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Modifications of the double arrow space and related Banach spaces $C(K)$

by

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Abstract. We consider the class of compact spaces K_A which are modifications of the well known double arrow space. The space K_A is obtained from a closed subset K of the unit interval $[0,1]$ by "splitting" points from a subset $A \subset K$. The class of all such spaces coincides with the class of separable linearly ordered compact spaces. We prove some results on the topological classification of K_A spaces and on the isomorphic classification of the Banach spaces $C(K_A)$.

1. Introduction. Let A be an arbitrary subset of a closed subset K of the unit interval $I = [0,1]$. Recall the construction of the compact space used in [Ka1] (actually, we use a slightly different description). Put

$$K_A = (K \times \{0\}) \cup (A \times \{1\})$$

and equip this set with the order topology given by the lexicographical order (i.e., $(s,i) \prec (t,j)$ if either $s < t$, or $s = t$ and $i < j$).

For $K = I$ and $A = (0,1)$ the space $\mathbb{K} = I_{(0,1)}$ is the well known double arrow space. Some authors use this name for the space I_I; others call the space I_I the split interval. The class of I_A spaces (again, described in a slightly different way) was investigated by van Douwen in [vD]. Since the Banach spaces $C(I_A)$ of real-valued continuous functions on I_A (for $A \subset (0,1)$) have recently provided many interesting examples in Banach space theory (see [Ka1], [Ka2], [KMS], and [MS]), Godefroy has posed the problems of the topological classification of I_A spaces and of the isomorphic classification of the Banach spaces $C(I_A)$ (see http://www.fields.utoronto.ca/audio/02-03/banach/godefroy/). In this paper we give some partial results in this

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direction. We concentrate on the spaces I_A for $A \subset (0,1)$. However, in many cases we investigate the larger class of all spaces of the form K_A. This more general approach is worth some extra effort, since it is known that the class of all K_A spaces coincides with the class of separable linearly ordered compact spaces. Namely, the following is a reformulation of the characterization due to Ostaszewski [Os]:

Result 1.1 (Ostaszewski). *The space L is a separable compact linearly ordered space if and only if L is homeomorphic to K_A for some closed set $K \subset I$ and a subset $A \subset K$.*

The results concerning the topological classification of K_A spaces are included in Section 2. In Section 3 we investigate which function spaces $C(K_A)$ can be homeomorphic when equipped with the weak topology. This allows us to give some negative results on the existence of isomorphisms between the Banach spaces $C(K_A)$. In particular, we show that there exist 2^{2^ω} pairwise non-isomorphic $C(I_A)$ spaces. In Section 4 we prove that for some classes of K_A spaces, the function spaces $C(K_A)$ are isomorphic to $C(\mathbb{K})$.

Given $a, b \in K_A$ with $a \preceq b$, we denote the open, closed and halfopen intervals (with respect to the order $\prec$) with endpoints a and b by (a,b), $\langle a,b\rangle$, $(a,b\rangle$, and $\langle a,b)$, respectively.

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2. Homeomorphisms of spaces K_A. The following observation was made by van Douwen in [vD]. We include its proof for the reader's convenience.

Proposition 2.1. *Let K and L be closed subsets of I, and let $A \subset K$ and $B \subset L$ be such that the spaces K_A and L_B are homeomorphic. Then there exist countable sets $C \subset K$ and $D \subset L$ such that $K \setminus (A \cup C)$ and $L \setminus (B \cup D)$ are homeomorphic.*

Proof. Observe that on $(K \setminus A) \times \{0\}$, the subspace topology inherited from K_A coincides with the Euclidean topology. On the other hand, the subspaces $A \times \{0\}$ and $A \times \{1\}$ are embeddable in the Sorgenfrey line S. It remains to recall the well known fact that each metrizable subset of S is countable. ∎

Since every subset X of I is homeomorphic to at most continuum many sets $Y \subset I$, the above proposition, together with some routine cardinal calculations, implies the following.

Corollary 2.2 (van Douwen). *There are 2^{2^ω} pairwise nonhomeomorphic I_A spaces.*

We also give another family of pairwise nonhomeomorphic I_A spaces constructed in a more effective way.

Example 2.3. *There exists a family $\{A_\alpha : \alpha < 2^\omega\}$ of open subsets of $(0,1)$ such that the spaces I_{A_α} and I_{A_β} are not homeomorphic for $\alpha \neq \beta$.*

The construction. For $n \ge 1$, let $g_n : \omega^n + 1 \to (1/(3n+3), 1/(3n+2)]$ be a topological embedding of the ordinal space $\omega^n + 1$ such that g_n is order preserving and $g_n(\omega^n) = 1/(3n+2)$. Let $h_n : \omega^n + 1 \to [1/(3n+1), 1/(3n))$ be a topological embedding such that h_n is order reversing and $h_n(\omega^n) = 1/(3n+1)$. Put

$$F_n = \bigcup\{[g_n(\alpha), g_n(\alpha+1)] : \alpha < \omega^n,\ \alpha \text{ even}\} \cup \bigcup\{[h_n(\alpha+1), h_n(\alpha)] : \alpha < \omega^n,\ \alpha \text{ even}\} \cup \left[\frac{1}{3n+2}, \frac{1}{3n+1}\right].$$

Next, for every subset X of the set $\mathbb{N}$ of positive integers, we define $F_X = \bigcup\{F_n : n \in X\}$ and $U_X = (0,1) \setminus F_X$. Clearly, each U_X is open in $(0,1)$. To see that, for distinct $X, Y \subset \mathbb{N}$, the spaces I_{U_X} and I_{U_Y} are topologically distinct, note the following properties of the space I_{U_X}:

(1) The nontrivial components of I_{U_X} are $J \times \{0\}$, where J is an interval which is a component of F_X.
(2) The components of the form $[1/(3n+2), 1/(3n+1)] \times \{0\}$ for $n \in X$ are distinguished by the property that both their endpoints are accumulation points of the set E of endpoints of all nontrivial components of I_{U_X}.
(3) For each $n \in X$, the endpoints of the component $[1/(3n+2), 1/(3n+1)] \times \{0\}$ have arbitrarily small neighborhoods V such that $V \cap E$ is homeomorphic to $\omega^n + 1$. ∎

The next proposition shows in particular that the property described in Proposition 2.1 is not sufficient for homeomorphy of K_A spaces.

Proposition 2.4. *Let P be the set of all irrational numbers in $(0,1)$. The spaces $\mathbb{K}$ and I_P are not homeomorphic.*

Proof. Assume towards a contradiction that $h : \mathbb{K} \to I_P$ is a homeomorphism. Let $Q = (0,1) \setminus P$ and $A = \mathbb{K} \setminus h^{-1}(Q \times 0)$. For a point $x = (t,0) \in \mathbb{K}$ and $n \ge 1$, the set $U_{x,n} = ((\max(0, t-1/n), 1), x\rangle = \{(s,i) \in \mathbb{K} : s \in (\max(0, t-1/n), t)\} \cup \{x\}$ is a basic neighborhood of x in $\mathbb{K}$. In a similar way we define basic neighborhoods $U_{x,n}$ of points x of the form $(t,1)$. For $i = 0,1$ and $n \ge 1$ we put

$$F(i,0,n)=\{x=(t,i)\in A: h(x)=(s,0) \text{ and } (\forall y\in U_{x,n})\ [h(y)\preceq h(x)]\},$$
$$F(i,1,n)=\{x=(t,i)\in A: h(x)=(s,1) \text{ and } (\forall y\in U_{x,n})\ [h(y)\succeq h(x)]\}.$$

From the continuity of h it follows that $A=\bigcup\{F(i,j,n): i,j=0,1,\ n\ge 1\}$. The subspace A, being a G_δ subset of $\mathbb{K}$, is a Baire space. Therefore there exist i_0,j_0 and n_0 such that the closure of $F_0=F(i_0,j_0,n_0)$ in A contains some nonempty open set $U\subset A$. We may assume that $U=\{x\in A: a\prec x\prec b\}$ for some $a,b\in\mathbb{K}$ such that $a=(s,0)$, $b=(t,0)$ and $0<t-s<1/n_0$.

First, consider the case when $i_0=j_0=0$. Then, for every $p,q\in F_0$ such that $a\prec p\prec q\prec b$, we have $p\in U_{q,n_0}$, hence $h(p)\prec h(q)$. Since $\langle a,b\rangle\cap F_0$ is dense in $\langle a,b\rangle$ we infer that h is increasing on $\langle a,b\rangle$. The image $h((a,b))$ is open in I_P, therefore we may find $c\in(a,b)$ such that $h(c)\in Q\times\{0\}$. If $c=(u,0)$ then the interval $(a,c\rangle$ is a neighborhood of c but its image $h((a,c\rangle)$ is contained in the interval $(h(a),h(c)\rangle$ which does not contain $h(c)$ in its interior, a contradiction. If $c=(u,1)$ we arrive at a contradiction by considering the interval $\langle c,b)$. The same argument works for other values of i_0 and j_0 (if $i_0\ne j_0$ then h is decreasing on $\langle a,b\rangle$). ■

Given a topological space X and an ordinal α, we use the standard notation $X^{(\alpha)}$ for the αth Cantor–Bendixson derivative of X.

Proposition 2.5. *Let A and B be subsets of $(0,1)$ such that*
$$((0,1)\setminus A)^{(\omega_1)}=((0,1)\setminus B)^{(\omega_1)}.$$
Then the spaces I_A and I_B are homeomorphic.

The key ingredient of the proof of the above proposition is the following.

Lemma 2.6. *Let A be a subset of $(0,1)$ and a be an isolated point of $(0,1)\setminus A$. Then, for each open interval J in I_A containing $(a,0)$, there is a homeomorphism h of I_A onto $I_{A\cup\{a\}}$ such that $h(x)=x$ for all $x\in I_A\setminus J$.*

Proof. Without loss of generality we may assume that $J=((b,0),(c,1))$, where $b\in(0,a)$, $c\in(a,1)$ and $[b,a)\cup(a,c]\subset A$. Take a strictly increasing sequence $(s_n)_{n\in\omega}$ in $[b,a)$ (resp. $(u_n)_{n\in\omega}$ in $[b,c)$) such that $s_0=b$ and $\lim_n s_n=a$ (resp. $u_0=b$ and $\lim_n u_n=c$). We also need a strictly decreasing sequence $(t_n)_{n\in\omega}$ in $(a,c]$ such that $t_0=c$ and $\lim_n t_n=a$. Now let $h: I_A\to I_{A\cup\{a\}}$ be a map such that

(1) $h(x)=x$ for all $x\in I_A\setminus J$,
(2) $h((a,0))=(c,0)$,
(3) h maps the interval $\langle(s_n,1),(s_{n+1},0)\rangle$ in I_A in a natural affine way onto the interval $\langle(u_{2n},1),(u_{2n+1},0)\rangle$ in $I_{A\cup\{a\}}$ for every $n\in\omega$,
(4) h maps the interval $\langle(t_{n+1},1),(t_n,0)\rangle$ in I_A in a natural affine way onto the interval $\langle(u_{2n+1},1),(u_{2n+2},0)\rangle$ in $I_{A\cup\{a\}}$ for every $n\in\omega$.

One can easily verify that h is the required homeomorphism. ■

LEMMA 2.7. *Let A be a subset of $(0,1)$ and let J be a clopen interval in I_A with endpoints (a,i), (b,j), $a<b$. Then, for every $\alpha \le \omega_1$ and $B = A \cup [((a,b)\setminus A)\setminus((a,b)\setminus A)^{(\alpha)}]$, there exists a homeomorphism $h: I_A \to I_B$ such that $h(x)=x$ for all $x \in I_A \setminus J$.*

Proof. Denote the set $(a,b)\setminus A$ by C. We will prove the lemma by induction on α.

The case $\alpha = 0$ is trivial. Suppose that $\alpha = 1$. Then $D = C \setminus C^{(1)}$ is the set of isolated points of C. Therefore, we can find a sequence of pairwise disjoint open intervals $J_n \subset J$, for $n < N \le \omega$, such that $\bigcup_{n<N} J_n$ covers $D \times \{0\}$ and each J_n contains exactly one point of $D \times \{0\}$. Let h_n be the homeomorphism given by Lemma 2.6 for the interval J_n. We can define the required homeomorphism h by declaring that $h|J_n = h_n|J_n$ for $n<N$, and $h(x) = x$ for $x \in I_A \setminus \bigcup_{n<N} J_n$.

Now, assume that $\alpha = \beta+1$ and the assertion holds true for $\beta \ge 1$. Set $D = C\setminus C^{(\beta+1)}$ and $E = C \setminus C^{(\beta)}$. By the inductive hypothesis there is a homeomorphism $h_1 : I_A \to I_{A\cup E}$ which is the identity on $I_A \setminus J$. Observe that

$$\begin{aligned} A\cup D &= (A\cup E)\cup (C^{(\beta)}\setminus C^{(\beta+1)})\\ &= A\cup E\cup[((a,b)\setminus(A\cup E))\setminus((a,b)\setminus(A\cup E))^{(1)}] \end{aligned}$$

and therefore, by applying the case $\alpha=1$ to $A\cup E$, we get a homeomorphism h_2 of the required type between $I_{A\cup E}$ and $I_{A\cup D}$. It remains to compose h_1 and h_2.

Finally, assume that α is a limit ordinal and the assertion holds true for all $\beta<\alpha$. Let $D = C\setminus C^{(\alpha)}$. Each point of D has a neighborhood U in C such that $U^{(\beta)} = \emptyset$ for some $\beta<\alpha$. Therefore we can find a sequence of pairwise disjoint clopen intervals $J_n \subset J$ for $n<N\le\omega$ such that, for every n, $(J_n \cap (C\times\{0\}))^{(\beta_n)} = \emptyset$ for some $\beta_n < \alpha$, and $\bigcup_{n<N} J_n$ covers $D\times\{0\}$ and is disjoint from $C^{(\alpha)}\times\{0\}$. We apply the inductive hypothesis for β_n to the intervals J_n and we define a homeomorphism h in the same way as in the case $\alpha = 1$. ■

Proposition 2.5 follows easily from Lemma 2.7 applied for the interval $J = I_A$ and $\alpha = \omega_1$.

3. Homeomorphisms of $(C(K_A), w)$ spaces. For a compact space K, we denote by $(C(K), w)$ (resp. $C_{\mathrm{p}}(K)$) the space $C(K)$ equipped with the weak topology (resp. the topology of pointwise convergence).

THEOREM 3.1. *There exist 2^{2^ω} pairwise nonhomeomorphic $(C(I_A), w)$ spaces.*

COROLLARY 3.2. *There exist 2^{2^ω} pairwise nonisomorphic $C(I_A)$ spaces.*

Theorem 3.1 follows easily from the following lemma.

Lemma 3.3. *For every $A \subset (0,1)$ there exist at most 2^ω sets $B \subset (0,1)$ such that the spaces $(C(I_A), w)$ and $(C(I_B), w)$ are homeomorphic.*

Proof. For each $A \subset (0,1)$ the space $C(I_A)$ is isometric to the subspace $E_A = \{f \in C(\mathbb{K}) : f((t,0)) = f((t,1))$ for all $t \in (0,1) \setminus A\}$ of $C(\mathbb{K})$. Observe that $E_A \neq E_B$ for $A, B \subset (0,1)$, $A \neq B$. Therefore it is enough to construct a family $\{\varphi_\alpha : \alpha < 2^\omega\}$ of functions (not necessarily continuous) such that:

(a) $\mathrm{dom}(\varphi_\alpha), \mathrm{ran}(\varphi_\alpha) \subset C(\mathbb{K})$ and φ_α is one-to-one for $\alpha < 2^\omega$,
(b) for every $A, B \subset (0,1)$ and any homeomorphism ψ of (E_A, w) onto (E_B, w), there is $\alpha < 2^\omega$ such that $\psi = \varphi_\alpha | E_A$.

Let $\{q_n : n \in \omega\}$ be a countable dense subset of $\mathbb{K}$. We denote by δ_n the Dirac measure on $\mathbb{K}$ concentrated at q_n. For a subset S of the dual space $C(\mathbb{K})^*$, we define $i_S : C(\mathbb{K}) \to \mathbb{R}^S$ by $i_S(f)(\mu) = \mu(f)$ for $f \in C(\mathbb{K})$ and $\mu \in S$. Clearly, if S contains $\{\delta_n : n \in \omega\}$, then i_S is injective.

Now, we take the family $\mathcal{F}$ of all functions φ satisfying the following condition: there is a countable set $S \subset C(\mathbb{K})^*$, a G_δ-set X in $\mathbb{R}^S$, and a homeomorphic embedding h of X into $\mathbb{R}^S$ such that

(i) $\{\delta_n : n \in \omega\} \subset S$,
(ii) $\mathrm{dom}(\varphi) = i_S^{-1}(h^{-1}(i_S(C(\mathbb{K}))))$,
(iii) $\varphi(f) = i_S^{-1} \circ h \circ i_S(f)$ for $f \in \mathrm{dom}(\varphi)$.

It is well known that the dual space $C(\mathbb{K})^*$ has cardinality 2^ω (see [KMS] or [GT]). Therefore we can repeat the argument from [Ma1, proof of Lemma 4.3] to verify that $\mathcal{F}$ has cardinality 2^ω and satisfies conditions (a) and (b). ∎

Remark 3.1. The same argument shows that Theorem 3.1 also holds true for the function spaces $C(I_A)$ equipped with the topology of pointwise convergence (here, one should consider only sets $S \subset C(\mathbb{K})^*$ consisting of Dirac measures on $\mathbb{K}$).

In the remaining part of this section we will discuss a more effective way of distinguishing $(C(K_A), w)$ spaces. We can do this for K_A which are Rosenthal compacta (see [Ma3]). For a separable metrizable space M, we denote by $B_1(M)$ the space of all real-valued functions on M of the first Baire class, endowed with the topology of pointwise convergence. Recall that a compact space K is *Rosenthal compact* if K embeds into $B_1(\omega^\omega)$, where ω^ω is the space of irrationals.

For a dense subset D of a compact space K, $C_D(K)$ is $C(K)$ equipped with the topology of pointwise convergence on D. Hence, we can identify $C_D(K)$ with the subset $\{f|D : f \in C(K)\}$ of the product $\mathbb{R}^D$.

RESULT 3.4 (Godefroy, [Go]). *A compact separable space K is Rosenthal compact if and only if, for every countable dense set $D \subset K$, the space $C_D(K)$ is analytic.*

Recall that a metrizable space A is *analytic* if it is a continuous image of the irrationals ω^ω.

It is easy to describe which K_A spaces are Rosenthal compact:

REMARK 3.2. K_A is Rosenthal compact if and only if A is analytic.

The above fact is well-known (see [Go]; the necessity follows from Theorem 3.4 and Lemma 3.6, the sufficiency can be obtained using the same theorem and the argument from the proof of Theorem 3.7).

Let us recall the definition of the index $\eta(K)$ of a separable Rosenthal compact space K introduced in [Ma2]:

$$\eta(K) = \begin{cases} \min\{\alpha : \text{there exists a countable dense } D \subset K \text{ such that} & \\ \qquad C_D(K) \text{ is a Borel set of class } \alpha \text{ in } \mathbb{R}^D\} & \text{if such } D \text{ exist,} \\ \omega_1 & \text{otherwise.} \end{cases}$$

The index η can be useful for the classification of function spaces on separable Rosenthal compacta.

RESULT 3.5 ([Ma2]). *Let K and L be separable Rosenthal compacta. If the Banach spaces $C(K)$ and $C(L)$ are isomorphic (or homeomorphic with respect to weak or pointwise topology) then $1+\eta(K) \geq \eta(L)$ and $1+\eta(L) \geq \eta(K)$ (if $\eta(K) \geq \omega$ or $\eta(L) \geq \omega$, then $\eta(K) = \eta(L)$).*

We denote by p_1 the restriction to K_A of the projection of $K \times \{0,1\}$ onto the first factor, i.e., $p_1((t,i)) = t$ for $(t,i) \in K_A$.

LEMMA 3.6. *Let A be a subset of a closed set $K \subset I$ and D be a countable dense subset of K_A. There exists a countable set $E \subset K_A$ such that $D \subset E$ and the space $C_E(K_A)$ contains a closed copy of $A \setminus p_1(E)$.*

Proof. Let $a = (a',0) = \min K_A$ and $b = (b',i) = \max K_A$. Consider the set G of all pairs (s,t) such that $s,t \in K$, $s < t$ and $(s,t) \cap K = \emptyset$ (intervals (s,t) are the components of $[a',b'] \setminus K$). Clearly, the set G is countable. Put $H = \bigcup\{\{s,t\} : (s,t) \in G\}$. Let $F = p_1(D) \cup H \cup \{a',b'\}$ and $E = F \times \{0\} \cup (F \cap A) \times \{1\}$. Consider the set

$$\begin{aligned} S = \{f \in C_E(K_A) \cap \{0,1\}^E : {} & f(a) = 0,\ f(b) = 1, \\ & (\forall x,y \in E)\ [x \preceq y \Rightarrow f(x) \leq f(y)], \\ & (\forall (s,t) \in G)\ [f((s,0)) = f((t,0))] \text{ and} \\ & (\forall t \in F \cap A)\ [f((t,0)) = f((t,1))]\}. \end{aligned}$$

Clearly S is closed in $C_E(K_A)$. One can easily verify that

$$S = \{\chi_{\{e \in E\,:\,(t,1) \preceq e\}} : t \in A \setminus F\}$$

and the mapping $t \mapsto \chi_{\{e\in E\,:\,(t,1)\preceq e\}}$ is a homeomorphism of $A\backslash F = A\backslash p_1(E)$ onto S. ∎

THEOREM 3.7. *Let K be a closed subset of I. If A is a Borel subset of K of exact multiplicative class $\alpha > 1$, then $\alpha \le \eta(K_A) \le 3+\alpha$. In particular, if $\alpha \ge \omega$ then $\eta(K_A) = \alpha$.*

Proof. First, we will show that $\eta(K_A) \le 3+\alpha$. Lemma 5.7 from [DM] says that, for every countable dense set $D \subset \mathbb{K}$, the space $C_D(\mathbb{K})$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^D$. Exactly the same reasoning demonstrates that the same is true for the space K_K.

Let q_A be the quotient map of K_K onto K_A, obtained by identifying points $(t,1)$ and $(t,0)$ for every $t \in K \setminus A$. As in the proof of the previous lemma we take the set $H = \bigcup\{\{s,t\} : s < t,\, (s,t) \cap K = \{s,t\}\}$. Fix a countable dense set $D' \subset K$ containing $H \cup \{\min K, \max K\}$ and put $D = D' \times \{0,1\}$ and $E = q_A(D)$. We will prove that $C_E(K_A)$ is a Borel subset of $\mathbb{R}^E$ of class $\le 3+\alpha$. Let

$$S_A = \{f \in C_{\mathrm{p}}(K_K) : (\forall t \in K \setminus A)\ [f((t,0)) = f((t,1))]\}.$$

The space S_A is the canonical topological copy of $C_{\mathrm{p}}(K_A)$ in $C_{\mathrm{p}}(K)$, and $T_A = \{f|D : f \in S_A\} \subset C_D(K_K)$ can be easily identified with $C_E(K_A)$. For a metrizable space X, we denote by $[X]^{<\omega}$ the space of all finite subsets of X equipped with the Vietoris topology (metrizable by the Hausdorff metric). Given $\varepsilon > 0$, define $\varphi_\varepsilon : C_{\mathrm{p}}(K_K) \to [K]^{<\omega}$ by

$$\varphi_\varepsilon(f) = \{t \in K : |f((t,0)) - f((t,1))| > \varepsilon\}$$

for $f \in C_{\mathrm{p}}(K_K)$. Observe that the continuity of f implies that φ_ε is well-defined, i.e., the set $\{t \in K : |f((t,0)) - f((t,1))| > \varepsilon\}$ is always finite. For $f \in C_{\mathrm{p}}(K_K)$ we have

$$f \in S_A \ \Leftrightarrow\ (\forall \varepsilon > 0)\ [\varphi_\varepsilon(f) \in [A]^{<\omega}].$$

Hence we can describe the subspace S_A as $\bigcap_{n\ge1} \varphi_{1/n}^{-1}([A]^{<\omega})$.

We also have $T_A = \bigcap_{n\ge1} \psi_{1/n}^{-1}([A]^{<\omega})$, where $\psi_\varepsilon : C_D(K) \to [K]^{<\omega}$ is defined by $\psi_\varepsilon(f|D) = \varphi_\varepsilon(f)$ for $f \in C_{\mathrm{p}}(K_K)$.

We will show that ψ_ε is Borel of the third class, i.e., $\psi_\varepsilon^{-1}(U)$ is an $F_{\sigma\delta\sigma}$-set for every open $U \subset [K]^{<\omega}$.

First consider the family $\mathcal{U}$ of open sets of the form $U = \{F \in [K]^{<\omega} : F \cap J \ne \emptyset\}$, where J is an open interval in I. Given J we will check that

$$\begin{aligned}\psi_\varepsilon(f) \cap J \ne \emptyset \ \Leftrightarrow\ & (\exists m \ge 1)(\forall n \ge 1)(\exists q, r \in D' \cap J)\\ & [|q-r| < 1/n \ \&\ |f((q,0)) - f((r,1))| > \varepsilon + 1/m].\end{aligned}$$

($\Rightarrow$) Suppose that $\psi_\varepsilon(f) \cap J \ne \emptyset$ and pick $s \in \psi_\varepsilon(f) \cap J$. Then $|f((s,0)) - f((s,1))| > \varepsilon + 2/m$ for some $m \ge 1$. If $s \in (H \cup \{\min K, \max K\}) \subset D'$ then $q = r = s$ satisfy the right-hand condition for all n. If not, then we can find an increasing sequence (q_k) in $D' \cap J$, and a decreasing sequence (r_k) in $D' \cap J$, both converging to s. Then $f((q_k, 0)) \to f((s, 0))$ and $f((r_k, 1)) \to f((s, 1))$. One can easily verify that the right-hand condition holds true.

($\Leftarrow$) Suppose that the right-hand condition holds true. Then, for some $m \geq 1$, we can find sequences (q_n) and (r_n) in $D' \cap J$ such that $|q_n - r_n| < 1/n$ and $|f((q_n, 0)) - f((r_n, 1))| > \varepsilon + 1/m$.

Without loss of generality we may assume that $(q_n, 0) \to (s, \epsilon_1)$ and $(r_n, 1) \to (s, \epsilon_2)$ for some s in the closure of J and $\epsilon_1, \epsilon_2 \in \{0, 1\}$. If $\epsilon_1 = \epsilon_2$ we obtain a contradiction with the continuity of f at (s, ϵ_1). If $\epsilon_1 \neq \epsilon_2$, then necessarily $s \in J$ and by the continuity of f we have $|f((s, 0)) - f((s, 1))| \geq \varepsilon + 1/m$, so $s \in \psi_\varepsilon(f)$.

The above formula shows that $\psi_\varepsilon^{-1}(U)$ is a $G_{\delta\sigma}$-set for $U \in \mathcal{U}$. Let $\mathcal{V}$ be the family of open sets in $[K]^{<\omega}$ of the form $V = \{F \in [K]^{<\omega} : F \subset W\}$, where W is an open subset of K. Each $V \in \mathcal{V}$ is a countable union of complements of finite unions of sets $U \in \mathcal{U}$. Indeed, if W is open in K, then $K \setminus W$ is a compact G_δ-subset of I, hence $K \setminus W$ can be written as $\bigcap_n U_n$, where each U_n is a finite union of open intervals in I. We may additionally require that, for every n, $U_{n+1} \subset U_n$, therefore, a finite set F in K is contained in W if and only if F is disjoint from some U_n.

It follows that $\psi_\varepsilon^{-1}(V)$ is an $F_{\sigma\delta\sigma}$-set for every $V \in \mathcal{V}$. The family $\mathcal{U} \cup \mathcal{V}$ forms a subbase of the topology of $[K]^{<\omega}$, hence $\psi_\varepsilon^{-1}(U)$ is an $F_{\sigma\delta\sigma}$-set for every open $U \subset [K]^{<\omega}$.

Now, the estimate for the Borel class of $C_E(K_A)$ follows from the formula $T_A = \bigcap_{n\geq 1} \psi_{1/n}^{-1}([A]^{<\omega})$ and the fact that $[A]^{<\omega}$ is a Borel subset of $[K]^{<\omega}$ of multiplicative class α ($[A]^{<\omega}$ is a countable union of closed subsets homeomorphic to closed subspaces of products A^n, see [DM, Prop. 3.8]).

Finally, we will bound $\eta(K_A)$ from below. Let D be a countable dense subset of K_A such that $\eta(K_A)$ is equal to the Borel class of $C_D(K_A)$. Let E be a countable dense set in K_A given by Lemma 3.6. Then the Borel class of $C_E(K_A)$ is $\geq \alpha$. From Theorem 2.2 of [Ma2], it follows that $\eta(K_A) \geq \alpha$. ∎

Corollary 3.8. *Let K and L be closed subsets of I, and $A \subset K$ and $B \subset L$ be Borel sets of exact multiplicative classes $\alpha > \beta \geq \omega$, respectively. Then the spaces $(C(K_A), w)$ and $(C(L_B), w)$ (and $C_{\mathrm{p}}(K_A)$ and $C_{\mathrm{p}}(L_B)$) are not homeomorphic; in particular, $C(K_A)$ and $C(L_B)$ are not isomorphic.*

Corollary 3.9. *There exists a family $\{A_\alpha : \alpha < \omega_1\}$ of Borel subsets of $(0, 1)$ such that the spaces $(C(I_{A_\alpha}), w)$ and $(C(I_{A_\beta}), w)$ are not homeomorphic for $\alpha \neq \beta$.*

Using the same argument as in the proof of Theorem 3.7 and a slight modification of the proofs of Theorems 3.1 and 3.5 from [Ma2], one can also prove the following:

Theorem 3.10. *Let K and L be closed subsets of I, and $A \subset K$ and $B \subset L$ be projective sets of different projective classes. Then the spaces $(C(K_A), w)$ and $(C(L_B), w)$ (and $C_{\mathrm{p}}(K_A)$ and $C_{\mathrm{p}}(L_B)$) are not homeomorphic.*

Let us finish this section with the following simple remark. The weight of the space K_A is equal to $\max(|A|, \omega)$ (for K infinite). Recall that the weight of a compact space K is equal to the density of $(C(K), w)$ and to the network weight of $C_{\mathrm{p}}(K)$ (see [Ar1]). Therefore, for infinite sets $A \subset K$ and $B \subset L$, if the spaces $(C(K_A), w)$ and $(C(L_B), w)$ (or $C_{\mathrm{p}}(K_A)$ and $C_{\mathrm{p}}(L_B)$) are homeomorphic, then $|A| = |B|$.

4. Isomorphisms of $C(K_A)$ spaces. Let E and F be Banach spaces. We write $E \approx F$ if E and F are isomorphic. We say that F is a *factor* of E, and write $F \mid E$, if there is a Banach space F' such that $E \approx F \times F'$.

For a closed subset A of a compact space L, we denote by $C(L, A)$ the subspace $\{f \in C(L) : f|A \equiv 0\}$ of $C(L)$.

Lemma 4.1. *Let A be a subset of an infinite closed set $K \subset I$. Then $C(K_A)$ is isomorphic to $C(K_A) \times \mathbb{R}$. For every finite set $F \subset K_A$, the spaces $C(K_A)$ and $C(K_A, F)$ are isomorphic.*

Proof. Since K is infinite and K_A is first countable, it contains a nontrivial convergent sequence. It follows that $C(K_A)$ is isomorphic to $C(K_A) \times \mathbb{R}$ (see [Ar2, Section 4]). It is standard that this implies the last statement of the lemma. ∎

Corollary 4.2. *Let K be an infinite closed subset of I, and let A and B be subsets of K such that the symmetric difference $A \triangle B$ is finite. Then the spaces $C(K_A)$ and $C(K_B)$ are isomorphic.*

Proof. Clearly, it is enough to prove the corollary for $B = A \cup \{a\}$. One can easily verify that $C(K_B)$ is isomorphic to $C(K_A) \times \mathbb{R} \approx C(K_A)$. ∎

Lemma 4.3. *Let A be a subset of a closed set $K \subset I$. For each nonempty closed subset H of K_A, the space $C(K_A)$ is isomorphic to $C(H) \times C(K_A, H)$.*

Proof. We will show that there exists an extension operator $e : C(H) \to C(K_A)$, i.e., a continuous linear map such that $e(f)|H = f$ for each $f \in C(H)$. Then $C(K_A) \approx C(H) \times C(K_A, H)$ (see [Se, §21.4]). Let $a = \inf H$ and $b = \sup H$ (with respect to the order $\prec$). For each $x \in \langle a, b\rangle \setminus H$ put $a_x = (s_x, i_x) = \sup\{y \in H : y \prec x\}$ and $b_x = (t_x, j_x) = \inf\{y \in H : y \succ x\}$. Now, we can define the operator e as follows:

$$e(f)(x) = \begin{cases} f(x) & \text{if } x \in H, \\ f(a) & \text{if } x \prec a, \\ f(b) & \text{if } x \succ a, \\ \dfrac{t_x - t}{t_x - s_x}\, f(a_x) + \dfrac{t - s_x}{t_x - s_x}\, f(b_x) & \text{if } x = (t,i) \in \langle a, b\rangle \setminus H, \end{cases}$$

for $f \in C(H)$ and $x \in K_A$. A routine verification shows that e has the required properties. ∎

For a sequence $(E_n)_{n\in\omega}$ of Banach spaces, $(E_0 \times E_1 \times \cdots)_0$ is the standard c_0-product of the spaces E_n equipped with the supremum norm. We denote $(E \times E \times \cdots)_0$ by $(E)_0^\omega$.

LEMMA 4.4. *The space $C(\mathbb{K})$ is isomorphic to $(C(\mathbb{K}))_0^\omega$.*

Proof. Let $K_n = \langle (1/(n+1), 1), (1/n, 0)\rangle \subset \mathbb{K}$ for $n \geq 1$. Clearly, each K_n is a topological copy of $\mathbb{K}$. By Lemma 4.1 we have $C(\mathbb{K}) \approx C(\mathbb{K}, \{(0,0)\})$. It remains to observe that $C(\mathbb{K}, \{(0,0)\})$ is isometric to the c_0-product $(C(K_1) \times C(K_2) \times \cdots)_0$. ∎

Let $C \subset I$ be the standard Cantor middle-thirds set. We denote the space C_C by $\mathbb{L}$.

LEMMA 4.5. *The space $C(\mathbb{K})$ is isomorphic to $C(\mathbb{L})$.*

Proof. Since $(0,1)$ contains a copy of the Cantor set C, the space $\mathbb{L}$ embeds into $\mathbb{K}$. Hence, from Lemma 4.3 we infer that $C(\mathbb{L}) \,|\, C(\mathbb{K})$. We can also embed $\mathbb{K}$ into $\mathbb{L}$: Let $\{J_n = (a_n, b_n) : n \in \omega\}$ be the family of all components of the complement $I \setminus C$. Set

$$H = \mathbb{L} \setminus [(\{b_n : n \in \omega\} \times \{0\}) \cup (\{0,1\} \cup \{a_n : n \in \omega\} \times \{1\})].$$

A routine verification shows that H is homeomorphic to $\mathbb{K}$ (recall that gluing together points a_n and b_n in C, for $n \in \omega$, one obtains a topological copy of I). Again, Lemma 4.3 implies that $C(\mathbb{K}) \,|\, C(\mathbb{L})$. Since $C(\mathbb{K}) \approx (C(\mathbb{K}))_0^\omega$ by Lemma 4.4, we can apply the Pełczyński decomposition scheme (see [Pe], [Se]) to conclude that $C(\mathbb{K}) \approx C(\mathbb{L})$. ∎

LEMMA 4.6. *For each nonempty metrizable compact space M, the spaces $C(\mathbb{K})$ and $C(\mathbb{K}) \times C(M)$ are isomorphic.*

Proof. First, we will show that $C(2^\omega)$ is a factor of $C(\mathbb{L})$.

Recall that $p_1 : \mathbb{L} \to C$ is the projection. Fix a homeomorphism h of C onto $C_1 \times C_2$, the product of two copies of 2^ω. Let $\pi_i : C_1 \times C_2 \to C_i$ be the projection and $h_i = \pi_i \circ h$, $i = 1, 2$. We will show that the map $\varphi = \pi_1 \circ h \circ p_1 : \mathbb{L} \to C_1$ admits a regular averaging operator, i.e., there exists a positive linear operator $T : C(\mathbb{L}) \to C(C_1)$ such that $T(1_{\mathbb{L}}) = 1_{C_1}$ and $T(g \circ \varphi) = g$ for all $g \in C(C_1)$. The existence of such T implies that $C(C_1) \,|\, C(\mathbb{L})$ (see [Pe, Proposition 8.2]).

Denote the standard product measure on C_2 (the copy of 2^ω) by μ. We define the operator T as follows:

$$T(f)(x) = \int_{C_2} f((h^{-1}(x,\cdot),0))\,d\mu$$

for $f \in C(\mathbb{L})$ and $x \in C_1$. Observe that, for $f \in C(\mathbb{L})$, the function $f' : C \to \mathbb{R}$ defined by $f'(t) = f((t,0))$ is bounded and has at most countably many points of discontinuity (see the proof of Theorem 3.7), therefore $T(f)(x)$ is well defined. It is clear that T is positive and $T(1_{\mathbb{L}}) = 1_{C_1}$. For $g \in C(C_1)$, $x \in C_1$, and $y \in C_2$ we have $g \circ \varphi((h^{-1}(x,y),0)) = g(x)$, hence $T(g \circ \varphi) = g$. We shall check that $T(f)$ is continuous on C_1 for each $f \in C(\mathbb{L})$. Suppose that $x_n \to x$ in C_1. Then $h^{-1}(x_n,y) \to h^{-1}(x_n,y)$ for each $y \in C_2$. Let A be the countable set of points of discontinuity of the function $f' : C \to \mathbb{R}$ defined above, and let $B = \pi_2 \circ h(A)$. For every $y \in C_2 \setminus B$ we have $f((h^{-1}(x_n,y),0)) \to f((h^{-1}(x,y),0))$. Hence $f((h^{-1}(x_n,y),0)) \to f((h^{-1}(x,y),0))$ μ-a.e. on C_2. Therefore from the Lebesgue dominated convergence theorem it follows that $T(f)(x_n) \to T(f)(x)$.

Let M be a nonempty metrizable compact space. Since $C(M)$ is a factor of $C(2^\omega)$ (see [Se]), we infer that $C(M)$ is a factor of $C(\mathbb{L})$, hence of $C(\mathbb{K})$ by Lemma 4.5. Thus $C(\mathbb{K}) \times C(M)$ is a factor of $C(\mathbb{K}) \times C(\mathbb{K})$ and consequently of $C(\mathbb{K})$ by Lemma 4.4. The same lemma together with the Pełczyński decomposition scheme implies that $C(\mathbb{K}) \times C(M)$ and $C(\mathbb{K})$ are isomorphic. ∎

THEOREM 4.7. *For every uncountable, open or closed subset A of a closed set $K \subset I$, the spaces $C(K_A)$ and $C(\mathbb{K})$ are isomorphic.*

Proof. First, consider the case when A is an uncountable open subset of K.

Since A is an uncountable Borel subset of I, we can embed the Cantor set C in A. In the same way as in the proof of Lemma 4.5, we conclude that $C(\mathbb{L}) \mid C(K_A)$, hence, by the same lemma, $C(\mathbb{K}) \mid C(K_A)$.

Next, we will show that $C(K_A) \mid C(\mathbb{K})$. Let B be an open subset of I such that $A = K \cap B$. Then we can consider K_A as a subset of I_B, and Lemma 4.3 implies that $C(K_A) \mid C(I_B)$. For $H = I \setminus B$, we have $C(I_B) \approx C(H) \times C(I_B, H \times \{0\})$, by Lemma 4.3. The space $C(I_B, H \times \{0\})$ is isomorphic either to a finite product or to a c_0-product of spaces of the form $C(\overline{J}_J, (\overline{J} \cap H) \times \{0\})$, where J is a component of B, and $\overline{J}$ is the closure of J in I. Clearly, $\overline{J} \cap H$ is finite—it can contain only the endpoints of J. Therefore, each $C(\overline{J}_J, (\overline{J} \cap H) \times \{0\})$ is isomorphic to $C(\mathbb{K})$, by Lemma 4.1. From Lemma 4.4 we infer that $C(I_B, H \times \{0\}) \approx C(\mathbb{K})$, and $C(I_B) \approx C(H) \times C(\mathbb{K}) \approx C(\mathbb{K})$, by Lemma 4.6.

Again, we conclude that $C(K_A) \approx C(\mathbb{K})$ using the Pełczyński decomposition scheme.

Secondly, assume that A is an uncountable closed subset of K. As in the first case, $C(\mathbb{K})$ is a factor of $C(K_A)$. It remains to prove that $C(K_A) \mid C(\mathbb{K})$. Let $F = K_A \setminus ((K \setminus A) \times \{0\})$. Then F is a closed subset of K_A which can be identified with A_A. From Lemma 4.3 it follows that $C(K_A) \approx C(F) \times C(K_A, F)$. Observe that no points of $K_A \setminus F$ are split, so we may identify $C(K_A, F)$ with $C(K, A)$. Using Lemmas 4.3 and 4.6 we obtain

$$\begin{aligned} C(\mathbb{K}) &\approx C(\mathbb{K}) \times C(K) \approx C(A_A) \times C(\mathbb{K}, A_A) \times C(A) \times C(K, A) \\ &\approx C(F) \times C(K_A, F) \times C(\mathbb{K}, A_A) \times C(A) \\ &\approx C(K_A) \times C(\mathbb{K}, A_A) \times C(A), \end{aligned}$$

which gives us the required factorization. ∎

Corollary 4.8. *Let A be a nonempty open or uncountable closed subset of* $(0,1)$. *Then the spaces $C(I_A)$ and $C(\mathbb{K})$ are isomorphic.*

Let us note that by Example 2.3 we have 2^ω pairwise nonhomeomorphic I_A spaces such as in the above corollary.

In Section 2 we proved that the spaces $\mathbb{K}$ and I_P are not homeomorphic, where P is the set of all irrational numbers in $(0,1)$. We do not know if the corresponding Banach spaces $C(\mathbb{K})$ and $C(I_P)$ are isomorphic.

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