Lacunary series in $Q_K$ spaces

by

HASI WULAN (Shantou) and KEHE ZHU (Albany, NY, and Shantou)

Abstract. Under mild conditions on the weight function $K$ we characterize lacunary series in the so-called $Q_K$ spaces.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Green’s function for $\mathbb{D}$ is given by

$$g(z, w) = \log \frac{1}{|\sigma_w(z)|} = \log \frac{|1 - \overline{w}z|}{|w - z|},$$

where

$$\sigma_w(z) = \frac{w - z}{1 - \overline{w}z}$$

is a Möbius transformation of $\mathbb{D}$.

Given a function $K : (0, \infty) \to [0, \infty)$, we consider the space $Q_K$ of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{Q_K} = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, w)) \, dA(z) < \infty,$$

where $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$ and $dA$ is the Euclidean area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. It is easy to check that $\| \|_{Q_K}$ is a complete seminorm on $Q_K$ and it is Möbius invariant, that is,

$$\|f \circ \sigma\|_{Q_K} = \|f\|_{Q_K}, \quad \sigma \in \text{Aut}(\mathbb{D}),$$

where $\text{Aut}(\mathbb{D})$ is the group of all Möbius maps of the unit disk. Earlier studies on $Q_K$ spaces can be found in [8], [9], [15]–[18].

It is clear that each $Q_K$ contains all constant functions. If $Q_K$ consists of just the constant functions, we say that it is trivial. It follows from the general theory of Möbius invariant function spaces (see [2] for example) that

2000 Mathematics Subject Classification: Primary 30B10, 30H05; Secondary 46E15.

Key words and phrases: $Q_K$ spaces, lacunary series, $Q_p$ spaces.

The first author supported by NSF-China.

The second author partially supported by NSF-USA.
$Q_K$ is nontrivial if and only if it contains the coordinate function $z$, and in this case, $Q_K$ contains all polynomials.

From a change of variables we see that the coordinate function $z$ belongs to $Q_K$ if and only if

$$\sup_{w \in D} \left( \frac{(1-|w|^2)^2}{|1-wz|^4} K \left( \frac{1}{|z|} \right) \right) dA(z) < \infty.$$  

Simplifying the above integral in polar coordinates, we conclude that $Q_K$ is nontrivial if and only if

$$\sup_{t \in (0,1)} \left( \frac{(1-t)^2}{(1-tr^2)^3} K \left( \frac{1}{r} \right) \right) r dr < \infty. \quad (1)$$

Throughout the paper we always assume that condition (1) above is satisfied, so that the space $Q_K$ we study is nontrivial. Another standing assumption we make for the rest of the paper is that the weight function $K$ is nondecreasing.

An important tool in the study of $Q_K$ spaces is the auxiliary function $\varphi_K$ defined by

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$  

The following condition has played a crucial role in the study of $Q_K$ spaces during the last few years:

$$\int_1^\infty \varphi_K(s) \frac{ds}{s^2} < \infty. \quad (2)$$

See [9], [17], [18] for example. This condition will be crucial for us here as well. The main result of the paper is the following.

**Main Theorem.** If $K$ satisfies condition (2), then a lacunary series

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k}$$

belongs to $Q_K$ if and only if

$$\sum_{k=1}^\infty n_k |a_k|^2 K \left( \frac{1}{n_k} \right) < \infty.$$  

Recall that a function

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k}$$
is called a lacunary series if
\[ \lambda = \inf_k \frac{n_{k+1}}{n_k} > 1. \]
Such series are often used to construct examples of analytic functions in various function spaces.

A special case is worth mentioning. When \( K(t) = t^p, \) \( 0 \leq p < \infty, \) the resulting \( Q_K \) space is usually denoted by \( Q_p. \) It is well known that \( Q_p \) coincides with BMOA if \( p = 1, \) and \( Q_p \) is the Bloch space \( B \) if \( p > 1. \) We remind the reader that \( B \) consists of analytic functions \( f \) in \( D \) such that
\[ \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty. \]
The most interesting case is when \( 0 < p < 1; \) such \( Q_p \) spaces are distinct Möbius invariant Banach spaces that are strictly contained in BMOA. See [19] for the relatively new theory of \( Q_p \) spaces.

It is well known that a lacunary series belongs to BMOA if and only if it is in the Hardy space \( H^2; \) see [5] for example. It is also well known that a lacunary series is in the Bloch space if and only if its Taylor coefficients are bounded; see [20] for example. Lacunary series in \( Q_p \) are characterized in [4]. More specifically, if \( 0 \leq p \leq 1, \) then a lacunary series
\[ f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \]
is in \( Q_p \) if and only if
\[ \sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty. \]
Since the function \( K(t) = t^p \) satisfies condition (2) if and only if \( p < 1, \) our main result covers \( Q_p \) spaces for \( 0 \leq p < 1, \) but it misses the classical case of BMOA (corresponding to \( p = 1). \) Nevertheless, it should be clear from these remarks that condition (2) is very sharp.

2. Preliminaries on weight functions. The function theory of \( Q_K \) obviously depends on the properties of \( K. \) Given two weight functions \( K_1 \) and \( K_2, \) we write \( K_1 \lesssim K_2 \) if there exists a constant \( C > 0, \) independent of \( t, \) such that \( K_1(t) \leq CK_2(t) \) for all \( t. \) The notation \( K_1 \gtrsim K_2 \) is used in a similar fashion. When \( K_1 \lesssim K_2 \lesssim K_1, \) we write \( K_1 \approx K_2. \)

It is clear that \( K_1 \lesssim K_2 \) implies \( Q_{K_2} \subset Q_{K_1} \). In particular, \( K_1 \) and \( K_2 \) give rise to the same \( Q_K \) space whenever \( K_1 \approx K_2. \) The converse is false in general, as is demonstrated by the fact that \( Q_p \) equals the Bloch space for all \( p > 1. \)

In this section we collect several results about the weight functions that are needed for subsequent sections and are of some independent interest.
Although a few of the results in this section are buried in [8] and [9], we include proofs here for the sake of completeness and ease of reference.

**Lemma 1.** If

\[
K_1(t) = \begin{cases} 
K(t), & 0 < t \leq 1, \\
K(1), & 1 \leq t < \infty,
\end{cases}
\]

then \(Q_K = Q_{K_1}\).

**Proof.** Since \(K\) is nondecreasing, we have \(K_1 \leq K\), so \(Q_K \subset Q_{K_1}\). In particular, both spaces are nontrivial Möbius invariant spaces.

Since \(|K(\log(1/|z|))|\) is a radial function, integration in polar coordinates shows that \(f' \mapsto f'(0)\) is a bounded linear functional on any nontrivial \(Q_K\) space. By [12], each such space \(Q_K\) is contained in the Bloch space.

Fix a function \(f \in Q_{K_1}\) and consider the integrals

\[
I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z).
\]

We must show that \(I(a)\) is bounded for \(a \in \mathbb{D}\). To this end, we write \(I(a) = I_1(a) + I_2(a)\), where

\[
I_1(a) = \int_{|\varphi_a(z)| > e^{-1}} |f'(z)|^2 K(g(z, a)) \, dA(z),
\]

\[
I_2(a) = \int_{|\varphi_a(z)| \leq e^{-1}} |f'(z)|^2 K(g(z, a)) \, dA(z).
\]

It is clear that

\[
I_1(a) \leq \int_{\mathbb{D}} |f'(z)|^2 K_1(g(z, a)) \, dA(z),
\]

so there exists a positive constant \(C_1\) such that \(I_1(a) \leq C_1\) for all \(a \in \mathbb{D}\).

By a change of variables, we have

\[
I_2(a) = \int_{|\varphi_a(z)| \leq e^{-1}} |f'(\varphi_a(z))|^2 K\left(\log \frac{1}{|\varphi_a(z)|}\right) \, dA(z)
\]

\[
= \int_{|z| \leq e^{-1}} |f'(\varphi_a(z))|^2 K\left(\log \frac{1}{|z|}\right) \cdot \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} \, dA(z)
\]

\[
= \int_{|z| \leq e^{-1}} \frac{|f'(\varphi_a(z))|^2 (1 - |\varphi_a(z)|^2)^2}{(1 - |z|^2)^2} \cdot K\left(\log \frac{1}{|z|}\right) \, dA(z).
\]

Since \(f\) is in the Bloch space, we can find a constant \(C_2 > 0\) such that

\[
I_2(a) \leq C_2 \int_{|z| \leq e^{-1}} K\left(\log \frac{1}{|z|}\right) \, dA(z) \leq C_2 \int_{\mathbb{D}} K\left(\log \frac{1}{|z|}\right) \, dA(z).
\]
By condition (1), the last integral above is convergent, so there exists a constant $C_3 > 0$ such that $J_2(a) \leq C_3$ for all $a \in \mathbb{D}$. This shows that $I(a)$ is bounded in $a$, or equivalently, $f$ belongs to $\mathcal{Q}_K$. ■

The significance of Lemma 1 is that the space $\mathcal{Q}_K$ only depends on the behavior of $K(t)$ for $t$ close to 0. In particular, when studying $\mathcal{Q}_K$ spaces, we can always assume that $K(t) = K(1)$ for $t \geq 1$. However, we do not make this assumption in our main theorems.

**Lemma 2.** If $K$ satisfies condition (2), then the function

$$K^*(t) = t \int_1^\infty K(s) \frac{ds}{s^2}, \quad 0 < t < \infty,$$

has the following properties:

1. $K^*$ is nondecreasing on $(0, \infty)$.
2. $K^*(t)/t$ is nonincreasing on $(0, \infty)$.
3. $K^*(t) \geq K(t)$ for all $t \in (0, \infty)$.
4. $K^* \lesssim K$ on $(0, 1]$.

If $K(t) = K(1)$ for $t \geq 1$, then we also have

(v) $K^*(t) = K^*(1) = K(1)$ for $t \geq 1$, so $K^* \approx K$ on $(0, \infty)$.

**Proof.** If $t \in (0, 1]$, then a change of variables gives

$$K^*(t) = t \int_1^\infty K(s) \frac{ds}{s^2} = \int_1^\infty K(ts) \frac{ds}{s^2} = K(t) \int_1^\infty \frac{K(ts) ds}{s^2} \leq K(t) \int_1^\infty \varphi_K(s) \frac{ds}{s^2}.$$

So condition (2) implies that $K^*(t) \lesssim K(t)$ for $t \in (0, 1]$. This yields property (iv) and shows that $K^*(t)$ is well defined for all $t > 0$.

Since

$$\frac{K^*(t)}{t} = \int_t^\infty K(s) \frac{ds}{s^2}$$

and $K$ is nonnegative, we see that the function $K^*(t)/t$ is decreasing. This proves (ii). Property (v) follows from a direct calculation.

Using the assumption that $K$ is nondecreasing again, we obtain

$$K^*(t) = t \int_t^\infty K(s) \frac{ds}{s^2} \geq tK(t) \int_t^\infty \frac{ds}{s^2} = K(t)$$

for all $0 < t < \infty$. This proves property (iii).
It remains for us to show that $K^*$ is nondecreasing. To this end, we fix $0 < t < T < \infty$ and consider the difference

$$D = K^*(T) - K^*(t) = T \int_T^\infty \frac{K(s) \, ds}{s^2} - t \int_t^\infty \frac{K(s) \, ds}{s^2},$$

$$= (T - t) \int_T^\infty \frac{K(s) \, ds}{s^2} - t \int_t^T \frac{K(s) \, ds}{s^2}.$$

Since $K$ is nondecreasing and nonnegative, we have

$$D \geq (T - t)K(T) \int_T^\infty \frac{ds}{s^2} - tK(T) \int_t^T \frac{ds}{s^2} = 0.$$

This proves property (i) and completes the proof of the lemma. ■

Note that condition (2) is critically needed only in the proof of (iv). Without condition (2), properties (i), (ii), and (iii) remain valid, provided that $K^*$ is allowed to be identically infinite.

**Corollary 3.** If $K$ satisfies condition (2), then there exists a constant $C > 0$ such that $K(2t) \leq CK(t)$ for all $0 \leq 2t \leq 1$.

**Proof.** For any $t > 0$, we have

$$\frac{K^*(2t)}{K^*(t)} = 2 \int_t^{2t} \frac{K(s) \, ds}{s^2} \leq 2.$$

The desired estimate now follows from parts (iii) and (iv) of Lemma 2. ■

If we started out with a weight function $K$ with the property that $K(t) = K(1)$ for $t \geq 1$, then the conclusion of Corollary 3 could be strengthened to $K(2t) \approx K(t)$ for $t > 0$.

**Proposition 4.** If $K$ satisfies condition (2), then we can find another nonnegative weight function $K^*$ such that $Q_K = Q_{K^*}$ and that the new weight function $K^*$ has the following properties:

(a) $K^*$ is nondecreasing on $(0, \infty)$.
(b) $K^*$ satisfies condition (1).
(c) $K^*$ satisfies condition (2).
(d) $K^*(2t) \approx K^*(t)$ on $(0, \infty)$.
(e) $K^*$ is differentiable (up to any given order) on $(0, \infty)$.
(f) $K^*$ is concave on $(0, \infty)$.
(g) $K^*(t) = K^*(1)$ for $t \geq 1$. 
(h) $K^*(t)/t$ is nonincreasing on $(0, \infty)$.

(i) $K^*(t) \approx K(t)$ on $(0, 1]$.

**Proof.** By Lemma 1, we may assume that $K(t) = K(1)$ for all $t \geq 1$. Under this assumption, the function $K^*$ from Lemma 2 then satisfies $K^* \approx K$ on $(0, \infty)$. Moreover, properties (a), (b), (c), (g), (h), and (i) all hold.

Property (d) follows from the proof of Corollary 3.

If we repeat the construction $K \mapsto K^*$, then we can make the new weight function differentiable up to any desired order. So property (e) holds.

If the function $K$ is differentiable, which we may assume by property (e), then

$$
\frac{d}{dt} K^*(t) = \int K(s) \frac{ds}{s^2} - \frac{K(t)}{t} \quad \text{and} \quad \frac{d^2}{dt^2} K^*(t) = -\frac{K'(t)}{t} \leq 0.
$$

This shows that $K^*$ is concave on $(0, \infty)$ and completes the proof of the proposition. \qed

**Theorem 5.** If $K$ satisfies condition (2), then for any $\alpha > 0$ and $0 \leq \beta < 1$ we have

$$
\int_0^1 r^{\alpha-1} \left( \log \frac{1}{r} \right)^{-\beta} K \left( \log \frac{1}{r} \right) dr \approx C(\beta) \left( \frac{1-\beta}{\alpha} \right)^{1-\beta} K \left( \frac{1-\beta}{\alpha} \right),
$$

where $C(\beta)$ is a constant depending on $\beta$ alone.

**Proof.** Let

$$
I = \int_0^1 r^{\alpha-1} \left( \log \frac{1}{r} \right)^{-\beta} K \left( \log \frac{1}{r} \right) dr.
$$

By a change of variables,

$$
I = \int_0^{\infty} e^{-\alpha t} t^{-\beta} K(t) dt.
$$

We write $I = I_1 + I_2$, where

$$
I_1 = \int_0^{(1-\beta)/\alpha} e^{-\alpha t} t^{-\beta} K(t) dt, \quad I_2 = \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t} t^{-\beta} K(t) dt.
$$

Since $K$ is nondecreasing, we have

$$
I_1 \leq K \left( \frac{1-\beta}{\alpha} \right)^{(1-\beta)/\alpha} \int_0^{(1-\beta)/\alpha} e^{-\alpha t} t^{-\beta} dt.
$$
Making the change of variables $t = (1 - \beta)s/\alpha$, we obtain

$$I_1 \leq \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K\left(\frac{1 - \beta}{\alpha}\right) \int_0^1 e^{-(1-\beta)s}s^{-\beta} ds$$

$$= C(\beta) \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K\left(\frac{1 - \beta}{\alpha}\right).$$

By part (iii) of Lemma 2, we have

$$I_2 \leq \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t}t^{1-\beta} \frac{K^*(t)}{t} dt.$$  

According to part (ii) of Lemma 2, the function $K^*(t)/t$ is decreasing on $(0, \infty)$, so

$$I_2 \leq \frac{K^*((1 - \beta)/\alpha)}{(1 - \beta)/\alpha} \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t}t^{1-\beta} dt.$$  

A change of variables $(t = (1 - \beta)s/\alpha)$ in the integral above leads to

$$I_2 \leq \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K^*\left(\frac{1 - \beta}{\alpha}\right) \int_0^\infty e^{-(1-\beta)s}s^{1-\beta} ds.$$  

This together with part (iv) of Lemma 2 shows that

$$I_2 \lesssim C(\beta) \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K\left(\frac{1 - \beta}{\alpha}\right).$$

Combining this with what was proved in the previous paragraph, we have

$$I \lesssim C(\beta) \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K\left(\frac{1 - \beta}{\alpha}\right).$$

On the other hand,

$$I \geq \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t}t^{-\beta} K(t) dt.$$  

The assumption that $K$ is nondecreasing gives

$$I \geq K\left(\frac{1 - \beta}{\alpha}\right) \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t}t^{-\beta} dt.$$  

Make a change of variables according to $t = (1 - \beta)s/\alpha$. Then

$$I \geq C(\beta) \left(\frac{1 - \beta}{\alpha}\right)^{1-\beta} K\left(\frac{1 - \beta}{\alpha}\right).$$

This completes the proof of the theorem.
3. Lacunary series in $Q_K$. We begin with an estimate of the weighted Dirichlet integral in terms of Taylor coefficients.

**Theorem 6.** If $K$ satisfies condition (2) and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$\int_\mathbb{D} |f'(z)|^2 K\left( \log \frac{1}{|z|} \right) dA(z) \approx \sum_{n=1}^{\infty} n |a_n|^2 K\left( \frac{1}{n} \right).$$

**Proof.** Write

$$I(f) = \int_\mathbb{D} |f'(z)|^2 K\left( \log \frac{1}{|z|} \right) dA(z).$$

Integrating in polar coordinates leads to

$$I(f) = 2 \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^{\log \frac{1}{r}} r^{2n-1} K\left( \log \frac{1}{r} \right) dr.$$

We apply Theorem 5 with $\beta = 0$ and $\alpha = 2n$ to obtain

$$I(f) \approx \sum_{n=1}^{\infty} n |a_n|^2 K\left( \frac{1}{2n} \right).$$

The desired result then follows from Corollary 3.

We are now ready to prove the main result of the paper.

**Theorem 7.** If $K$ satisfies condition (2), then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $Q_K$ if and only if

$$\sum_{k=1}^{\infty} n_k |a_k|^2 K\left( \frac{1}{n_k} \right) < \infty. \quad (3)$$

**Proof.** First assume that

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

is a lacunary series in $Q_K$. Then

$$\int_\mathbb{D} |f'(z)|^2 K\left( \log \frac{1}{|z|} \right) dA(z) = \int_\mathbb{D} |f'(z)|^2 K(g(z,0)) dA(z) < \infty,$$

which, according to Theorem 6, implies condition (3).
Next assume that condition (3) holds. We proceed to estimate the integral
\[ I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z), \quad a \in \mathbb{D}. \]
As the first step, we show that for any \( a \in \mathbb{D} \),
\[ I(a) \leq 2^{\frac{1}{2}} \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \, dr. \]  
(4)
To this end, we write \( z = re^{i\theta} \) in polar form and observe that
\[ |f'(z)| \leq \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1}. \]
It follows that
\[ I(a) \leq 2^{\frac{1}{2}} \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 r \, dr \cdot \frac{1}{2\pi} \int_{0}^{2\pi} K(g(re^{i\theta}, a)) \, d\theta. \]
By Proposition 4, we may as well assume that \( K \) is concave. Then
\[ \frac{1}{2\pi} \int_{0}^{2\pi} K(g(re^{i\theta}, a)) \, d\theta \leq K \left( \frac{1}{2\pi} \int_{0}^{2\pi} g(re^{i\theta}, a) \, d\theta \right). \]
By Jensen’s formula, the integral
\[ \frac{1}{2\pi} \int_{0}^{2\pi} g(re^{i\theta}, a) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| \frac{1-\overline{a}re^{i\theta}}{re^{i\theta}-a} \right| \, d\theta \]
is equal to \( \log(1/|a|) \) for \( 0 < r \leq |a| \) and \( \log(1/r) \) for \( |a| < r < 1 \). In particular,
\[ \frac{1}{2\pi} \int_{0}^{2\pi} g(re^{i\theta}, a) \, d\theta \leq \log \frac{1}{r}. \]
From this we deduce inequality (4).

Our second step is to prove that inequality (4) implies
\[ I(a) \lesssim \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^{\frac{1}{2}} \frac{1}{2^n} K \left( \frac{1}{2^n} \right), \]  
(5)
where
\[ I_n = \{ k : 2^n \leq k < 2^{n+1}, \, k \in \mathbb{N} \}. \]
To this end, we combine the elementary estimates
\[ \sum_{n=0}^{\infty} 2^{n/2} r^{2^n} \leq \sqrt{2} \sum_{n=0}^{\infty} 2^{n+1} \int_{0}^{2^n} t^{-1/2} r^{t/2} \, dt \]
\[ \leq \sqrt{2} \int_{0}^{\infty} t^{-1/2} r^{t/2} \, dt = 2 \Gamma \left( \frac{1}{2} \right) \left( \log \frac{1}{r} \right)^{-1/2} \]
with the Cauchy–Schwarz inequality to produce
\[
\left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k} \right]^2 = \left[ \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k| r^{n_k} \right]^2 \leq \left[ \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k| r^{2n_k} \right]^2
\]
\[
\leq \left[ \sum_{n=0}^{\infty} 2^{n/2} r^{2n} \right] \left[ \sum_{n=0}^{\infty} 2^{-n/2} r^{2n} \left( \sum_{n_k \in I_n} n_k |a_k| \right)^2 \right]
\]
\[
\leq \frac{2 \Gamma(1/2)}{(\log(1/r))^{1/2}} \sum_{n=0}^{\infty} 2^{-n/2} r^{2n} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2.
\]
This together with (4) and Theorem 5 and Corollary 3 gives
\[
I(a) \leq 2 \int_0^1 \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k} \right]^2 K\left( \log \frac{1}{r} \right) dr
\]
\[
\lesssim \sum_{n=0}^{\infty} 2^{-n/2} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2 \int_0^{\frac{r^{2n-1}}{2}} \left( \log \frac{1}{r} \right)^{-1/2} K\left( \log \frac{1}{r} \right) dr
\]
\[
\lesssim \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2 \frac{1}{2^n} K\left( \frac{1}{2^n} \right).
\]
Thus, inequality (5) holds.

If \( n_k \in I_n \), then \( n_k < 2^{n+1} \). It follows from the monotonicity of \( K \) and Corollary 3 that
\[
\frac{1}{n_k} K\left( \frac{1}{n_k} \right) \geq \frac{1}{2^{n+1}} K\left( \frac{1}{2^{n+1}} \right) \geq \frac{1}{2^n} K\left( \frac{1}{2^n} \right).
\]
Combining this with (5), we obtain
\[
I(a) \lesssim \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \sqrt{\frac{1}{n_k} K\left( \frac{1}{n_k} \right)} \right]^2.
\]
(6)

Note that everything so far in the proof works for an arbitrary analytic function, not just for a lacunary series. Our final step, though, does make use of the fact that \( f \) is a lacunary series. More specifically, if
\[
\frac{n_{k+1}}{n_k} \geq \lambda > 1
\]
for all \( k \), then the Taylor series of \( f(z) \) has at most \( \lfloor \log_\lambda 2 \rfloor + 1 \) terms \( a_k z^{n_k} \) such that \( n_k \in I_n \) for \( n \in \mathbb{N} \). By (6) and Hölder’s inequality,
\[
I(a) \lesssim (\lfloor \log_\lambda 2 \rfloor + 1) \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k|^2 K\left( \frac{1}{n_k} \right)
\]
\[
= (\lfloor \log_\lambda 2 \rfloor + 1) \sum_{k=1}^{\infty} n_k |a_k|^2 K\left( \frac{1}{n_k} \right).
\]
This shows that condition (3) implies \( f \in Q_K \). The proof of the theorem is now complete. ■

4. Lacunary series in \( Q_{K,0} \). Let \( Q_{K,0} \) denote the subspace of \( Q_K \) consisting of functions \( f \) with

\[
\lim_{|a| \to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z) = 0.
\]

The following result together with Theorem 7 characterizes lacunary series in \( Q_{K,0} \).

**Theorem 8.** Let

\[
f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}
\]

be a lacunary series. If \( K \) satisfies condition (2), then \( f \in Q_K \) if and only if \( f \in Q_{K,0} \).

**Proof.** Suppose the lacunary series \( f \) belongs to \( Q_K \). We must show that \( I(a) \to 0 \) as \( |a| \to 1^- \), where

\[
I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z), \quad a \in \mathbb{D}.
\]

From the proof of Theorem 7, we know that \( f \in Q_K \) implies

\[
\frac{1}{r} \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \, dr < \infty.
\]

Thus for any given \( \varepsilon > 0 \) there exists some \( \sigma \in (0, 1) \) such that

\[
2 \frac{1}{\sigma} \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \, dr < \varepsilon.
\]

We may assume that

\[
\lim_{|a| \to 1^-} K \left( \log \frac{1}{|a|} \right) = 0.
\]

Otherwise, \( Q_K \) coincides with the Dirichlet space \( D \) (see [8]), and the desired result is obvious.

We write \( I(a) = I_1(a) + I_2(a) \), where

\[
I_1(a) = \int_{|z| < \sigma} |f'(z)|^2 K(g(z, a)) \, dA(z),
\]

\[
I_2(a) = \int_{\sigma \leq |z| < 1} |f'(z)|^2 K(g(z, a)) \, dA(z).
\]
By arguments used in the second paragraph of the proof of Theorem 7, we have
\[ I_1(a) \leq 2K \left( \log \frac{1}{|a|} \right) \int_0^\sigma \left[ \sum_{k=1}^\infty n_k |a_k|^r^{n_k-1} \right]^2 r \, dr \]
whenever \( \sigma < |a| < 1 \), because in this case
\[ \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) \, d\theta = \log \frac{1}{|a|}. \]
In particular, \( I_1(a) \to 0 \) as \( |a| \to 1^- \). Similarly, we have
\[ I_2(a) \leq 2 \left[ \sum_{k=1}^\infty n_k |a_k|^r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) r \, dr < \varepsilon. \]
It follows that
\[ \limsup_{|a| \to 1^-} I(a) \leq \varepsilon. \]
Since \( \varepsilon \) is arbitrary, we conclude that \( I(a) \to 0 \) as \( |a| \to 1^- \). So \( f \in \mathcal{Q}_{K,0} \) and the proof is complete.

Carefully checking the proof of Theorems 7 and 8, we also obtain the following sufficient condition for a function to be in \( \mathcal{Q}_{K,0} \) (and hence in \( \mathcal{Q}_K \)) in terms of Taylor coefficients.

**Theorem 9.** If \( K \) satisfies condition (2), and if
\[ f(z) = \sum_{n=0}^\infty a_n z^n \]
satisfies the condition
\[ \sum_{n=0}^\infty \left[ \sum_{k \in I_n} k |a_k| \right]^2 \frac{1}{2^n} K \left( \frac{1}{2^n} \right) < \infty, \]
then \( f \in \mathcal{Q}_{K,0} \).

**Proof.** We leave the details to the interested reader.

**References**


Department of Mathematics
Shantou University
Shantou, China
E-mail: wulan@stu.edu.cn

Department of Mathematics
SUNY
Albany, NY 12222, U.S.A.
E-mail: kzhu@math.albany.edu

Received October 31, 2005
Revised version December 4, 2006
(5788)