to consider $M_\infty(T_n Dh, r_{m_{n+1}}) w(r_{m_{n+1}})$. We have

$$(T_n Dh)(z) = \frac{m_{n+1}}{r_{m_{n+1}}} (U_{-1}(\mathrm{id} - \sigma_{m_{n+1}}) U_1 T_n U_{-1} h)(z)$$

if $|z| = r_{m_{n+1}}$. The operators U_k, $k = \pm 1$, $T_n U_{-1}$ and $\sigma_{m_{n+1}}$ are uniformly bounded with respect to $M_\infty(\cdot, r)$ for all r and the operator norms do not depend on r. Hence there is a universal constant c with

$$M_\infty(T_n Dh, r_{m_{n+1}})w(r_{m_{n+1}}) \le c\,\frac{m_{n+1}}{r_{m_{n+1}}}\,M_\infty(h, r_{m_{n+1}})w(r_{m_{n+1}}).$$

On the other hand we have $(r^{m_{n+1}}w(r))'|_{r=r_{m_{n+1}}} = 0$ since $r_{m_{n+1}}$ is a global maximum point of $r^{m_{n+1}}w(r)$. This implies $m_{n+1}w(r_{m_{n+1}})/r_{m_{n+1}} = -w'(r_{m_{n+1}})$. Fix some $r_0 > 0$. A change of v and w on $[0, r_0]$ does not affect $Hv(\Omega)$ and $Hw(\Omega)$. Therefore we can assume that there is $d > 0$ with $-w'(r)/v(r) \le d$ for all r. Then we obtain

$$M_\infty(T_n Dh, r_{m_{n+1}})w(r_{m_{n+1}}) \le cdM_\infty(h, r_{m_{n+1}})v(r_{m_{n+1}}) \le cd\|h\|_v.$$

By (7), D is bounded. The proof for the case (5) is the same.

(b) Fix $r > 0$. According to our general assumption r is a global maximum point for some function $r^n v(r)$. Hence we have $s_{[n]} \le r \le s_{[n]+1}$ with $n = -rv'(r)/v(r)$. Assume that r is so large that $1 \le [n]$. Consider $h(z) = z^{[n]}$. We have

$$\|h\|_v = s_{[n]}^{[n]} v(s_{[n]}) \le \left(\frac{r}{s_{[n]}}\right)^{n-[n]} r^{[n]} v(r)$$

and

$$[n] r^{[n]-1} w(r) \le \|Dh\|_w \le \|D\| \cdot \|h\|_v \le \|D\| \left(\frac{r}{s_{[n]}}\right)^{n-[n]} r^{[n]} v(r).$$

Hence

$$\frac{n}{r}\,\frac{w(r)}{v(r)} \le \frac{n}{[n]}\,\|D\| \left(\frac{r}{s_{[n]}}\right)^{n-[n]}$$

and therefore

$$-\frac{v'(r)}{v^2(r)}\,w(r) \le \frac{n}{[n]}\,\|D\| \left(\frac{r}{s_{[n]}}\right)^{n-[n]} \le \frac{n}{[n]}\,\|D\| \left(\frac{s_{[n]+1}}{s_{[n]}}\right)^{n-[n]}.$$

For $r = s_{[n]}$ we obtain the first assertion of (b). If $\sup_n s_{[n]+1}/s_{[n]} < \infty$ then the second assertion of (b) follows. ∎

Recall that $\limsup_{n\in\mathbb{Z}_+,\, n\to\infty} s_{n+1}/s_n < \infty$ always holds if $\Omega = \mathbb{D}$.

Let I be the integration operator, i.e. for a holomorphic function h we put

$$(Ih)(z) = \int_0^z h(u)\,du$$

To decide whether the differentiation operator is surjective we prove

2.2. PROPOSITION.

(a) *Let $Hw(\Omega)$ be isomorphic to l_∞. Moreover, assume that*

$$\limsup_{r\to a}\left(-\frac{w^2(r)}{w'(r)v(r)}\right)<\infty.$$

Then $I : Hv(\Omega)\to Hw(\Omega)$ is bounded.

(b) *Let s_n be a global maximum point of $r^n v(r)$. If $I : Hv(\Omega)\to Hw(\Omega)$ is bounded then*

$$\limsup_{n\in\mathbb{Z}_+,\,n\to\infty}\left(-\frac{w(s_n)}{v'(s_n)}\right)<\infty.$$

If, moreover, $\limsup_{n\in\mathbb{Z}_+,\,n\to\infty} s_{n+1}/s_n<\infty$ then also

$$\limsup_{r\to a}\left(-\frac{w(r)}{v'(r)}\right)<\infty.$$

(*In* (b), *$Hw(\Omega)$ need not be isomorphic to l_∞.*)

Proof. (a) We use (7) again. (The proof for the case (5) is the same.) Fix n and consider $h\in Hv(\Omega)$. We have

$$(T_nIh)(re^{i\varphi})=\int_0^r (T_nU_1h)(se^{i\varphi})\,ds.$$

Using [16, Lemma 3.1(a)], we see that, for any $s\in[r_{m_{n-1}},r_{m_{n+1}}]$,

$$M_\infty(T_nU_1h,s)=M_\infty(U_{-1}T_nU_1h,s)\le\left(\frac{s}{r_{m_{n-1}}}\right)^{m_{n+1}}M_\infty(T_nU_1h,r_{m_{n-1}}).$$

In particular,

$$\begin{aligned}
&M_\infty(T_nU_1h,r_{m_{n+1}})w(r_{m_{n+1}})\\
&\qquad\le\left(\frac{r_{m_{n+1}}}{r_{m_{n-1}}}\right)^{m_{n+1}}\frac{w(r_{m_{n+1}})}{w(r_{m_{n-1}})}\,M_\infty(T_nU_1h,r_{m_{n-1}})w(r_{m_{n-1}})\\
&\qquad=\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}}\frac{w(r_{m_{n+1}})}{w(r_{m_n})}\left(\frac{r_{m_n}}{r_{m_{n-1}}}\right)^{m_n}\frac{w(r_{m_n})}{w(r_{m_{n-1}})}\\
&\qquad\quad\cdot\left(\frac{r_{m_n}}{r_{m_{n-1}}}\right)^{m_{n+1}-m_n}M_\infty(T_nU_1h,r_{m_{n-1}})w(r_{m_{n-1}})\\
&\qquad\le dM_\infty(T_nU_1h,r_{m_{n-1}})w(r_{m_{n-1}})
\end{aligned}$$

for some universal constant d. (As before, the last inequality follows from (3), (3′) and (6).) Hence in view of (7) it suffices to consider the right-hand side of the preceding inequality. We have, using [16, Lemma 3.1(b)],

$$M_\infty(T_nU_1h,s)\le2\left(\frac{s}{r_{m_{n-1}}}\right)^{m_{n-1}-1}M_\infty(T_nU_1h,r_{m_{n-1}})$$

if $s \le r_{m_{n-1}}$. (Recall that $M_\infty(T_nU_1h, r) = M_\infty(U_{-1}T_nU_1h, r)$ for any r, and $U_{-1}T_nU_1h$ has the form $(U_{-1}T_nU_1h)(re^{i\varphi}) = \sum_{k\ge m_{n-1}-1} \alpha_k r^k e^{ik\varphi}$ for some α_k.) This implies

$$\begin{aligned} M_\infty(T_nIh, r_{m_{n-1}})w(r_{m_{n-1}}) &\le \int_0^{r_{m_{n-1}}} M_\infty(T_nU_1h, s)\,ds\,w(r_{m_{n-1}}) \\ &\le 2M_\infty(T_nU_1h, r_{m_{n-1}}) \int_0^{r_{m_{n-1}}} \left(\frac{s}{r_{m_{n-1}}}\right)^{m_{n-1}-1} ds\,w(r_{m_{n-1}}) \\ &\le cM_\infty(h, r_{m_{n-1}})\,\frac{r_{m_{n-1}}}{m_{n-1}}\,w(r_{m_{n-1}}) \end{aligned}$$

where c is a universal constant. Since $r_{m_{n-1}}/m_{n-1} = -w(r_{m_{n-1}})/w'(r_{m_{n-1}})$ we conclude that

$$M_\infty(T_nIh, r_{m_{n-1}})w(r_{m_{n-1}}) \le cM_\infty(h, r_{m_{n-1}})\left(-\frac{w^2(r_{m_{n-1}})}{w'(r_{m_{n-1}})}\right).$$

Our assumptions yield a universal constant c_1 and $r_0 > 0$ with $-w^2(r)/w'(r) \le c_1v(r)$ for all $r \ge r_0$. We may assume again $r_0 = 0$ (and perhaps change v and w on $[0, r_0]$). Then $\|Ih\|_w \le d_1\|h\|_v$ for some universal constant d_1.

(b) Fix $r > 0$. Then there is $n > 0$ such that r is a global maximum point of $r^nv(r)$. We have $s_{[n]} \le r \le s_{[n]+1}$. With $h(z) = z^{[n]}$ we obtain $\|h\|_v = s_{[n]}^{[n]}v(s_{[n]})$ and

$$\frac{1}{[n]+1}\,r^{[n]+1}w(r) \le \|Ih\|_w \le \|I\|s_{[n]}^{[n]}v(s_{[n]}) \le \|I\|\left(\frac{r}{s_{[n]}}\right)^{n-[n]} r^{[n]}v(r).$$

This yields

$$\frac{r}{n}\,\frac{w(r)}{v(r)} \le \frac{[n]+1}{n}\left(\frac{r}{s_{[n]}}\right)^{n-[n]}\|I\| \le \frac{[n]+1}{n}\left(\frac{s_{[n]+1}}{s_{[n]}}\right)^{n-[n]}\|I\|$$

and hence

$$-\frac{w(r)}{v'(r)} \le \frac{[n]+1}{n}\left(\frac{r}{s_{[n]}}\right)^{n-[n]}\|I\| \le \frac{[n]+1}{n}\left(\frac{s_{[n]+1}}{s_{[n]}}\right)^{n-[n]}\|I\|,$$

which implies (b). ∎

2.3. Corollary. *Assume that $Hv(\Omega)$ and $Hw(\Omega)$ are isomorphic to l_∞. If*

$$\limsup_{r\to a}\left(-\frac{w'(r)}{v(r)}\right) < \infty \quad \textit{and} \quad \limsup_{r\to a}\left(-\frac{v^2(r)}{v'(r)w(r)}\right) < \infty$$

then $D : Hv(\Omega) \to Hw(\Omega)$ is bounded and surjective.

Proof. The boundedness follows from Theorem 2.1. According to Proposition 2.2. the integration operator $I : Hw(\Omega) \to Hv(\Omega)$ is bounded, which yields the surjectivity of D. ∎

We deduce that, in view of the open mapping theorem, under the assumptions of Corollary 2.3 there are universal constants c and d such that $c\|h\|_v \le \|h'\|_w \le d\|h\|_v$ whenever $h \in Hv(\Omega)$ and $h(0) = 0$.

3. The differentiation operator on holomorphic functions over the unit disk. Here we consider $\Omega = \mathbb{D}$. First we show that D is never a bounded endomorphism $Hv(\mathbb{D}) \to Hv(\mathbb{D})$.

3.1. Proposition. *For any weight v there exists a function $h \in Hv(\mathbb{D})$ such that $h' \notin Hv(\mathbb{D})$.*

Proof. Otherwise we would have $D(Hv(\mathbb{D})) \subset Hv(\mathbb{D})$ and, in view of the closed graph theorem, $D : Hv(\mathbb{D}) \to Hv(\mathbb{D})$ would be bounded. If r_n is a global maximum point of $r^n v(r)$ we would obtain $n r_n^{n-1} v(r_n) \le \|D\| r_n^n v(r_n)$ for any $n \in \mathbb{Z}_+$. Hence $n/\|D\| \le r_n \le 1$ for all n, a contradiction. ∎

If we consider $w(r) = (1-r)v(r)$ we obtain positive results. We extend Theorem 3.1 of [13].

3.2. Theorem. *Let $v : [0,1[\ \to\ ]0,\infty[$ be a weight and put $w(r) = (1-r)v(r)$. Then the following are equivalent*:

(i) *If $h \in Hv(\mathbb{D})$ then $h' \in Hw(\mathbb{D})$.*
(ii) *$D : Hv(\mathbb{D}) \to Hw(\mathbb{D})$ is bounded.*
(iii) $\limsup_{r\to 1} \left(-\frac{(1-r)v'(r)}{v(r)} \right) < \infty$.
(iv) *$v(r)/(1-r)^\alpha$ is increasing on $[r_0, 1[$ for some $\alpha > 0$ and $r_0 > 0$.*
(v) $\sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty$.

Proof. (i)⇒(ii) follows from the closed graph theorem; (ii)⇒(i) is obvious; (ii)⇒(iii) is a consequence of Theorem 2.1(b).

(iii)⇔(iv): Consider $f(r) = \log(v(r)(1-r)^{-\alpha})$. Then

$$f'(r) = \left(\alpha + (1-r)\frac{v'(r)}{v(r)} \right) \frac{1}{1-r},$$

which proves the claim.

(iv)⇒(v) follows from [10, Lemma 1(a)], and (v)⇒(i) from [13, Theorem 3.1]. ∎

Property (iv) of the preceding theorem is known as *property* (U) (see [10]).

To round out the discussion we mention the following result which was essentially proved in [13].

3.3. THEOREM. *Let $v : [0,1[\to]0,\infty[$ be a weight and put $w(r) = (1-r)v(r)$. Then the following are equivalent*:

(i) $h \in Hv(\mathbb{D})$ *if and only if* $h' \in Hw(\mathbb{D})$.

(ii) $0 < \liminf_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) < \limsup_{r\to 1}\left(-\frac{(1-r)v'(r)}{v(r)}\right) < \infty$.

(iii) $v(r)/(1-r)^\alpha$ *is increasing and* $v(r)/(1-r)^\beta$ *is decreasing on* $[r_0,1[$ *for some* $\alpha > 0$, $\beta > 0$ *and* $r_0 > 0$.

(iv) $\sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty$ *and* $\limsup_n \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1$ *for some* $k \in \mathbb{Z}_+$.

Proof. (ii)⇔(iii): Put

$$f(r) = \log\left(\frac{v(r)}{(1-r)^\alpha}\right) \quad \text{and} \quad g(r) = \log\left(\frac{v(r)}{(1-r)^\beta}\right).$$

Then

$$f'(r) = \left(\alpha + (1-r)\,\frac{v'(r)}{v(r)}\right)\frac{1}{1-r} \quad \text{and} \quad g'(r) = \left(\beta + (1-r)\frac{v'(r)}{v(r)}\right)\frac{1}{1-r}.$$

From this we derive the claim.

(iii)⇔(iv) is [10, Lemma 1].

(iv)⇒(i) is [13, Theorem 3.1].

(i)⇒(iv): According to Theorem 3.2 we have

$$\sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty.$$

Then [13, Theorem 3.1] yields (iv). ∎

Weights v with property (iii) of the preceding theorem are called *normal* (see [19]). Note that here $\|h\|_v$ is equivalent to $|h(0)| + \sup_{0\le r<1} M_\infty(h',r) \times (1-r)v(r)$.

EXAMPLES. $v(r) = (1-r)^\alpha$ for some $\alpha > 0$ satisfies the assumptions of Theorem 3.3; $v(r) = (1-\log(1-r))^{-1}$ satisfies the assumptions of Theorem 3.2 but not of Theorem 3.3; $v(r) = \exp(-(1-r)^{-1})$ does not even satisfy the assumptions of Theorem 3.2.

4. The differentiation operator on entire functions. In contrast to $\mathbb{D}$, for $\Omega = \mathbb{C}$, we may have $DHv(\Omega) \subset Hv(\Omega)$. We characterize these weights. To this end we recall some facts for general weights v. Proposition 5.2 and Lemma 5.3 of [16] imply that there are constants $c_1, c_2 > 0$, integers $0 \le k_1 < k_2 < \cdots$, radii $0 < t_1 < t_2 < \cdots$ and numbers $s_{n,j} > 0$

such that the operators T_n with

$$(T_n h)(z) = \sum_{k_{n-2}<j\le k_{n+1}} s_{n,j}\alpha_j z^j \quad \text{for } h(z) = \sum_j \alpha_j z^j$$

satisfy

$$c_1 \sup_n M_\infty(T_n h, t_n)v(t_n) \le \|h\|_v \le c_2 \sup_n M_\infty(T_n h, t_n)v(t_n) \tag{8}$$

for all $h \in Hv(\Omega)$. Moreover

$$\|h\|_v \le c_2 M_\infty(h, t_n)v(t_n) \quad \text{whenever} \quad h \in T_n Hv(\Omega) \tag{9}$$

and $n = 1, 2, \ldots$ (see also [16, Corollary 4.4]). Finally, the numbers $s_{n,j}$ are such that the shifts $U_{\pm 1}$ satisfy

(10)

$$M_\infty((T_n - U_{-k}T_nU_k)h, r) \le \gamma M_\infty(h, r) \quad \text{for all } n,\ h,\ r \text{ and } k = \pm 1$$

where $\gamma > 0$ is a universal constant. (See the operators of [16, Lemma 5.3].)

4.1. Theorem. *Let $v : [0,\infty[\to]0,\infty[$ be a weight. Then the following are equivalent*:

(i) $D : Hv(\mathbb{C}) \to Hv(\mathbb{C})$ *is bounded*,
(ii) $\limsup_{r\to\infty}(-v'(r)/v(r)) < \infty$,
(iii) *There are $\beta, r_0 > 0$ such that $v(r)e^{\beta r}$ is increasing on $[r_0,\infty[$.*

Proof. Let r_n be a global maximum point of $r^n v(r)$.

(i)⇒(ii): Theorem 2.1(b) implies $n/r_n = -v'(r_n)/v(r_n) \le \beta$ for some β and large enough $n \in \mathbb{Z}_+$. Now fix some r, say $r_n \le r \le r_{n+1}$ for some $n \in \mathbb{Z}_+$. According to our assumptions on v there is $m \in [n, n+1]$ such that r is a maximum point for the function $s^m v(s)$. We have

$$-n\,\frac{v(r_n)}{v'(r_n)} = r_n \le r = -m\,\frac{v(r)}{v'(r)}.$$

This implies

$$-\frac{v'(r)}{v(r)} \le \frac{m}{n}\,\beta \le \frac{n+1}{n}\,\beta$$

and hence (ii).

(ii)⇔(iii): Put $f(r) = \log(v(r)e^{\beta r})$. Then $f'(r) = v'(r)/v(r) + \beta$. Hence $v(r)e^{\beta r}$ is increasing if and only if (ii) holds. This proves the claim.

(ii)⇒(i): We proceed as in the proof of 2.1 to show that $D : Hv(\mathbb{C}) \to Hv(\mathbb{C})$ is bounded. Fix n and consider the operator T_n of (8). We have

$$(T_nDh)(z) = \frac{k_{n+1}}{t_n}\,(U_{-1}(\mathrm{id} - \sigma_{k_{n+1}})U_1T_nU_{-1}h)(z) \tag{11}$$

if $|z| = t_n$. We claim that k_{n+1}/t_n is uniformly bounded. First, (ii) implies

$$\frac{n}{r_n} = -\frac{v'(r_n)}{v(r_n)} \le \beta$$

for all $n > 0$ and some $\beta > 0$. Hence $n/\beta \le r_n$. We may take β so large that it satisfies the assertion of (iii) as well. Now fix n and assume $k_{n+1}/\beta \ge t_n$. Using (9) with $h(z) = z^{k_{n+1}}$ we see that

$$c_2 \ge \left(\frac{r_{k_{n+1}}}{t_n}\right)^{k_{n+1}} \frac{v(r_{k_{n+1}})}{v(t_n)} \ge \left(\frac{k_{n+1}}{\beta t_n}\right)^{k_{n+1}} \frac{v(k_{n+1}/\beta)}{v(t_n)}$$
$$\ge \left(\frac{k_{n+1}}{\beta t_n}\right)^{k_{n+1}} \exp\left(\beta\left(t_n - \frac{k_{n+1}}{\beta}\right)\right).$$

Here the second inequality follows by comparing the function $r^{k_{n+1}}v(r)$ at $r = k_{n+1}/\beta$ and at the maximum point $r = r_{k_{n+1}}$. Using (iii), since $k_{n+1}/\beta \ge t_n$ we obtain the last inequality.

Hence
$$\frac{k_{n+1}}{t_n} \le \beta c_2^{1/k_{n+1}} \exp(1).$$

If $k_{n+1}/\beta \le t_n$ then $k_{n+1}/t_n \le \beta$. So (11) together with (10) imply

$$\begin{aligned} M_\infty(T_n Dh, t_n)v(t_n) &\le \widetilde{c}\,\frac{k_{n+1}}{t_n}\, M_\infty(U_1 T_n U_{-1}h, t_n)v(t_n) \\ &\le \widetilde{c}\,\frac{k_{n+1}}{t_n}\,(M_\infty(T_n h, t_n) \\ &\quad + M_\infty(T_n - U_1 T_n U_{-1}h, t_n)v(t_n) \\ &\le c\beta(M_\infty(T_n h, t_n) + M_\infty(h, t_n))v(t_n) \end{aligned}$$

where $\widetilde{c}$ and c are universal constants. Here we used again the fact that U_k and $\sigma_{k_{n+1}}$ are uniformly bounded with respect to $M_\infty(\cdot, t_n)$ and the operator norms do not depend on n. Finally, (8) shows that D is bounded. ∎

4.2. Theorem. *Let $v : [0,\infty[\, \to\,]0,\infty[$ be a weight. Then the following are equivalent*:

(i) *$h \in Hv(\mathbb{C})$ if and only if $h' \in Hv(\mathbb{C})$.*
(ii) *$D : Hv(\mathbb{C}) \to Hv(\mathbb{C})$ is bounded and surjective.*
(iii) *There are $c_1, c_2 > 0$ such that, for all $h \in Hv(\mathbb{C})$,*
$$c_1(\|h'\|_v + |h(0)|) \le \|h\|_v \le c_2(\|h'\|_v + |h(0)|).$$
(iv) $0 < \liminf_{r\to\infty}(-v'(r)/v(r)) \le \limsup_{r\to\infty}(-v'(r)/v(r)) < \infty$.
(v) *There are $\alpha, \beta, r_0 > 0$ such that $v(r)e^{\alpha r}$ is decreasing and $v(r)e^{\beta r}$ is increasing on $[r_0, \infty[$.*

Proof. The first three items are equivalent by the closed graph theorem and the open mapping theorem.

Let r_n be a global maximum point of $r^n v(r)$.

(iv)⇔(v): Put $f(r) = \log(v(r)e^{\alpha r})$ and $g(r) = \log(v(r)e^{\beta r})$. Then we have $f'(r) = v'(r)/v(r) + \alpha$ and $g'(r) = v'(r)/v(r) + \beta$. Hence $v(r)e^{\alpha r}$ is decreasing if and only if $-v'(r)/v(r) \geq \alpha$, and $v(r)e^{\beta r}$ is increasing if and only if $-v'(r)/v(r) \leq \beta$. This proves the claim.

(ii)⇒(iv): That $\limsup_{r\to\infty}(-v'(r)/v(r)) < \infty$ follows from Theorem 4.1. Proposition 2.2(b) yields $0<\liminf_{n\in\mathbb{Z}_+,n\to\infty}(-v'(r_n)/v(r_n))$. Fix $r>0$ and $n \in \mathbb{Z}_+$ such that $r_{n-1} \leq r \leq r_n$. Then r is a global maximum point of $t^m v(t)$ for some $m \in [n-1, n]$. We have

$$-n\,\frac{v(r_n)}{v'(r_n)} = r_n \geq r = -m\,\frac{v(r)}{v'(r)},$$

which implies

$$-\frac{v'(r)}{v(r)} \geq -\frac{m}{n}\,\frac{v'(r_n)}{v(r_n)} \geq -\frac{n-1}{n}\,\frac{v'(r_n)}{v(r_n)}.$$

This proves (iv).

(iv)⇒(ii): According to Theorem 4.1, D is bounded. Finally, we show that the integration operatotion $I : Hv(\mathbb{C}) \to Hv(\mathbb{C})$ is bounded. Fix n and consider the operator T_n of (8). By Lemma 3.1(b) of [16], we have

$$M_\infty(T_nU_1h, s) \leq 2\bigg(\frac{s}{t_n}\bigg)^{k_{n-2}-1} M_\infty(T_nU_1h, t_n) \quad \text{ if } t_n \geq s.$$

Hence

$$\begin{aligned}
(12)\qquad M_\infty(T_nIh, t_n)v(t_n) &\leq \int_0^{t_n} M_\infty(T_nU_1h, s)\,ds\,v(t_n)\\
&\leq 2M_\infty(T_nU_1h, t_n)\int_0^{t_n}\bigg(\frac{s}{t_n}\bigg)^{k_{n-2}-1} ds\,v(t_n)\\
&\leq 2M_\infty(U_{-1}T_nU_1h, t_n)\,\frac{t_n}{k_{n-1}}\,v(t_n)\\
&\leq c(M_\infty(T_nh, t_n)\\
&\qquad + M_\infty(T_n - U_{-1}T_nU_1h, t_n))\,\frac{t_n}{k_{n-2}}\,v(t_n)\\
&\leq c(M_\infty(T_nh, t_n) + M_\infty(h, t_n))\,\frac{t_n}{k_{n-2}}\,v(t_n)
\end{aligned}$$

where c is a universal constant. (We have used (10).)

We claim that t_n/k_{n-2} is uniformly bounded. Fix n and let α be the constant of (v). Using (9) with $h(z) = z^k$ where $k = k_{n-2} + 1$ we see that, in view of (iv),

$$\begin{aligned}
c_2 &\geq \bigg(\frac{r_k}{t_n}\bigg)^k \frac{v(r_k)}{v(t_n)} \geq \bigg(\frac{t_n}{2t_n}\bigg)^k \frac{v(t_n/2)}{v(t_n)}\\
&\geq \bigg(\frac{1}{2}\bigg)^k \exp(\alpha(t_n - t_n/2)) = \exp(-k\log 2 + \alpha t_n/2)
\end{aligned}$$

For the last inequality we have used $v(t_n/2)e^{\alpha t_n/2} \geq v(t_n)e^{\alpha t_n}$, which holds according to (v). So, for large n we have $t_n \leq 3\alpha^{-1}k \log 2$ and hence t_n/k_{n-2} is uniformly bounded. Now (12) together with (8) shows that I is bounded. ■

Of course the standard example for Theorem 4.1 is $v(r) = e^{-r}$. Moreover, for $v(r) = \exp(-\log^2 r)$ the differentiation operator $D : Hv(\mathbb{C}) \to Hv(\mathbb{C})$ is bounded (in view of Theorem 2.1) but not surjective. For $v(r) = \exp(-e^r)$ the differentiation operator is unbounded since

$$\lim_{n\to\infty} -v'(s_n)/v(s_n) = \lim_{n\to\infty} e^{s_n} = \infty$$

for the global maximum points s_n of $r^n v(r)$ (Lemma 1.2).

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