STUDIA MATHEMATICA 184 (3) (2008)

Regularity of the symbolic calculus in Besov algebras

by

GÉRARD BOURDAUD (Paris) and
MASSIMO LANZA DE CRISTOFORIS (Padova)

Abstract. We consider Besov and Lizorkin–Triebel algebras, that is, the real-valued function spaces $B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ for all $s > 0$. To each function $f : \mathbb{R} \to \mathbb{R}$ one can associate the composition operator T_f which takes a real-valued function g to the composite function $f \circ g$. We give necessary conditions and sufficient conditions on f for the continuity, local Lipschitz continuity, and differentiability of any order of T_f as a map acting in Besov and Lizorkin–Triebel algebras. In some cases, such as for $n = 1$, such conditions turn out to be necessary and sufficient.

1. INTRODUCTION

The Superposition Operator Problem (S.O.P.) for a given real-valued function space E consists in the full characterization of those functions $f : \mathbb{R} \to \mathbb{R}$ such that the superposition operator $T_f : g \mapsto f \circ g$ takes E to itself. Such a function f is also said to *act on* E by superposition. In case E is a normed space, we say that f acts *boundedly* on E if the mapping T_f is bounded on every bounded subset of E. The superposition operator can as well take a given function space E to another space F. Then we say that *f acts from E to F*. We could also consider spaces of V-valued functions, for a given finite-dimensional vector space V, and superposition operators defined by mappings $f : V \to V$. Part of our results have extensions to this more general framework.

We consider the S.O.P. for the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and the Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$ (see Section 2 for the definition). Unless otherwise specified,

> *in all statements of this paper we assume p, q to be a priori fixed numbers, with $p, q \in [1, \infty]$ in the case of Besov spaces, and with $q \in [1, \infty]$ and $p \in [1, \infty[$ in the case of Lizorkin–Triebel spaces.*

2000 *Mathematics Subject Classification*: 46E35, 47H30.
Key words and phrases: Lizorkin–Triebel spaces, Besov spaces, continuity and differentiability of superposition operators.

We set $E^s_{p,q}(\mathbb{R}^n) := B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$, when there is no need to distinguish the B spaces and the F ones. In this context, only a small part of the S.O.P. has been solved so far.

The first remarkable property is the possible existence of an interval of s for which no nontrivial superposition operator exists. More precisely, we have the following (see [2, 3, 13, 22, 23]).

THEOREM 1. *Let $1 + 1/p < s < n/p$. Then f acts on $E^s_{p,q}(\mathbb{R}^n)$ if and only if f is a linear function. The same triviality result holds in the critical case $1 + 1/p = s < n/p$, provided that $q > 1$ in the case of Besov spaces, and $p > 1$ in the case of Lizorkin–Triebel spaces.*

REMARK 1. The existence of nontrivial functions acting on $B^{1+1/p}_{p,1}(\mathbb{R}^n)$ in case $n > p + 1$, and on $F^2_{1,q}(\mathbb{R}^n)$ in case $n > 2$, are open questions.

Since the triviality phenomenon is connected with the existence of unbounded functions in the relevant function spaces, it is natural to consider the spaces

$$\mathcal{B}^s_{p,q}(\mathbb{R}^n) := B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n), \quad \mathcal{F}^s_{p,q}(\mathbb{R}^n) := F^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n).$$

We denote the above spaces by $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ if there is no need to distinguish between B and F spaces. As usual, $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ is endowed with the natural norm

$$\|f\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} := \|f\|_{E^s_{p,q}(\mathbb{R}^n)} + \|f\|_\infty.$$

By contrast with the spaces $E^s_{p,q}(\mathbb{R}^n)$, there are always nontrivial superposition operators on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ for $s > 0$.

PROPOSITION 1. *Assume that $s > 0$.*

(i) *$\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ is a Banach algebra for the pointwise product.*
(ii) *Any $f \in C^\infty(\mathbb{R})$ such that $f(0) = 0$ acts boundedly on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$.*

Proof. See [23, 4.6.4, 5.3.4]. ∎

REMARK 2. Proposition 1 has a counterpart for complex-valued Besov and Lizorkin–Triebel spaces (see [23, 5.5.1]).

Another necessary condition is that f must be locally Lipschitz continuous.

THEOREM 2. *Let $s > 0$.*

(i) *If a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ acts from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to $B^s_{p,\infty}(\mathbb{R}^n)$, then*

(A) *f is locally Lipschitz continuous.*

(ii) *Let $\mathbb{B}$ be a ball in $\mathbb{R}^n$. Let K be a compact subset of $\mathbb{R}$. Then there exist $r_1, r_2 \in]0, \infty[$ such that the Lipschitz constant of the restriction* *of f to K is less than or equal to*

$$r_1 \sup\{\|f\circ g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \,:\, g\in\mathcal{D}(\mathbb{B}),\ \|g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)}\le r_2\}$$

for all Borel measurable functions $f:\mathbb{R}\to\mathbb{R}$ such that T_f acts boundedly from $(\mathcal{D}(\mathbb{B}),\|-\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$ to $B^s_{p,\infty}(\mathbb{R}^n)$.

Remark 3. Assume that $0<s<1$. It is well known that any locally Lipschitz continuous function acts boundedly in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. We can conclude from Theorem 2 that the S.O.P. is solved for $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ in that case.

A third necessary condition is almost immediate: if f acts on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ then

(B) f belongs locally to $E^s_{p,q}(\mathbb{R})$.

We believe that condition (B) is also sufficient in case $s>1+1/p$ (see also Remark 5 below).

Conjecture. *Assume that $s>1+1/p$. Let $f:\mathbb{R}\to\mathbb{R}$ be any Borel measurable function such that $f(0)=0$. Then the following properties are equivalent*:

1. *f acts on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$,*
2. *f acts boundedly on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$,*
3. *f belongs locally to $E^s_{p,q}(\mathbb{R})$.*

We denote by $\mathcal{I}_{n,B}$ and $\mathcal{I}_{n,F}$ the set of triples (s,p,q) with $s>1+1/p$, $p,q\in[1,\infty]$ ($p<\infty$ in the F-case) for which the above conjecture holds true in $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$, respectively.

The sets $\mathcal{I}_{1,B}$ and $\mathcal{I}_{1,F}$ are known to be large, in some sense. Thus for instance,

- $\mathcal{I}_{1,B}$ contains all (s,p,q) such that $s>1+1/p$ and $4/3<p\le q$,
- $\mathcal{I}_{1,F}$ contains all (s,p,q) such that $s>1+1/p$, $4/3<p<\infty$ and $q\in[1,\infty]$.

See [6, 11, 12] for more details. On the contrary, very little is known for $n>1$. The only triples known to be in $\mathcal{I}_{n,B}$ for $n>1$ are

$$s\,\text{integer}\ \ge 2,\qquad p=q=2,$$

and the triples known to be in $\mathcal{I}_{n,F}$ for $n>1$ are

$$s\,\text{integer}\ \ge 2,\qquad 1<p<\infty,\qquad q=2.$$

In the above two cases, $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ coincides with the classical Sobolev algebra $W^s_p(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$.

Remark 4. A weaker version of the Conjecture has been established for Besov algebras in case $n>1$, and for a "substantial" set of triples (s,p,q). Namely, every $f\in B^{s+\varepsilon}_{p,\infty}(\mathbb{R})_{\text{loc}}$ ($\varepsilon>0$) such that $f(0)=0$ acts boundedly in $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ (see [7] for more details).

Remark 5. In case $1 \leq s \leq 1 + 1/p$ the S.O.P. turns out to be more mysterious. Indeed, we suspect that conditions (A) and (B), together with $f(0) = 0$, *are not* sufficient for f to act on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. A typical example is the Zygmund class $B^1_{\infty,\infty}(\mathbb{R}^n)$, for which a full description of the S.O.P. has been given in [8]. In that case, the necessary and sufficient condition for f to act in $B^1_{\infty,\infty}(\mathbb{R}^n)$ is stronger than (A) and (B) combined.

This paper is mostly devoted to the regularity—i.e., continuity and differentiability of all orders, and Lipschitz continuity on bounded sets—for the operator T_f in Besov and Lizorkin–Triebel algebras.

Plan of the paper. This paper is organized as follows. In Section 2 we recall some relevant properties of Besov and Lizorkin–Triebel spaces. The proof of Theorem 2 is given in Section 3. Section 4 is devoted to the Lipschitz continuity of T_f on bounded sets. Concerning the global Lipschitz continuity of T_f we prove a degeneracy result: this property can occur only if f is an affine function. In Section 5 we give general sufficient conditions and necessary conditions for the continuity and differentiability of T_f. Then we exploit the above results, in order to characterize regularity of T_f for parameters (s, p, q) in $\mathcal{I}_{n,B}$ or in $\mathcal{I}_{n,F}$. Section 6 is an appendix devoted to more or less classical results on distribution spaces, which have been exploited in the paper.

Notation. We denote by $\mathbb{N}$ the set of all natural numbers, including 0. We denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$, by Q the unit cube $[-1/2, 1/2]^n$, and by φ an even C^∞ function on $\mathbb{R}^n$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ on Q, and $\varphi(x) = 0$ outside $2Q$.

We introduce the translation operator τ_h and the difference operator Δ_h, defined on functions (or distributions) by $(\tau_h f)(x) := f(x - h)$ and $\Delta_h f := \tau_{-h} f - f$. If g belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, we denote by $g(D)$ the pseudodifferential operator with symbol g, defined by

$$\widehat{g(D)f} := g\widehat{f} \quad \forall f \in \mathcal{S}'(\mathbb{R}^n),$$

where $\widehat{f}$ denotes the Fourier transform of f.

In a given metric space, we denote by $\mathbb{B}(a, r)$ the open ball with centre a and radius r. In a topological space E, $\mathrm{cl}_E(A)$ is the closure of a subset A. The symbol $\mathbb{B}$ will also be used for a general ball in $\mathbb{R}^n$. If necessary, we write $\mathbb{B}_n$ instead of $\mathbb{B}$.

If E is any normed real-valued function space, we set

$$\Phi(E) := \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is Borel measurable and acts boundedly on } E,\ f(0) = 0\}.$$

We endow $\Phi(E)$ with the seminorms

$$\nu_r(f) := \sup\{\|f \circ g\|_E \,:\, \|g\|_E \le r\} \quad \forall r \in]0,\infty[.$$

The restriction $f(0) = 0$ is just a technical convenience, and does not imply a loss of generality. The reduction to that case follows by adding a suitable constant to f.

If E is any distribution space, and if $r \in \mathbb{N}$, we denote by $W^r(E)$ the Sobolev space built on E, i.e., the set of distributions whose derivatives up to order r belong to E. As usual, we set

$$W_p^r(\mathbb{R}^n) := W^r(L_p(\mathbb{R}^n)).$$

We denote by p' the conjugate exponent of p, i.e., $p' := p/(p-1)$. As usual, $c, c_1, \ldots$ are strictly positive constants and depend only on the fixed parameters n, s, p, q, and on auxiliary functions, unless otherwise specified. Their values can change from a line to another.

Unless otherwise specified, all functions are assumed to be real-valued (see Remark 12).

2. DEFINITIONS AND PROPERTIES OF BESOV SPACES

2.1. The classical Littlewood–Paley framework. We need to recall the definition of Besov and Lizorkin–Triebel spaces in the Littlewood–Paley setting. Let

$$\gamma(x) := \varphi(x) - \varphi(2x) \quad \forall x \in \mathbb{R}^n.$$

Then $\gamma \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and the following identity holds:

$$\varphi(x) + \sum_{j\ge 1} \gamma(2^{-j}x) = 1 \quad \forall x \in \mathbb{R}^n. \tag{1}$$

The functions φ and γ clearly depend on n. In case we deal with several values of n, we shall denote them as φ_n and γ_n, respectively. We define the operators Q_j on $\mathcal{S}'(\mathbb{R}^n)$ by setting

$$Q_j := \gamma(2^{-j}D) \quad (j \ge 1), \quad Q_0 := \varphi(D).$$

Remark 6. Let $j \ge 1$. We note that $\varphi(2^{-j}D) = \sum_{k=0}^{j} Q_k$ and that the operator $\varphi(2^{-j}D)$ coincides with the convolution operator with the function $v_j(x) := 2^{nj}(\mathcal{F}^{-1}\varphi)(2^j x)$; accordingly $\varphi(2^{-j}D)$ acts boundedly in $L_p(\mathbb{R}^n)$ and the norm of $\varphi(2^{-j}D)$ has an upper bound independent of j. Also, $\{v_j\}_{j\in\mathbb{N}}$ is well known to be an approximate identity of convolution.

We also introduce even functions $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)$ and $\widetilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that

$$\widetilde{\varphi}\varphi = \varphi \quad \text{and} \quad \widetilde{\gamma}\gamma = \gamma. \tag{2}$$

The operators $\widetilde{Q}_j$ are defined accordingly.

DEFINITION 1. For any $s \in \mathbb{R}$, $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are the sets of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \Big(\sum_{j\geq 0}(2^{sj}\|Q_j f\|_p)^q\Big)^{1/q} < \infty,$$

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \Big\|\Big(\sum_{j\geq 0}(2^{sj}|Q_j f|)^q\Big)^{1/q}\Big\|_p < \infty,$$

respectively.

PROPOSITION 2. *For any $s \in \mathbb{R}$, we can define a continuous bilinear form on $E^s_{p,q}(\mathbb{R}^n) \times E^{-s}_{p',q'}(\mathbb{R}^n)$ by setting*

$$\lhd f, g \rhd := \sum_{j\geq 0} \int_{\mathbb{R}^n} Q_j f(x)\widetilde{Q}_j g(x)\, dx. \tag{3}$$

The restriction of $\lhd -, - \rhd$ to $E^s_{p,q}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ coincides with the canonical bilinear form on $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$.

Proof. By the Nikol'skiĭ representation method (see Runst and Sickel [23, Prop. 2.3.2(1), p. 59] or Yamazaki [26]), there exists $c > 0$ such that

$$\Big(\sum_{j\geq 0}(2^{sj}\|\widetilde{Q}_j f\|_p)^q\Big)^{1/q} \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)},$$

$$\Big\|\Big(\sum_{j\geq 0}(2^{sj}|\widetilde{Q}_j f|)^q\Big)^{1/q}\Big\|_p \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $f \in E^s_{p,q}(\mathbb{R}^n)$, respectively. By applying twice the Hölder inequality, we can infer that there exists $c > 0$ such that

$$|\lhd f, g \rhd| \leq c\|f\|_{E^s_{p,q}(\mathbb{R}^n)}\|g\|_{E^{-s}_{p',q'}(\mathbb{R}^n)} \quad \forall f \in E^s_{p,q}(\mathbb{R}^n),\ \forall g \in E^{-s}_{p',q'}(\mathbb{R}^n). \tag{4}$$

By (2) and by the Plancherel identity, we deduce that

$$\langle Q_j f, \widetilde{Q}_j g\rangle = \langle f, Q_j g\rangle \quad \forall f \in E^s_{p,q}(\mathbb{R}^n),\ \forall g \in \mathcal{S}(\mathbb{R}^n),\ \forall j \in \mathbb{N}. \tag{5}$$

Hence,

$$\lhd f, g \rhd = \langle f, g\rangle \quad \forall f \in E^s_{p,q}(\mathbb{R}^n),\ \forall g \in \mathcal{S}(\mathbb{R}^n). \ \blacksquare \tag{6}$$

2.2. A variant of the Littlewood–Paley decomposition. In some cases, it is useful to replace the standard functions γ and φ by tensor product functions. We work out explicitly the construction with respect to the decomposition $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. Similar constructions hold for all decompositions $\mathbb{R}^{n_1+\cdots+n_m} = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$.

For all functions f, g defined on $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively, we set

$$(f \otimes g)(t, x) := f(t)g(x) \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Now we set

$$u_0 := \varphi_1 \otimes \varphi_{n-1}, \qquad u_1 := \varphi_1(2\,\cdot) \otimes \gamma_{n-1}, \qquad u_2 := \gamma_1 \otimes \varphi_{n-1}.$$

Then we have

$$u_0(t,x) - u_0(2t,2x) = u_1(t,x) + u_2(t,x) \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

We define the operators U_0 and $U_{m,j}$ ($j \ge 1, m = 1,2$) on $\mathcal{S}'(\mathbb{R}^n)$ by setting

$$U_{m,j} := u_m(2^{-j}D), \qquad U_0 := u_0(D).$$

PROPOSITION 3. *Let $n > 1$ and $s \in \mathbb{R}$. Then a tempered distribution f belongs to $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$ if and only if*

$$\|U_0 f\|_p + \sum_{m=1,2} \Big(\sum_{j\ge 1} (2^{sj}\|U_{m,j} f\|_p)^q\Big)^{1/q} < \infty,$$

or

$$\|U_0 f\|_p + \sum_{m=1,2} \Big\|\Big(\sum_{j\ge 1} (2^{sj}|U_{m,j} f|)^q\Big)^{1/q}\Big\|_p < \infty,$$

respectively. Moreover, the above expressions are equivalent norms on $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively.

Proof. The above statements can be proved by minor modifications of classical results (cf. e.g. Triebel [24, Prop. 2.3.2/1, p. 46], [25, Ch. 2] or Peetre [21, Ch. 8]). ∎

PROPOSITION 4. *Let $n > 1$ and $s > 0$. Then the following statements hold.*

(i) *There exists $c > 0$ such that $f \otimes g \in E^s_{p,q}(\mathbb{R}^n)$ and*

(7) $$\|f \otimes g\|_{E^s_{p,q}(\mathbb{R}^n)} \le c\|f\|_{E^s_{p,q}(\mathbb{R})}\,\|g\|_{E^s_{p,q}(\mathbb{R}^{n-1})}$$

for all $f \in E^s_{p,q}(\mathbb{R})$ and $g \in E^s_{p,q}(\mathbb{R}^{n-1})$.

(ii) *Let $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be a measurable function such that*

- $0 < \|g\|_{L_p(\mathbb{R}^{n-1})} < \infty$ *in the Besov case,*
- $0 < \|g\|_{L_\infty(\mathbb{R}^{n-1})} < \infty$ *and g is uniformly continuous in the Lizorkin–Triebel case.*

Then there exists a constant $c(g) > 0$ such that for all measurable functions $f : \mathbb{R} \to \mathbb{R}$ with $f \otimes g \in E^s_{p,q}(\mathbb{R}^n)$ we have $f \in E^s_{p,q}(\mathbb{R})$ and

(8) $$\|f\|_{E^s_{p,q}(\mathbb{R})} \le c(g)\|f \otimes g\|_{E^s_{p,q}(\mathbb{R}^n)}.$$

Proof. We endow the space $E^s_{p,q}(\mathbb{R}^n)$ with the equivalent norms of Proposition 3, and we divide our proof into two steps.

STEP 1: *the Besov case.* We have

$$\|f \otimes g\|_{B^s_{p,q}(\mathbb{R}^n)} = \|Q_0 f\|_p \|Q_0 g\|_p + \Big(\sum_{j\geq 1} (2^{sj} \|\varphi_1(2^{1-j}D)f\|_p \|Q_j g\|_p)^q\Big)^{1/q} + \Big(\sum_{j\geq 1} (2^{sj} \|\varphi_{n-1}(2^{-j}D)g\|_p \|Q_j f\|_p)^q\Big)^{1/q}.$$

Since the imbedding

$$E^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \quad \forall s > 0 \tag{9}$$

is continuous and the operators $\varphi(2^{-j}D)$ are bounded on L_p uniformly with respect to j (cf. Remark 6), we can obtain inequality (7) for Besov spaces. Now let $0 < \|g\|_{L_p(\mathbb{R}^{n-1})} < \infty$. By Remark 6, we have

$$\lim_{j\to\infty} \varphi_{n-1}(2^{-j}D)g = g \quad \text{in } L_p(\mathbb{R}^{n-1}).$$

Hence, there exists j_0 such that

$$\|\varphi_{n-1}(2^{-j}D)g\|_p \geq \frac{1}{2}\|g\|_p \quad \forall j > j_0.$$

Hence,

$$\frac{1}{2}\|g\|_p \Big(\sum_{j>j_0} (2^{sj}\|Q_j f\|_p)^q\Big)^{1/q} \leq \|f \otimes g\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Since the operators Q_j are bounded on $L_p(\mathbb{R}^{n-1})$ uniformly with respect to j, the imbedding (9) implies that there exist $c_1, c_2 > 0$ such that

$$\|Q_j f\|_p \leq c_1 \|f\|_p \leq \frac{c_2}{\|g\|_p} \|f \otimes g\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Since $(\sum_{j=0}^{j_0} 2^{sjq})^{1/q} \leq \frac{2^s}{(2^{sq}-1)^{1/q}} 2^{sj_0}$, we have

$$\Big(\sum_{j=0}^{j_0} (2^{sj}\|Q_j f\|_p)^q\Big)^{1/q} \leq \frac{c 2^{sj_0}}{\|g\|_p} \|f \otimes g\|_{B^s_{p,q}(\mathbb{R}^n)},$$

and inequality (8) follows in the Besov case.

STEP 2: *the Lizorkin–Triebel case.* We have

$$\|f \otimes g\|_{F^s_{p,q}(\mathbb{R}^n)} = \|Q_0 f\|_p \|Q_0 g\|_p + \Big\| \Big(\sum_{j\geq 1} |2^{sj} \varphi_1(2^{1-j}D)f \otimes Q_j g|^q\Big)^{1/q} \Big\|_p + \Big\| \Big(\sum_{j\geq 1} |2^{sj} Q_j f \otimes \varphi_{n-1}(2^{-j}D)g|^q\Big)^{1/q} \Big\|_p. \tag{10}$$

By the equality $\varphi_1(2^{1-j}D) = \sum_{m=0}^{j-1} Q_m$, the Hölder inequality, and the condition $s > 0$, there exists $c > 0$ such that

$$|(\varphi_1(2^{1-j}D)f)(t)| \le c\Big(\sum_{m\ge 0} |2^{sm} Q_m f(t)|^q\Big)^{1/q} \quad \forall t \in \mathbb{R},\ \forall j \ge 1.$$

Hence,

$$\Big\|\Big(\sum_{j\ge 1} |2^{sj}\varphi_1(2^{1-j}D)f \otimes Q_j g|^q\Big)^{1/q}\Big\|_p \le c\|f\|_{F^s_{p,q}(\mathbb{R})}\|g\|_{F^s_{p,q}(\mathbb{R}^{n-1})}.$$

Arguing similarly for the other terms in (10), we obtain inequality (7) in the Lizorkin–Triebel case.

Now let g be uniformly continuous and satisfy $0 < \|g\|_{L_\infty(\mathbb{R}^{n-1})} < \infty$. Since

$$\lim_{j\to\infty} \varphi_{n-1}(2^{-j}D)g = g \quad \text{uniformly on } \mathbb{R}^{n-1},$$

there exist a ball $\mathbb{B}_{n-1}$ in $\mathbb{R}^{n-1}$, a number $r > 0$, and an integer j_0 such that

$$|\varphi_{n-1}(2^{-j}D)g(x)| \ge r \quad \forall x \in \mathbb{B}_{n-1},\ \forall j > j_0.$$

Hence,

$$\Big\|\Big(\sum_{j>j_0} |2^{sj} Q_j f \otimes \varphi_{n-1}(2^{-j}D)g|^q\Big)^{1/q}\Big\|_p \ge r|\mathbb{B}_{n-1}|^{1/p}\Big\|\Big(\sum_{j>j_0} |2^{sj}Q_j f|^q\Big)^{1/q}\Big\|_p.$$

Then by (9) and by arguing as for Besov spaces, we obtain (8). ■

REMARK 7. Inequality (7) is classical (cf. e.g. Franke [14]).

2.3. Besov spaces as dual spaces. One of the useful properties of $E^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ is to be dual BDS's (see the Appendix for the definition). More precisely, we have the following (cf. e.g. Triebel [24, 2.11]).

PROPOSITION 5. *Let $s \in \mathbb{R}$. Then $E^s_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathcal{D}'(\mathbb{R}^n)$ such that there exists $A > 0$ satisfying*

$$(11) \qquad |\langle f, g\rangle| \le A\|g\|_{E^{-s}_{p',q'}(\mathbb{R}^n)} \quad \forall g \in \mathcal{D}(\mathbb{R}^n).$$

Moreover, the least constant A such that (11) *holds is an equivalent norm in $E^s_{p,q}(\mathbb{R}^n)$.*

Then we have the following.

PROPOSITION 6. *Let $s \in \mathbb{R}$. Endow $L_1(\mathbb{R}^n) + E^{-s}_{p',q'}(\mathbb{R}^n)$ with its natural norm, i.e., the infimum of the numbers*

$$\|f_1\|_1 + \|f_2\|_{E^{-s}_{p',q'}(\mathbb{R}^n)}$$

for all decompositions $f = f_1 + f_2$ *with* $f_1 \in L_1(\mathbb{R}^n)$ *and* $f_2 \in E^{-s}_{p',q'}(\mathbb{R}^n)$. *Then* $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ *is the set of* $f \in \mathcal{D}'(\mathbb{R}^n)$ *such that there exists* $A > 0$ *satisfying*

$$|\langle f, g\rangle| \le A\|g\|_{L_1(\mathbb{R}^n)+E^{-s}_{p',q'}(\mathbb{R}^n)} \qquad \forall g \in \mathcal{D}(\mathbb{R}^n). \tag{12}$$

Moreover, *the least constant* A *such that* (12) *holds is an equivalent norm in* $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$.

Proof. Step 1. Assume that (12) holds. Then taking the trivial decompositions $g = 0 + g = g + 0$, we obtain the inequalities

$$|\langle f, g\rangle| \le A\|g\|_1 \qquad \forall g \in \mathcal{D}(\mathbb{R}^n), \tag{13}$$

$$|\langle f, g\rangle| \le A\|g\|_{E^{-s}_{p',q'}(\mathbb{R}^n)} \qquad \forall g \in \mathcal{D}(\mathbb{R}^n). \tag{14}$$

Inequality (13) implies classically that $f \in L_\infty(\mathbb{R}^n)$ and $\|f\|_\infty \le A$. By Proposition 5, inequality (14) implies that $f \in E^s_{p,q}(\mathbb{R}^n)$ with a norm less than or equal to cA.

Step 2. Assume that $f \in \mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Let $g \in \mathcal{D}(\mathbb{R}^n)$ and $g = g_1 + g_2$, where $g_1 \in L_1(\mathbb{R}^n)$ and $g_2 \in E^{-s}_{p',q'}(\mathbb{R}^n)$. Since $f \in L_\infty(\mathbb{R}^n)$, the bracket $\langle f, g_1\rangle$ has the usual meaning. Now we prove that

$$\langle f, g\rangle = \langle f, g_1\rangle + \lhd f, g_2 \rhd. \tag{15}$$

Since $g_2 = g - g_1 \in L_1(\mathbb{R}^n)$, Remark 6 implies that

$$\lim_{N\to\infty} \sum_{j=0}^{N} Q_j g_2 = g_2 \quad \text{in } L_1(\mathbb{R}^n).$$

By the density of $\mathcal{S}(\mathbb{R}^n)$ in L_1, identity (5) also holds for $f \in L_\infty \subset B^0_{\infty,\infty}$ and $g \in L_1$. Hence,

$$\lhd f, g_2 \rhd = \lim_{N\to\infty} \sum_{j=0}^{N} \langle Q_j f, \widetilde{Q}_j g_2\rangle = \lim_{N\to\infty} \Big\langle f, \sum_{j=0}^{N} Q_j g_2 \Big\rangle = \langle f, g_2\rangle,$$

which proves the formula (15). Hence, (4) implies that

$$|\langle f, g\rangle| \le \|f\|_\infty \|g_1\|_1 + c\|f\|_{E^s_{p,q}(\mathbb{R}^n)} \|g_2\|_{E^{-s}_{p',q'}(\mathbb{R}^n)}. \ \blacksquare$$

Proposition 6 has an important consequence, the *Fatou property.*

Corollary 1. *Let* $(f_k)_{k\ge 0}$ *be a bounded sequence in* $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, *which converges to* f *in* $\mathcal{S}'(\mathbb{R}^n)$. *Then* $f \in \mathcal{E}^s_{p,q}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} \le \liminf_{k\to\infty} \|f_k\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)}. \tag{16}$$

Remark 8. In Corollary 1, we assume that $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ is normed as in Proposition 6. In case we use another (more usual) equivalent norm in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, a constant $c > 1$ independent of f may appear on the right hand side of (16).

REMARK 9. All spaces $E^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ are $\mathcal{D}(\mathbb{R}^n)$-modules (see the Appendix). Then by Proposition 10 of the Appendix, the operator which takes a function in $E^s_{p,q}(\mathbb{R}^n)$ or in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to its product with a fixed test function is continuous in $E^s_{p,q}(\mathbb{R}^n)$ or in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, respectively.

2.4. Regular functions in Besov spaces. We denote by $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathring{F}^s_{p,q}(\mathbb{R}^n)$ the closures of $\mathcal{D}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively. For simplicity, we denote by $\mathring{E}^s_{p,q}(\mathbb{R}^n)$ either $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ or $\mathring{F}^s_{p,q}(\mathbb{R}^n)$. The following property is classical [24, 2.3.3].

$$\mathring{E}^s_{p,q}(\mathbb{R}^n) = E^s_{p,q}(\mathbb{R}^n) \quad \text{if both } p < \infty \text{ and } q < \infty. \tag{17}$$

On the other hand, for the density of $C^\infty(\mathbb{R}^n) \cap E^s_{p,q}(\mathbb{R}^n)$ in $E^s_{p,q}(\mathbb{R}^n)$, we have a slightly different result, which also holds for $p = \infty$ if E is a Besov space.

PROPOSITION 7. *Let $1 \le q < \infty$ and $s > 0$. Let $p \in [1,\infty]$ for Besov spaces and $p \in [1,\infty[$ for Lizorkin–Triebel spaces. Then the following statements hold.*

(i) *$C^\infty(\mathbb{R}^n) \cap E^s_{p,q}(\mathbb{R}^n)$ is dense in $E^s_{p,q}(\mathbb{R}^n)$.*
(ii) *$C^\infty(\mathbb{R}^n)$ is dense in $E^s_{p,q}(\mathbb{R}^n)_{\mathrm{loc}}$.*

Proof. We first prove (i). Let $f \in E^s_{p,q}(\mathbb{R}^n)$. We will prove that the sequence of functions

$$f_j := \sum_{k=0}^{j} Q_k f \quad \forall j \in \mathbb{N}$$

approximates f in $E^s_{p,q}(\mathbb{R}^n)$. By the Paley–Wiener theorem, the functions f_j are of class C^∞. Since

$$\Big(\sum_{k=0}^{j} (2^{ks}\|Q_k f\|_p)^q\Big)^{1/q} \le \|f\|_{B^s_{p,q}(\mathbb{R}^n)} \quad \forall j \in \mathbb{N},$$

Nikol'skiĭ's method implies that $f_j \in B^s_{p,q}(\mathbb{R}^n)$ whenever $f \in B^s_{p,q}(\mathbb{R}^n)$ (cf. Runst and Sickel [23, §2.3.2, Prop. 1(i), p. 59] or Yamazaki [26]). Similarly, $f_j \in F^s_{p,q}(\mathbb{R}^n)$ if $f \in F^s_{p,q}(\mathbb{R}^n)$. Again by Nikol'skiĭ's method, there exists $c > 0$ such that

$$\|f - f_j\|_{B^s_{p,q}(\mathbb{R}^n)} = \Big\|\sum_{k>j} Q_k f\Big\|_{B^s_{p,q}(\mathbb{R}^n)} \le c\Big(\sum_{k>j} (2^{ks}\|Q_k f\|_p)^q\Big)^{1/q} \quad \forall j \in \mathbb{N}$$

for all $f \in B^s_{p,q}(\mathbb{R}^n)$. Since $q < \infty$, we have

$$\lim_{j\to\infty} \sum_{k>j} (2^{ks}\|Q_k f\|_p)^q = 0.$$

Hence, $\lim_{j\to\infty} f_j = f$ in $B^s_{p,q}(\mathbb{R}^n)$. Similarly, if $f \in F^s_{p,q}(\mathbb{R}^n)$, we have

$$\|f - f_j\|_{F^s_{p,q}(\mathbb{R}^n)} \le c\Big\|\Big(\sum_{k>j}(2^{ks}|Q_k f|)^q\Big)^{1/q}\Big\|_p. \tag{18}$$

We now prove that the right hand side above tends to 0 as $j \to \infty$. Since $f \in F^s_{p,q}(\mathbb{R}^n)$, we have

$$\sum_{k\ge 0}(2^{ks}|Q_k f(x)|)^q < \infty$$

for almost every $x \in \mathbb{R}^n$. Since $q < \infty$, we have

$$\lim_{j\to\infty}\sum_{k>j}(2^{ks}|Q_k f(x)|)^q = 0,$$

for almost every $x \in \mathbb{R}^n$. By the dominated convergence theorem, the right hand side of (18) tends to 0 as $j \to \infty$. Statement (ii) follows by Proposition 12 of the Appendix. ■

Remark 10. By arguing as in the previous proof, one could prove that if $f \in E^s_{p,\infty}(\mathbb{R}^n)$ satisfies the condition

- $\lim_{j\to\infty} 2^{sj}\|Q_j f\|_p = 0$ in the Besov case,
- $\lim_{j\to\infty} \|\sup_{k>j} 2^{ks}|Q_k f|\|_p = 0$ in the Lizorkin–Triebel case,

then f belongs to the closure of $C^\infty(\mathbb{R}^n) \cap E^s_{p,\infty}(\mathbb{R}^n)$ in $E^s_{p,\infty}(\mathbb{R}^n)$.

2.5. A concrete characterization. As we shall see, most of our results rely on the properties of the $E^s_{p,q}$ spaces proved in Subsections 2.1–2.4. In some specific cases, we need a concrete description of $B^s_{p,\infty}(\mathbb{R}^n)$ by means of integral moduli of continuity. Hence, for each $m \in \mathbb{N}\setminus\{0\}$ and $p \in [1,\infty]$, we set

$$\omega_{p,m}(f;t) := \sup_{|h|\le t}\Big(\int_{\mathbb{R}^n} |\Delta^m_h f(x)|^p\,dx\Big)^{1/p}$$

for all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ and all $t \in\,]0,\infty[$. Then the following statement is well known (cf. e.g. Triebel [25, Thm. 2.6.1, p. 140]).

Proposition 8. *Let $0 < s < m$ and f be a distribution on $\mathbb{R}^n$. Then f belongs to $B^s_{p,\infty}(\mathbb{R}^n)$ if and only if $f \in L_p(\mathbb{R}^n)$ and*

$$N_{p,m}(f) := \sup_{0<t\le 1} t^{-s}\omega_{p,m}(f;t) < \infty.$$

Moreover, $\|f\|_p + N_{p,m}(f)$ is an equivalent norm in $B^s_{p,\infty}(\mathbb{R}^n)$.

3. LOCAL LIPSCHITZ CONTINUITY AS A NECESSARY CONDITION

As shown in [4], if f acts on $E^s_{p,q}(\mathbb{R}^n)$, then f must be locally Lipschitz continuous. We shall see that the proof of [4] can be easily modified so as to prove Theorem 2. The following preliminary result will be our main tool.

LEMMA 1. *Let $s > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Assume that f acts from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to $B^s_{p,\infty}(\mathbb{R}^n)$. If $a \in \mathbb{R}$, then there exists a nonlinear operator $U_a : \mathcal{E}^s_{p,q}(\mathbb{R}^n) \to B^s_{p,\infty}(\mathbb{R}^n)$ and $\delta_1, \delta_2 > 0$ such that*

$$U_a g(x) = f(a + g(x)) - f(a) \quad \forall x \in Q \tag{19}$$

and

$$\|U_a g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \le \delta_2$$

for any $g \in \mathcal{D}(\mathbb{R}^n)$ with support in Q and satisfying

$$\|g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} \le \delta_1.$$

Proof of Lemma 1. We first define the nonlinear operator $V_a : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}^{\mathbb{R}^n}$ by setting

$$V_a g(x) := \varphi(x)(f(a + g(x)) - f(a)) \quad \forall x \in \mathbb{R}^n, \forall g \in \mathcal{D}(\mathbb{R}^n).$$

Then

$$V_a g(x) = \varphi(x)(f((a + g(x))\varphi(x/2)) - f(a\varphi(x/2))) \quad \forall x \in \mathbb{R}^n. \tag{20}$$

Hence, V_a maps $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to $B^s_{p,\infty}(\mathbb{R}^n)$ and a standard argument (see [4, proof of Lemme 1], [9, proof of Lemma 3]) shows that there exist a cube $Q' \subset Q$ and $\delta'_1, \delta'_2 > 0$ such that

$$\|g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} \le \delta'_1 \;\Rightarrow\; \|V_a g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \le \delta'_2$$

for any $g \in \mathcal{D}(\mathbb{R}^n)$ with support in Q'. Now let $r > 0$ and $b \in \mathbb{R}^n$ be such that $Q' = rQ + b$, and

$$U_a g(x) := V_a(g(r^{-1}(\cdot - b))(rx + b) \quad \forall x \in \mathbb{R}^n.$$

Then

$$U_a g(x) = \varphi(rx + b)(f(a + g(x)) - f(a)) \quad \forall x \in \mathbb{R}^n.$$

By the inclusion $Q' \subset Q$, we have $\varphi(rx + b) = 1$ on Q. Hence, U_a has all the required properties. ∎

Proof of Theorem 2. We first prove statement (i). Since $B^s_{p,1}(\mathbb{R}^n)$ is continuously imbedded into $E^s_{p,q}(\mathbb{R}^n)$, it suffices to assume that f acts from $\mathcal{B}^s_{p,1}(\mathbb{R}^n)$ to $B^s_{p,\infty}(\mathbb{R}^n)$. We now fix an arbitrary real number a which remains fixed throughout the proof of (i), and prove that f is Lipschitz continuous in a neighbourhood of a by estimating $|f(a + b) - f(a + b')|$ in terms of $|b - b'|$, with b, b' in a neighbourhood of 0 to be determined below. In order to estimate $|f(a+b) - f(a+b')|$ we fix, by now arbitrarily, two real numbers b, b'. Then we consider an integer $N \ge 1$, to be specified below depending on b, b', and we introduce the set

$$A_N := \{(k_1, \dots, k_n) \in \mathbb{Z}^n : |k_j| \le N, \forall j = 1, \dots, n\},$$

and we define the real numbers

$$\varkappa := \frac{1}{2m+1}, \qquad r := \frac{1}{6N},$$

where $m := [s] + 1$. We now test U_a on the function g defined by

$$g(x) := (b' - b) \sum_{k \in A_N} \varphi\left(\frac{1}{\varkappa}\left(\frac{x}{r} - k\right)\right) + b\varphi(2x) \qquad \forall x \in \mathbb{R}^n.$$

Since $\varkappa < 1/2$, the cubes $r(2\varkappa Q + k)$, $k \in \mathbb{Z}^n$, are pairwise disjoint. By definition of r, we have $r(2\varkappa Q + k) \subset r(Q + k) \subset Q/2$ for all $k \in A_N$. Hence,

$$U_a g(x) = f(a + b') - f(a) \qquad \text{if } x \in r(\varkappa Q + k) \text{ for some } k \in A_N, \tag{21}$$

$$U_a g(x) = f(a + b) - f(a) \qquad \text{if } x \in (Q/2) \setminus \bigcup_{k \in A_N} r(2\varkappa \operatorname{int}(Q) + k), \tag{22}$$

where $\operatorname{int}(Q)$ denotes the interior of Q. By the classical atomic characterization of Besov spaces [15, Thm. 3.1, p. 785], there exists $c_1 > 0$ such that

$$\left\| \sum_{k \in A_N} \varphi\left(\frac{1}{\varkappa}\left(\frac{\cdot}{r} - k\right)\right) \right\|_{B^s_{p,1}(\mathbb{R}^n)} \le c_1 r^{n/p-s} N^{n/p}. \tag{23}$$

Since $r = (6N)^{-1}$, we obtain

$$\|g\|_{\mathcal{B}^s_{p,1}(\mathbb{R}^n)} \le c_2(N^s |b' - b| + |b|). \tag{24}$$

Now we assume that

$$\max(|b|, |b - b'|) \le \frac{\delta_1}{2c_2}, \qquad b \ne b', \tag{25}$$

and we define N as follows:

$$N^s \le \frac{\delta_1}{2c_2|b - b'|} < (N+1)^s.$$

We note that the definition of N implies that

$$N^s \ge \frac{\delta_1}{2^{s+1} c_2 |b - b'|}. \tag{26}$$

If (25) holds, then the definition of N implies that $\|g\|_{\mathcal{B}^s_{p,1}(\mathbb{R}^n)} \le \delta_1$. Since the support of g is included in Q, Lemma 1 ensures that

$$\|U_a g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \le \delta_2. \tag{27}$$

Let $Q^+ := [0, 1/2]^n$. For any $x \in r(\varkappa Q^+ + k)$, we have

$$\begin{aligned} x + jr\varkappa e_1 &\in r(Q + k), & \forall j &= 0, \ldots, m, \\ x + jr\varkappa e_1 &\notin r(2\varkappa \operatorname{int}(Q) + k), & \forall j &= 1, \ldots, m. \end{aligned}$$

If $x \in r(\varkappa Q^+ + k)$, equalities (21) and (22) and formula (4.1) of Bennett and Sharpley [1, p. 332] for an mth order difference imply that

$$|\Delta^m_{r\varkappa e_1}(U_a g)(x)| = |f(a+b') - f(a+b)|.$$

By Proposition 8, there exist $c_3, c_4, c_5 > 0$ such that

$$\|U_a g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \geq c_3 (r\varkappa)^{-s} \Big(\sum_{k\in A_N} \int_{r(\varkappa Q^+ + k)} |\Delta^m_{r\varkappa e_1}(U_a g)(x)|^p \, dx \Big)^{1/p}$$
$$\geq c_4 |f(a+b') - f(a+b)|\, r^{-s} N^{n/p} r^{n/p} = c_5 N^s |f(a+b') - f(a+b)|.$$

By inequalities (26) and (27), we see that condition (25) implies that

$$|f(a+b) - f(a+b')| \leq \frac{2^{s+1}\delta_2 c_2}{c_5 \delta_1}\, |b-b'|,$$

which means that f is Lipschitz continuous in a neighbourhood of a.

We now prove (ii). If f acts boundedly from $(\mathcal{D}(\mathbb{B}), \| - \|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$ to $B^s_{p,\infty}(\mathbb{R}^n)$, we can define

$$\nu(R) := \sup\{\|f\circ g\|_{B^s_{p,\infty}(\mathbb{R}^n)} : g \in \mathcal{D}(\mathbb{B}),\ \|g\|_{\mathcal{B}^s_{p,1}(\mathbb{R}^n)} \leq R\} \quad \forall R > 0.$$

By an affine transformation, we can assume that $Q \subset \mathbb{B}$. We retain the same notation as in the proof of (i), except that we do not use Lemma 1. Let $\delta_1 := 2c_2$. By equality (20), the definition of ν, and Remark 9, there exist $c_6, c_7 > 0$ such that

$$\|V_a u\|_{B^s_{p,\infty}(\mathbb{R}^n)} \leq c_6 \nu(c_7(|a| + \|u\|_{\mathcal{B}^s_{p,1}(\mathbb{R}^n)})) \quad \forall u \in \mathcal{D}(\mathbb{B}).$$

Applying the above inequality to $u := g$ and arguing as for (i), we see that

$$|f(a+b) - f(a+b')| \leq \frac{2^s c_6}{c_5}\, \nu(c_7(R + 2c_2))|b-b'|$$

for any $|a| \leq R$ and any b, b' satisfying (25). ∎

REMARK 11. Up to a slight modification, the above also provides a new and simpler proof of the second assertion of the main theorem of [4].

REMARK 12. Theorem 2 remains valid, with the same proof, for complex-valued Besov and Lizorkin–Triebel spaces, and functions $f : \mathbb{C} \to \mathbb{C}$.

4. LOCAL LIPSCHITZ CONTINUITY PROPERTIES OF T_f

In this section, we analyze the conditions on f so that T_f is Lipschitz continuous on bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. By Theorem 2, any function in $\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$ is continuous. Hence, we can identify $\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$ with a space of distributions, more precisely with a subspace of $W^1_\infty(\mathbb{R})_{\mathrm{loc}}$. Moreover, we have the following.

PROPOSITION 9. *Let $s > 0$. The set $\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$ is a Fréchet space, continuously imbedded into $W^1_\infty(\mathbb{R})_{\mathrm{loc}}$.*

Proof. The continuity of the imbedding into $W^1_\infty(\mathbb{R})_{\rm loc}$ follows by Theorem 2. Thus it suffices to establish the completeness. Assume that $(f_k)_{k\ge 0}$ is a Cauchy sequence in $\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$. By the above imbedding, the sequence $(f_k)_{k\ge 0}$ has a limit f in $W^1_\infty(\mathbb{R})_{\rm loc}$, which we identify with its continuous representative. *A fortiori* $(f_k)_{k\ge 0}$ converges to f uniformly on every compact subset of $\mathbb{R}$. Assume that $g\in \mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Then the sequence $(f_k\circ g)_{k\ge 0}$ converges to $f\circ g$ in $L_\infty(\mathbb{R}^n)$. Moreover,

$$\sup_{k\ge 0}\|f_k\circ g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} \le \sup_{k\ge 0}\nu_{\|g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)}}(f_k)<\infty.$$

By the Fatou property of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, we obtain $f\circ g\in \mathcal{E}^s_{p,q}(\mathbb{R}^n)$ and the boundedness of T_f on bounded sets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Applying again the Fatou property, it is easily seen that

$$\lim_{k\to\infty}\nu_r(f_k-f)=0 \quad \text{for all } r>0. \blacksquare$$

By the above proposition, $\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$ can be identified with a Fréchet distribution space in $\mathbb{R}$. Then, for any $r\in\mathbb{N}$, the space $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$ can also be identified with a FDS (see the Appendix).

4.1. A sufficient condition for local Lipschitz continuity of T_f

THEOREM 3. *Let $s>0$. If $f:\mathbb{R}\to\mathbb{R}$ is a continuous function of class $W^1(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$, then T_f is Lipschitz continuous on any bounded set, as a mapping of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself.*

Proof. For simplicity, we set $E:=\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, $\Phi:=\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n))$, and we denote by $\|-\|$ the norm in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$.

Let $g,h\in E$. By Theorem 2, f is continuously differentiable. Thus

$$(f\circ(g+h)-f\circ g)(x)=\int_0^1 (f'\circ(g+th))(x)h(x)\,dt \quad \forall x\in\mathbb{R}^n. \tag{28}$$

We wish to interpret the above formula as a vector-valued integral in E. The difficulty here is to justify vector-valued measurability with respect to t. To overcome it, we shall exploit Proposition 6. We consider $u\in\mathcal{D}(\mathbb{R}^n)$ with norm equal to 1 in $L_1(\mathbb{R}^n)+E^{-s}_{p',q'}(\mathbb{R}^n)$. Then the Fubini theorem gives us the formula

$$\langle f\circ(g+h)-f\circ g,u\rangle=\int_0^1\langle (f'\circ(g+th))h,u\rangle\,dt.$$

Since E is a Banach algebra, we have

$$\|f\circ(g+h)-f\circ g\|\le c\nu_{\|g\|+\|h\|}(f')\|h\|, \tag{29}$$

which means that T_f is Lipschitz continuous on any ball of E. $\blacksquare$

4.2. A necessary condition for local Lipschitz continuity of T_f

THEOREM 4. *Let $s > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. If T_f is Lipschitz continuous from compact subsets of*

$$(\mathcal{D}(\mathbb{B}), \| - \|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$$

to $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ for all balls $\mathbb{B}$ of $\mathbb{R}^n$, then $f \in E^{s+1}_{p,q}(\mathbb{R})_{\text{loc}}$.

Proof. We divide our proof into two steps.

STEP 1. Assume that f has support in a compact interval $[a, b]$. Let $u \in \mathcal{D}(\mathbb{R})$ be such that $u(x) = 1$ on $[a-1, b+1]$ and $u(x) = 0$ outside $[a-2, b+2]$. Then

$$(\tau_t f - f)u = \tau_t f - f \quad \forall t \in [-1, 1]. \tag{30}$$

Let v be a nonzero function in $\mathcal{D}(\mathbb{R}^{n-1})$, with support in a ball $\mathbb{B}_{n-1}$. Let $g \in \mathcal{D}(\mathbb{R}^n)$ be such that $g(x) = x_1$ for $x \in [a-3, b+3] \times \mathbb{B}_{n-1}$. By (30), we have

$$(\tau_t f - f) \otimes v = (f \circ \tau_{te_1} g - f \circ g)\,(u \otimes v) \quad \forall t \in [-1, 1]. \tag{31}$$

Now by assumption on f, T_f is Lipschitz continuous on the set

$$\{\tau_{te_1} g \,:\, t \in [-1, 1]\}.$$

By Proposition 6 and by formula (44) of the Appendix, we deduce that $\|\tau_{te_1} g - g\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)} = O(|t|)$ as $|t| \to 0$. Then by (31), Proposition 4, and Remark 9,

$$\|\tau_t f - f\|_{E^s_{p,q}(\mathbb{R})} = O(|t|), \quad |t| \to 0.$$

Since $E^s_{p,q}(\mathbb{R})$ is the dual of a BDS, Proposition 14 of the Appendix implies that $f' \in E^s_{p,q}(\mathbb{R})$. *A fortiori*, $f' \in L_p(\mathbb{R})$. Hence, f equals almost everywhere a continuous function. Since f has compact support, we have $f \in L_p(\mathbb{R})$. By standard properties of Besov and Lizorkin–Triebel spaces, we know that $E^r_{p,q}(\mathbb{R}) = \{v \in L_p(\mathbb{R}) \,:\, v' \in E^{r-1}_{p,q}(\mathbb{R})\}$ for all $r > 0$. Hence, $f \in E^{s+1}_{p,q}(\mathbb{R})$.

STEP 2. We now turn to the general case. We want to prove that $uf \in E^{s+1}_{p,q}(\mathbb{R})$ for all $u \in \mathcal{D}(\mathbb{R})$. We can clearly assume that $f(0) = 0$. By Proposition 1(ii) and by Theorem 3 we know that $T_{u-u(0)-u'(0)\mathrm{id}_{\mathbb{R}}} = T_{u-u(0)} - T_{u'(0)\mathrm{id}_{\mathbb{R}}}$ is Lipschitz continuous on the bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Since the same holds for $T_{u'(0)\mathrm{id}_{\mathbb{R}}}$, the operator $T_{u-u(0)}$ is Lipschitz continuous on bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Since $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ is a Banach algebra, the identity

$$T_{(u-u(0))f}(g) = T_{u-u(0)}(g) T_f(g) \quad \forall g \in \mathcal{D}(\mathbb{R}^n) \tag{32}$$

and our assumptions on T_f imply that $T_{(u-u(0))f}$ is Lipschitz continuous from compact subsets of $(\mathcal{D}(\mathbb{B}), \|-\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$ to $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ for all balls $\mathbb{B}$. Then again our assumption on T_f implies that the same holds for $T_{uf} = u(0)T_f + T_{(u-u(0))f}$, and thus the conclusion follows by Step 1. ∎

4.3. A characterization of locally Lipschitz continuous superposition operators. We have the following necessary and sufficient condition on f for the Lipschitz continuity of T_f on bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$.

THEOREM 5. *Assume that $(s,p,q) \in \mathcal{I}_{n,E}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. Then T_f is Lipschitz continuous on bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ if and only if $f \in E^{s+1}_{p,q}(\mathbb{R})_{\mathrm{loc}}$.*

Proof. The necessity of the condition $f \in E^{s+1}_{p,q}(\mathbb{R})_{\mathrm{loc}}$ follows from Theorem 4.

We now turn to sufficiency. We assume that $f \in E^{s+1}_{p,q}(\mathbb{R})_{\mathrm{loc}}$ and that $f(0) = 0$. By the Sobolev imbedding theorem, f is of class C^1. Let $u := f - f'(0)\mathrm{id}_{\mathbb{R}}$. By the well known equality

$$E^{s+r}_{p,q}(\mathbb{R}^n) = W^r(E^s_{p,q}(\mathbb{R}^n)) \quad \forall r \in \mathbb{N}, \tag{33}$$

and by Proposition 13 of the Appendix, u and u' belong to $E^s_{p,q}(\mathbb{R})_{\mathrm{loc}}$. From $u(0) = u'(0) = 0$ and the assumption $(s,p,q) \in \mathcal{I}_{n,E}$, we obtain $u \in W^1(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$. By Theorem 3, T_u is Lipschitz continuous on bounded subsets of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. Since $T_f = T_u + f'(0)\mathrm{id}_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)}$, the same is true for T_f. ∎

REMARK 13. We note that partial results on the characterization of those f's for which T_f is locally Lipschitz continuous have been proved by Goebel and Sachweh [16] in the case of Schauder spaces on bounded intervals, and that such results would correspond here (on the whole space) to the case $n = 1$, $s > 1$, s noninteger, $p = q = \infty$. We also note that a necessary and sufficient condition for Lipschitz continuity has been proved by Goebel and Sachweh [16] in the case of the action of T_f on the Schauder space of continuously differentiable functions with Lipschitz continuous highest order derivatives on a bounded interval, which is not a Besov space.

4.4. A degeneracy result for uniform continuity. We *conjecture* that, except for the trivial case in which f is an affine function, the operator T_f cannot be uniformly continuous in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, for any $s > 0$. Such a degeneracy result holds at least for $s > 1/p$.

THEOREM 6. *Assume that $s > 1/p$. Let $\| - \|$ be a norm on $\mathcal{D}(\mathbb{R}^n)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If T_f is uniformly continuous from $(\mathcal{D}(\mathbb{R}^n), \| - \|)$ to $B^s_{p,\infty}(\mathbb{R}^n)_{\mathrm{loc}}$, then f is an affine function.*

Proof. We employ the argument of [10, Thm. 8, p. 505]. We first assume that $1 < p < \infty$. Without loss of generality, we can assume that $f(0) = 0$ and $1/p < s < 1$. By assumption, the nonlinear operator

$$S(g) := (f \circ g)\varphi$$

is uniformly continuous from $(\mathcal{D}(\mathbb{R}^n), \|-\|)$ to $B^s_{p,\infty}(\mathbb{R}^n)$. Define $g_{a,b} : \mathbb{R} \to \mathbb{R}$

by

$$g_{a,b}(x) := (ax_1 + b)\varphi(x)$$

for all real numbers a, b. Then there exists $\eta > 0$ such that

$$\|S(g_{a,b}) - S(g_{a,0})\|_{B^s_{p,\infty}(\mathbb{R}^n)} \le 1 \quad \forall b \in [-\eta, \eta], \forall a \in \mathbb{R}. \tag{34}$$

By exploiting the norm on $B^s_{p,\infty}(\mathbb{R}^n)$ of Proposition 8, it follows that there exists $c_1 > 0$ such that

$$\int_{Q/2} |f(a(x_1 + t) + b) - f(a(x_1 + t)) - f(ax_1 + b) + f(ax_1)|^p \, dx \le c_1 |t|^{sp}$$

for all $b \in [-\eta, \eta]$, $t \in [-1/4, 1/4]$ and $a \in \mathbb{R}$. By an obvious change of variables, we obtain

$$\int_{-a/4}^{a/4} |f(x + t + b) - f(x + t) - f(x + b) + f(x)|^p \, dx \le c_2 |t|^{sp} a^{1-sp}$$

for all $b \in [-\eta, \eta]$, $a > 0$ and $t \in [-a/4, a/4]$.

Now fixing $t \in \mathbb{R}$, letting $a \to \infty$, and exploiting the continuity of f, we deduce that

$$f(x + t + b) - f(x + t) - f(x + b) + f(x) = 0$$

for all $b \in [-\eta, \eta]$ and $x, t \in \mathbb{R}$. By taking $x = 0$, we obtain

$$f(t + b) = f(t) + f(b) \quad \forall b \in [-\eta, \eta], \forall t \in \mathbb{R}.$$

Then a standard argument shows that $f(t) = f(1)t$ for all $t \in \mathbb{R}$.

In case $p = \infty$, we can assume $0 < s < 1$. By inequality (34), we have

$$|f(a(x_1 + t) + b) - f(a(x_1 + t)) - f(ax_1 + b) + f(ax_1)| \le c_1 |t|^s$$

for all $b \in [-\eta, \eta]$, $t \in [-1/4, 1/4]$, $x \in Q/2$ and $a \in \mathbb{R}$. Hence,

$$|f(x + t + b) - f(x + t) - f(x + b) + f(x)| \le c_2 |t|^s a^{-s}$$

for all $b \in [-\eta, \eta]$, $a > 0$ and $x, t \in [-a/4, a/4]$. By letting $a \to \infty$, we can conclude as in case $p < \infty$.

In case $p = 1$, we take $1 < s < 2$ and we replace first order difference operators by second order ones in the above proof, to conclude that there exists $\eta > 0$ such that $\Delta_t^3 f = 0$ for all $t \in [-\eta, \eta]$. By formula (4.13) of Bennett and Sharpley [1, p. 335], we can easily deduce that

$$u''' = \lim_{t \to 0+} t^{-3} \Delta_t^3 u \quad \text{in } \mathcal{D}(\mathbb{R}) \quad \forall u \in \mathcal{D}(\mathbb{R}),$$

and we conclude that $f''' = 0$ in the sense of distributions. Hence, f is a polynomial of degree at most 2. If f is of degree 2, we deduce that the above operator S, with f replaced by $x \mapsto x^2$, is uniformly continuous from

$(\mathcal{D}(\mathbb{R}^n), \|-\|)$ to $B^s_{p,\infty}(\mathbb{R}^n)$. Arguing as above, we can show that there exists $\eta > 0$ such that

$$\|g^2_{a,\eta}\varphi - g^2_{a,0}\varphi\|_{B^s_{p,\infty}(\mathbb{R}^n)} \le 1 \qquad \forall a \in \mathbb{R}.$$

By setting $\psi(x) := x_1\varphi^3(x)$, we obtain

$$|a| \le \frac{1+\eta^2\|\varphi^3\|_{B^s_{p,\infty}(\mathbb{R}^n)}}{2\eta\|\psi\|_{B^s_{p,\infty}(\mathbb{R}^n)}} \qquad \forall a \in \mathbb{R},$$

a contradiction. ■

5. SUPERPOSITION OPERATORS OF CLASS C^r

5.1. A sufficient condition for regularity of the superposition operator. Let $r \in \mathbb{N}$. By Proposition 1, any function $f \in C^\infty(\mathbb{R})$ such that $f^{(j)}(0) = 0$ for $j = 0, \ldots, r$ belongs to $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$. Thus the main assumption of the following theorem makes sense.

THEOREM 7. *Assume that $r \in \mathbb{N}$ and $s > 0$. If a continuous function $f : \mathbb{R} \to \mathbb{R}$ belongs to the closure of $C^\infty(\mathbb{R}) \cap W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$ in $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$, then T_f is of class C^r as a mapping of $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself.*

Proof. We use the same notation as in the proof of Theorem 3. Since E is a Banach algebra, we can introduce the continuous linear mapping

$$M : E \to \mathcal{L}(E, E), \qquad u \mapsto \{v \mapsto uv\}.$$

We fix some $g \in E$ and assume that h is any function in E with $\|h\| \le 1$. Then we divide our proof into three steps.

STEP 1: *continuity of T_f, the case $f \in C^\infty(\mathbb{R}) \cap \Phi$.* We write

$$T_f = T_{f-f'(0)\mathrm{id}_\mathbb{R}} + f'(0)\mathrm{id}_E.$$

By Proposition 1(ii), we have $f - f'(0)\mathrm{id}_\mathbb{R} \in W^1(\Phi)$. Then the continuity of T_f follows by Theorem 3 applied to $f - f'(0)\mathrm{id}_\mathbb{R}$.

STEP 2: *continuity of T_f, the general case.* If f belongs to the closure of $C^\infty(\mathbb{R}) \cap \Phi$ in Φ, and $\varepsilon > 0$, then there exists $f_1 \in C^\infty(\mathbb{R}) \cap \Phi$ such that

$$\nu_{\|g\|+1}(f - f_1) \le \varepsilon.$$

Then by the triangle inequality, we have

$$\|f \circ (g+h) - f \circ g\| \le 2\varepsilon + \|f_1 \circ (g+h) - f_1 \circ g\|$$

and the continuity of T_f at g follows by Step 1 applied to f_1.

STEP 3. We now prove by induction on r that T_f is of class C^r if f belongs to the closure of $C^\infty(\mathbb{R}) \cap W^r(\Phi)$ in $W^r(\Phi)$. The case $r = 0$ has been considered in Step 2. We now assume that the statement holds for r, and we prove it for $r+1$. Assume that f belongs to the closure of $C^\infty(\mathbb{R}) \cap W^{r+1}(\Phi)$ in $W^{r+1}(\Phi)$. Since f is of class $C^{r+1}(\mathbb{R})$, we have

$$f \circ (g+h) - f \circ g - (f' \circ g)h = h \int_0^1 (f' \circ (g+th) - f' \circ g)\, dt. \tag{35}$$

Here the integral can be interpreted by duality, as in the proof of Theorem 3. By Step 2 we know that $T_{f'}$ is continuous on E. Since E is a Banach algebra, the same argument of the end of the proof of Theorem 3 and formula (35) yield the differentiability of T_f and the equality $dT_f = M \circ T_{f'}$. By the assumption on f, f' belongs to the closure of $C^\infty(\mathbb{R}) \cap W^r(\Phi)$ in $W^r(\Phi)$. Applying the inductive assumption, we conclude that $T_{f'}$ is of class C^r. Then dT_f is of class C^r as a mapping from E to $\mathcal{L}(E, E)$, which means that T_f is of class C^{r+1}. ■

Remark 14. Theorem 7 has been proved for $B^s_{\infty,\infty}(\mathbb{R}^n)$ in [8, 5]. The above proof is a simple transposition of [5, Subsection 7.1, pp. 69-70], which in turn is based on results of [17, pp. 467, 469–472], [19, pp. 927–932], and which introduces the notion of $\Phi(E)$. Theorem 7 could be deduced as well from an abstract result of [19, 18] by exploiting an argument of [8, Section 4].

Remark 15. In some cases, the sufficient condition of Theorem 7 turns out not to be necessary. Assume for instance that $0 < s < 1$ and $p < \infty$, $q < \infty$. Then by Proposition 9, any function which belongs to the closure of $C^\infty(\mathbb{R}) \cap \Phi(\mathcal{B}^s_{p,q}(\mathbb{R}^n))$ in $\Phi(\mathcal{B}^s_{p,q}(\mathbb{R}^n))$ is continuously differentiable. But the function $f(t) := |t|$ generates a continuous superposition operator on $\mathcal{B}^s_{p,q}(\mathbb{R}^n)$ (see [10, Subsection 2.1]).

5.2. A necessary condition for the regularity of the superposition operator

Theorem 8. *Let $r \in \mathbb{N}$ and $s > 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that T_f is of class C^r as a mapping from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself, then $f \in (\mathring{E}^{s+r}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$ (see Subsection 2.4).*

Proof. We argue as in [17, p. 474]. By Theorem 2, f is continuous. For convenience, we say that a nonlinear operator T has *property* $(\mathcal{P}_r)$ if T is a C^r mapping from $(\mathcal{D}(\mathbb{B}), \| - \|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$ to $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ for all balls $\mathbb{B}$ in $\mathbb{R}^n$. Then we have the following two lemmas.

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that T_f satisfies $(\mathcal{P}_1)$. Then f is continuously differentiable and*

$$dT_f(g).h = (f' \circ g)h \quad \forall g, h \in \mathcal{D}(\mathbb{B}(0, R)),\ \forall R > 0. \tag{36}$$

Proof of Lemma 2. By property $(\mathcal{P}_1)$, we have

$$dT_f(g).h = \lim_{t\to 0} \frac{f\circ(g+th) - f\circ g}{t} \tag{37}$$

in L_∞ norm. Then, by continuity of f, we deduce that $dT_f(g).h$ is a continuous function, and that the above convergence holds pointwise. Taking functions g, h such that $g(x) = x_1$ and $h(x) = 1$ on $\mathbb{B}(0, R/2)$, we obtain the existence and continuity of f' on $]-R/2, R/2[$, and thus on all of $\mathbb{R}$. Returning now to general functions g, h, we see that identity (37) implies equality (36). ∎

Lemma 3. *Let $s > 0$. A distribution $f \in \mathcal{D}'(\mathbb{R})$ belongs to $(\mathring{E}^{s+1}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$ if and only if both f and f' belong to $(\mathring{E}^{s}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$.*

Proof of Lemma 3. By Proposition 12 of the Appendix, we have

$$(\mathring{E}^{s}_{p,q}(\mathbb{R}))_{\mathrm{loc}} = \{f \in \mathcal{D}'(\mathbb{R}) : \lim_{x\to 0} \tau_x f = f \text{ in } E^s_{p,q}(\mathbb{R})_{\mathrm{loc}}\}. \tag{38}$$

By Proposition 13 of the Appendix, and by equality (33), we know that $E^{s+1}_{p,q}(\mathbb{R})_{\mathrm{loc}} = W^1(E^s_{p,q}(\mathbb{R})_{\mathrm{loc}})$ as Fréchet spaces. Then $f \in (\mathring{E}^{s+1}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$ if and only if $\lim_{x\to 0} \tau_x f = f$ in $E^{s+1}_{p,q}(\mathbb{R})_{\mathrm{loc}}$, a condition which holds if and only if both $\lim_{x\to 0} \tau_x f = f$ in $E^s_{p,q}(\mathbb{R})_{\mathrm{loc}}$ and $\lim_{x\to 0}(\tau_x f)' = f'$ in $E^s_{p,q}(\mathbb{R})_{\mathrm{loc}}$. Since $(\tau_x f)' = \tau_x(f')$, equality (38) implies the validity of the statement. ∎

We now go back to the proof of Theorem 8 and we claim the following. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that T_f has property $(\mathcal{P}_r)$, then f belongs locally to $\mathring{E}^{s+r}_{p,q}(\mathbb{R})$.*

We prove our claim by induction on r.

Step 1: *Case $r = 0$.* Assume that T_f has property $(\mathcal{P}_0)$.

Substep 1.1. Assume first that f has compact support. Then employing the same notation as in the proof of Theorem 4, and in particular formula (31), we obtain

$$\lim_{t\to 0} \|\tau_t f - f\|_{E^s_{p,q}(\mathbb{R})} = 0. \tag{39}$$

By (39), by the compactness of $\operatorname{supp} f$, and by Proposition 11 of the Appendix, we deduce that $f \in \mathring{E}^s_{p,q}(\mathbb{R})$.

Substep 1.2. If $\operatorname{supp} f$ is not necessarily compact, we argue as in Step 2 of the proof of Theorem 4. By Proposition 1(ii) and Theorem 7, $T_{u-u(0)}$ is continuous from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself. By identity (32) and the continuity of the product in $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$, $T_{(u-u(0))f}$ has property $(\mathcal{P}_0)$. Since $T_{uf} = T_{(u-u(0))f} + u(0)T_f$, the same is true for T_{uf}. By Substep 1.1, we conclude that $uf \in \mathring{E}^s_{p,q}(\mathbb{R})$.

STEP 2. Now we assume that our claim holds for an integer $r \in \mathbb{N}$, and that T_f is a C^{r+1} mapping from $(\mathcal{D}(\mathbb{B}), \|-\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)})$ to $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ for all balls $\mathbb{B}$ in $\mathbb{R}^n$.

We fix a ball $\mathbb{B}$, and we take $h \in \mathcal{D}(\mathbb{R}^n)$ such that $h(x) = 1$ on $\mathbb{B}$. By Lemma 2 applied to a ball larger than $\mathbb{B}$ and containing $\operatorname{supp} h$, we have

$$T_{f'-f'(0)}(g) = dT_f(g).h - f'(0)h \quad \forall g \in \mathcal{D}(\mathbb{B}).$$

By our assumption on f, the map $g \mapsto dT_f(g).h$ is of class C^r from $\mathcal{D}(\mathbb{B})$ equipped with the norm $\|-\|_{\mathcal{E}^s_{p,q}(\mathbb{R}^n)}$ to $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$. By the inductive assumption, we have $f' - f'(0) \in (\mathring{E}^{s+r}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$. Since constant functions belong to $(\mathring{E}^{s+r}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$, we see that both f and f' belong to $(\mathring{E}^{s+r}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$. By Lemma 3 we conclude that $f \in (\mathring{E}^{s+r+1}_{p,q}(\mathbb{R}))_{\mathrm{loc}}$. ∎

REMARK 16. In some cases, the necessary condition of Theorem 8 is not sufficient, as we now show by an example. Let $n = 1$, $0 < s < 1 + 1/p$, and $p, q < \infty$. Then $E^s_{p,q}(\mathbb{R}) = \mathring{E}^s_{p,q}(\mathbb{R})$. However, it is known that $E^s_{p,q}(\mathbb{R})$ contains functions which are not locally Lipschitz continuous, and by Theorem 2, such functions do not act on $\mathcal{E}^s_{p,q}(\mathbb{R})$ by superposition.

5.3. A characterization of C^r superposition operators

THEOREM 9. *Assume that $(s,p,q) \in \mathcal{I}_{n,E}$ and $r \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function such that $f(0) = 0$. Then the superposition operator T_f is a C^r map from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself if and only if f is continuous and belongs the closure of $C^\infty(\mathbb{R})$ in $E^{s+r}_{p,q}(\mathbb{R})_{\mathrm{loc}}$.*

Theorem 9 and Proposition 7 have the following consequence, which generalizes the corresponding result for Sobolev spaces W^1_p, obtained by Marcus and Mizel [20].

COROLLARY 2. *Assume that $(s,p,q) \in \mathcal{I}_{n,E}$ and $q < \infty$. Then, for any Borel measurable function f, the following three properties are equivalent*:

(i) *f acts on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$,*
(ii) *f acts boundedly on $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$,*
(iii) *T_f is a continuous operator from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself.*

Proof of Theorem 9. We divide our proof into three steps.

STEP 1. If T_f is a C^r mapping from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself, then Theorem 8 implies that $f \in (\mathrm{cl}_{E^{s+r}_{p,q}(\mathbb{R})}(\mathcal{D}(\mathbb{R})))_{\mathrm{loc}}$. By Proposition 12 of the Appendix, the latter space coincides with the closure of $C^\infty(\mathbb{R})$ in $E^{s+r}_{p,q}(\mathbb{R})_{\mathrm{loc}}$.

STEP 2. By assumption $s > 1/p$, we have the continuous imbedding

$$E^{s+r}_{p,q}(\mathbb{R})_{\mathrm{loc}} \hookrightarrow C^r(\mathbb{R}). \tag{40}$$

In particular, the set

$$\widetilde{E}^{s+r}_{p,q}(\mathbb{R}) := \{f \in E^{s+r}_{p,q}(\mathbb{R})_{\rm loc} : f(0) = \cdots = f^{(r)}(0) = 0\}$$

is a closed subspace of $E^{s+r}_{p,q}(\mathbb{R})_{\rm loc}$. By the assumption that $(s,p,q) \in \mathcal{I}_{n,E}$ and by Proposition 13 of the Appendix, $\widetilde{E}^{s+r}_{p,q}(\mathbb{R}) \subset W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$. By Proposition 9, $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$ is continuously imbedded in $W^r_\infty(\mathbb{R})_{\rm loc}$. Consequently, the closed graph theorem implies that $\widetilde{E}^{s+r}_{p,q}(\mathbb{R})$ is continuously imbedded in $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$.

Step 3. Let f be a continuous function of class $E^{s+r}_{p,q}(\mathbb{R})_{\rm loc}$ such that there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $C^\infty(\mathbb{R})$ with $f = \lim_{k\to\infty} f_k$ in $E^{s+r}_{p,q}(\mathbb{R})_{\rm loc}$. Since Taylor polynomials act on Banach algebras by superposition as operators of class C^∞, and since the imbedding (40) implies that $f^{(j)}(0) = \lim_{k\to\infty} f_k^{(j)}(0)$ for $j = 0,\ldots,r$, there is no loss of generality in assuming that $f, f_k \in \widetilde{E}^{s+r}_{p,q}(\mathbb{R})$. Then by Step 2, we have $f = \lim_{k\to\infty} f_k$ in $W^r(\Phi(\mathcal{E}^s_{p,q}(\mathbb{R}^n)))$. Since $f_k \in C^\infty(\mathbb{R})$ for all $k \in \mathbb{N}$, Theorem 7 implies that T_f is of class C^r from $\mathcal{E}^s_{p,q}(\mathbb{R}^n)$ to itself. ∎

6. APPENDIX

6.1. Properties of distribution spaces. A *distribution space* in $\mathbb{R}^n$ is a vector subspace of $\mathcal{D}'(\mathbb{R}^n)$. Let E be such a space. We say that E is a $\mathcal{D}(\mathbb{R}^n)$*-module* provided that $\psi f \in E$ for any $\psi \in \mathcal{D}(\mathbb{R}^n)$ and any $f \in E$. We say that E is a *topological distribution space* (a TDS) if E is endowed with a topology which renders E a topological vector space continuously imbedded in $\mathcal{D}'(\mathbb{R}^n)$. We say that E is a *Banach* or a *Fréchet distribution space* (a BDS or a FDS) if E is a Banach or a Fréchet TDS, respectively. A BDS $(E, \|-\|)$ is *translation invariant* if

$$\tau_x f \in E \quad \text{and} \quad \|\tau_x f\| = \|f\|$$

for all $f \in E$ and $x \in \mathbb{R}^n$. The following property follows from the closed graph theorem.

Proposition 10. *Let E be a FDS in $\mathbb{R}^n$, and a $\mathcal{D}(\mathbb{R}^n)$-module. If $\psi \in \mathcal{D}(\mathbb{R}^n)$, then the linear operator $f \mapsto \psi f$ from E to itself is continuous.*

In case E is a BDS, we denote by $\|\psi\|_{M(E)}$ the norm of the linear operator in E of Proposition 10.

We now recall the relation between the condition

$$\lim_{x\to 0} \|\tau_x f - f\|_E = 0 \tag{41}$$

and the approximability of f by smooth functions. We introduce a standard sequence $(\varrho_j)_{j\geq 1}$ of mollifiers, i.e.,

$$\varrho_j(x) := j^n \varrho(jx),$$

where ϱ is a positive smooth function on $\mathbb{R}^n$, with support in the unit ball, and such that $\int \varrho(x)\,dx = 1$. As is well known, the following proposition holds (cf. e.g. the Appendix of [10].)

PROPOSITION 11. *Let E be a translation invariant BDS.*

(i) *If $f \in E$ and f satisfies* (41), *then $f * \psi \in C^\infty(\mathbb{R}^n) \cap E$ for all $\psi \in \mathcal{D}(\mathbb{R}^n)$, and*

$$\lim_{j\to\infty} f * \varrho_j = f \quad \text{in } E.$$

(ii) *If $\mathcal{D}(\mathbb{R}^n) \subset E$, then any function f in the closure of $\mathcal{D}(\mathbb{R}^n)$ in E satisfies* (41).

If E is a distribution space in $\mathbb{R}^n$, we define E_{loc} as the set of $f \in \mathcal{D}'(\mathbb{R}^n)$ such that $f\psi \in E$ for all $\psi \in \mathcal{D}(\mathbb{R}^n)$. If E is a FDS, with a topology defined by a set of seminorms $\{N_k : k \in \mathbb{N}\}$, we endow E_{loc} with the seminorms

$$N_{\psi,k}(f) := N_k(\psi f)$$

for all $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Assume further that E is a $\mathcal{D}(\mathbb{R}^n)$-module. By using Proposition 10, it is easily seen that the set $\{N_{\psi,k} : \psi \in \mathcal{D}(\mathbb{R}^n),\ k \in \mathbb{N}\}$ can be replaced by a *countable* equivalent set of seminorms, for which E_{loc} turns out to be a FDS. Proposition 11 has a counterpart for the localized spaces.

PROPOSITION 12. *Let E be a translation invariant BDS in $\mathbb{R}^n$. Assume further that E is a $\mathcal{D}(\mathbb{R}^n)$-module and that $\mathcal{D}(\mathbb{R}^n) \subset E$. For any distribution f, the following properties are equivalent*:

(i) $f \in \mathrm{cl}_{E_{\text{loc}}}(C^\infty(\mathbb{R}^n))$,
(ii) $f \in (\mathrm{cl}_E \mathcal{D}(\mathbb{R}^n))_{\text{loc}}$,
(iii) $f \in (\mathrm{cl}_E(C^\infty(\mathbb{R}^n) \cap E))_{\text{loc}}$,
(iv) $\lim_{x\to 0} \tau_x f = f$ *in* E_{loc},
(v) $\lim_{j\to\infty} f * \varrho_j = f$ *in* E_{loc}.

Proof. Since $\mathcal{D}(\mathbb{R}^n) \subset E$, we have $C^\infty(\mathbb{R}^n) \subset E_{\text{loc}}$ and thus property (i) makes sense.

(i)⇒(ii). Let f satisfy (i), and let $\psi \in \mathcal{D}(\mathbb{R}^n)$. By assumption, there is a sequence (f_k) in $C^\infty(\mathbb{R}^n)$ which converges to f in E_{loc}. Accordingly, $\lim_{k\to\infty} \psi f_k = \psi f$ in E. Hence, $\psi f \in \mathrm{cl}_E \mathcal{D}(\mathbb{R}^n)$. So property (ii) holds.

(ii)⇒(iii) follows immediately from $\mathcal{D}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \cap E$.

(iii)⇒(iv). Let f satisfy (iii). We first prove that

$$\lim_{x\to 0} \|(\tau_x f)\psi - \tau_x(f\psi)\|_E = 0, \tag{42}$$

$$\lim_{x\to 0} \|\tau_x(f\psi) - f\psi\|_E = 0, \tag{43}$$

for every $\psi \in \mathcal{D}(\mathbb{R}^n)$, which implies (iv).

To prove (42), let $\theta \in \mathcal{D}(\mathbb{R}^n)$ be such that $\theta(x) = 1$ on $\operatorname{supp} \psi + \mathbb{B}(0,1)$. By translation invariance, we have

$$\|(\tau_x f)\psi - \tau_x(f\psi)\|_E = \|f\theta(\tau_{-x}\psi - \psi)\|_E \leq \|f\theta\|_E \|\tau_{-x}\psi - \psi\|_{M(E)}$$

for $|x| \leq 1$. Now by a standard argument (see [10, ineq. (36)]),

$$\lim_{x\to 0} \|\tau_x\psi - \psi\|_{M(E)} = 0.$$

To prove (43), note that by assumption, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^n) \cap E$ such that $\lim_{k\to\infty} g_k = f\psi$ in E. Then $\lim_{k\to\infty} g_k\theta = f\psi$ in E, which proves that $f\psi \in \mathrm{cl}_E\mathcal{D}(\mathbb{R}^n)$. Then (43) follows from Proposition 11(ii).

(iv)⇒(v). Let f satisfy (iv). Let $\psi, \theta \in \mathcal{D}(\mathbb{R}^n)$ be as above. By (iv), and arguing as in the preceding step, we obtain $\lim_{x\to 0} \tau_x(\theta f) = \theta f$ in E. By Propositions 10 and 11, we deduce that

$$\lim_{j\to\infty} \psi(\theta f * \varrho_j) = \psi f \quad \text{ in } E.$$

Then (v) follows from the identity $\psi(f * \varrho_j) = \psi(f\theta * \varrho_j)$.

(v)⇒(i). Since $f * \varrho_j \in C^\infty(\mathbb{R})$, the assertion is immediate. ∎

If E is a FDS, with a topology defined by a set of seminorms $\{N_k : k \in \mathbb{N}\}$, and if $r \in \mathbb{N}$, then the Sobolev space $W^r(E)$ endowed with the seminorms

$$N_{r,k}(f) := \sum_{|\alpha|\leq r} N_k(f^{(\alpha)}) \quad \forall k \in \mathbb{N}$$

is a FDS.

Proposition 13. *Let E be a FDS in $\mathbb{R}^n$, and $r \in \mathbb{N}$. If E is a $\mathcal{D}(\mathbb{R}^n)$-module, then so is $W^r(E)$ and*

$$W^r(E)_{\mathrm{loc}} = W^r(E_{\mathrm{loc}}),$$

with the same Fréchet topology.

Proof. All properties follow by an inductive argument on r based on the Leibniz formula and on the closed graph theorem. ∎

Let E be a BDS in $\mathbb{R}^n$ such that $\mathcal{D}(\mathbb{R}^n)$ is a dense subspace of E. Then the strong dual of E can be identified with a BDS in $\mathbb{R}^n$, namely the set of $f \in \mathcal{D}'(\mathbb{R}^n)$ such that there exists $A > 0$ satisfying

$$|\langle f, u\rangle| \leq A\|u\|_E \quad \forall u \in \mathcal{D}(\mathbb{R}^n).$$

If E is also translation invariant, then the BDS E' is also translation invariant. The following criterion gives an easy characterization of the Sobolev space $W^1(E')$.

PROPOSITION 14. *Let E be a translation invariant BDS in $\mathbb{R}^n$ such that $\mathcal{D}(\mathbb{R}^n)$ is a dense subspace of E. If $f \in \mathcal{D}'(\mathbb{R}^n)$, then the following two properties are equivalent.*

(i) *$\partial_j f$ belongs to E' for $j = 1, \ldots, n$.*
(ii) *$\tau_x f - f \in E'$ for all $x \in \mathbb{R}^n$, and $\|\tau_x f - f\|_{E'} = O(|x|)$ as $|x| \to 0$.*

Proof. Proposition 14 is an immediate consequence of the following formulas, which hold for an arbitrary function u in $\mathcal{D}(\mathbb{R}^n)$:

$$\langle \tau_x f - f, u\rangle = -\sum_{j=1}^{n} x_j \int_0^1 \langle \partial_j f, \tau_{-tx} u\rangle \, dt \qquad \forall x \in \mathbb{R}^n, \tag{44}$$

$$\langle \partial_j f, u\rangle = \lim_{t\to 0} \frac{1}{t} \langle f - \tau_{te_j} f, u\rangle. \ \blacksquare \tag{45}$$

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.
[2] G. Bourdaud, *Sur les opérateurs pseudo-différentiels à coefficients peu réguliers*, Thèse, Univ. Paris-Sud, Orsay, 1983.
[3] —, *La trivialité du calcul fonctionnel dans l'espace $H^{3/2}(\mathbb{R}^4)$*, C. R. Acad. Sci. Paris Sér. I 314 (1992), 187–190.
[4] —, *Fonctions qui opèrent sur les espaces de Besov et de Triebel*, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), 413–422.
[5] —, *Superposition operators in Zygmund and BMO spaces*, in: Function Spaces, Differential Operators and Nonlinear Analysis, D. Haroske et al. (eds.), Birkhäuser, Basel, 2003, 59–74.
[6] —, *Une propriété de composition dans l'espace H^s*, C. R. Math. Acad. Sci. Paris 340 (2005), 221–224.
[7] —, *Une propriété de composition dans l'espace H^s (II)*, ibid. 342 (2006), 243–246.
[8] G. Bourdaud and M. Lanza de Cristoforis, *Functional calculus in Hölder–Zygmund spaces*, Trans. Amer. Math. Soc. 354 (2002), 4109–4129.
[9] G. Bourdaud, M. Lanza de Cristoforis and W. Sickel, *Functional calculus in BMO and related spaces*, J. Funct. Anal. 189 (2002), 515–538.
[10] —, —, —, *Superposition operators and functions of bounded p-variation II*, Nonlinear Anal. 62 (2005), 483–517.
[11] G. Bourdaud, M. Moussai and W. Sickel, *An optimal symbolic calculus on Besov algebras*, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), 949–956.
[12] —, —, —, *Towards sharp superposition theorems in Besov and Lizorkin–Triebel spaces*, Nonlinear Anal. (2008), to appear.
[13] B. E. J. Dahlberg, *A note on Sobolev spaces*, in: Proc. Sympos. Pure Math. 35, Vol. 1, Amer. Math. Soc., 1979, 183–185.
[14] J. Franke, *On the spaces $F^s_{p,q}$ of Triebel–Lizorkin type: Pointwise multipliers and spaces on domains*, Math. Nachr. 125 (1986), 29–68.
[15] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 84 (1985), 777–799.

[16] M. Goebel and F. Sachweh, *On the autonomous Nemytskij operator in Hölder spaces*, Z. Anal. Anwendungen 18 (1999), 205–229.
[17] M. Lanza de Cristoforis, *Higher order differentiability properties of the composition and of the inversion operator*, Indag. Math. 5 (1994), 457–482.
[18] —, *Differentiability properties of a composition operator*, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998), 157–165.
[19] —, *Differentiability properties of an abstract autonomous composition operator*, J. London Math. Soc. 61 (2000), 923–936.
[20] M. Marcus and V. J. Mizel, *Every superposition operator mapping one Sobolev space into another is continuous*, J. Funct. Anal. 33 (1979), 217–229.
[21] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser. 1, Duke Univ., Durham, NC, 1976.
[22] T. Runst, *Mapping properties of non-linear operators in spaces of Triebel–Lizorkin and Besov type*, Anal. Math. 12 (1986), 313–346.
[23] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order*, *Nemytskiĭ Operators*, *and Nonlinear Partial Differential Equations*, de Gruyter, Berlin, 1996.
[24] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[25] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.
[26] M. Yamazaki. *A quasi-homogeneous version of paradifferential operators*, *I*: *Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), 131–174; *II: A symbolic calculus*, ibid. 33 (1986), 311–345.

Institut de Mathématiques de Jussieu
Projet d'analyse fonctionnelle
Case 186, 4 place Jussieu
75252 Paris Cedex 05, France
E-mail: bourdaud@ccr.jussieu.fr

Dipartimento di Matematica Pura
ed Applicata
Università di Padova
Via Trieste 63
35121 Padova, Italy
E-mail: mldc@math.unipd.it

Received May 2, 2007
Revised version October 9, 2007 (6150)