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# Moduli of smoothness of functions and their derivatives

by

Z. Ditzian (Edmonton) and S. Tikhonov (Pisa)

**Abstract.** Relations between moduli of smoothness of the derivatives of a function and those of the function itself are investigated. The results are for $L_p(T)$ and $L_p[-1,1]$ for $0<p<\infty$ using the moduli of smoothness $\omega^r(f,t)_p$ and $\omega^r_\varphi(f,t)_p$ respectively.

**1. Introduction.** For $f$, $f^{(k)} \in L_p(T)$, $1 \le p \le \infty$, the estimate (see [De-Lo, p. 46])

$$\omega^r(f,t)_p \le Ct^k\omega^{r-k}(f^{(k)},t)_p \quad \text{for } 1\le k\le r \tag{1.1}$$

and its weak inverse (see [De-Lo, p. 178]) given by

$$\omega^{r-k}(f^{(k)},t)_p \le C\int_0^t \frac{\omega^r(f,u)_p}{u^{k+1}}\,du \quad \text{for } 1\le k< r \tag{1.2}$$

are well-known. (We note that (1.2) is sometimes called a Marchaud-type inequality.) Here we extend the weak inverse (1.2) to the inequality, for $0<p<\infty$,

$$\omega^{r-k}(f^{(k)},t)_p \le C\biggl\{\int_0^t \frac{\omega^r(f,u)_p^q}{u^{qk+1}}\,du\biggr\}^{1/q}, \quad q=\min(p,2). \tag{1.3}$$

(For $p=\infty$ one still has only (1.2).) We recall that

$$\begin{gathered}\omega^r(f,t)_p = \sup_{|h|<t}\|\Delta^r_h f\|_p,\\ \Delta_h f(x)=f(x+h)-f(x),\quad \Delta^r_h f(x)=\Delta_h(\Delta^{r-1}_h f(x)).\end{gathered} \tag{1.4}$$

We note that (1.1) is not valid for $0<p<1$ (see [Pe-Po, p. 188]).

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For $1 < p \le 2$, Marcinkiewicz [Ma] proved

$$\|f'\|_p \le C\left\{\int_0^1 \frac{\omega^2(f,u)_p^p}{u^{p+1}}\,du\right\}^{1/p},$$

which is related to (1.3) and, as will be shown in Corollary 3.8, is a corollary of (1.3). For $1 < p < \infty$ the inequality (1.3) is related to the work of Besov [Be]. In the case of $L_p(T)$ our main result is when $0 < p < 1$ (which was not attempted earlier). We give the complete proof of (1.3) for $1 < p < \infty$ as well, since we use the same technique again for $\omega_\varphi^r(f,t)_p$ in Section 5 and we hope that it will have even further use.

The weighted $L_{p,w}[-1,1]$ is given by the norm or quasi-norm

$$\|f\|_{p,w} = \left\{\int_{-1}^1 |f(x)|^p w(x)^p\,dx\right\}^{1/p}, \quad 0 < p < \infty, \tag{1.5}$$

and

$$\|f\|_{\infty,w} = \operatorname*{ess\,sup}_{-1<x<1} |f(x)w(x)|.$$

The weighted moduli and main part moduli of smoothness $\omega_\varphi^r(f,t)_{p,w}$ and $\Omega_\varphi^r(f,t)_{p,w}$ (see also [Di-To]) are given for $\varphi(x)^2 = 1-x^2$ and $w(x) = \varphi(x)^\sigma$ $(\sigma \ge 0)$ by

$$\begin{aligned} \omega_\varphi^r(f,t)_{p,w} &\equiv \sup_{|h|\le t} \|\Delta_{h\varphi}^r f\|_{L_{p,w}[I]},\\ \Omega_\varphi^r(f,t)_{p,w} &\equiv \sup_{|h|\le t} \|\Delta_{h\varphi}^r f\|_{L_{p,w}[I(h,r)]} \end{aligned} \tag{1.6}$$

where

$$I(h,r) = [-1+2h^2r^2, 1-2h^2r^2], \qquad I = [-1,1],$$

and $\Delta_{h\varphi}^r f(x)$ is given by

$$\Delta_{h\varphi}^r f(x) = \begin{cases} \displaystyle\sum_{l=0}^r (-1)^l \binom{r}{l} f\left(x + \left(\frac{r}{2} - l\right)h\varphi(x)\right) & \text{for } x \pm (r/2)h\varphi(x) \in [-1,1],\\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

For $w(x) = 1$ $(\sigma = 0)$ we write

$$\omega_\varphi^r(f,t)_{p,1} \equiv \omega_\varphi^r(f,t)_p.$$

It is known (see [Di-To, Theorems 6.2.2 and 6.3.1]) that

$$\Omega_\varphi^r(f,t)_p \le Ct^k \omega_\varphi^{r-k}(f^{(k)},t)_{p,\varphi^k} \quad \text{for } 1 \le p \le \infty \tag{1.8}$$

and

$$\Omega_\varphi^{r-k}(f^{(k)},t)_{p,\varphi^k} \le C\Big[\int_0^t \frac{\Omega_\varphi^r(f,u)_p}{u^{k+1}}\,du\Big] \quad \text{for } 1\le p\le\infty. \tag{1.9}$$

For $0<p<\infty$ we will show

$$\Omega_\varphi^{r-k}(f^{(k)},t)_{p,\varphi^k} \le C\Big[\int_0^t \frac{\omega_\varphi^r(f,u)_p^q}{u^{qk+1}}\,du\Big]^{1/q}, \qquad q=\min(p,2). \tag{1.10}$$

(For $p=\infty$ one has (1.9) or (1.10) with $p=\infty$ and $q=1$.) The inequality (1.8) does not hold for $0<p<1$.

For $1\le p\le\infty$ the $k$th derivative $f^{(k)}$ can be given as a distributional derivative or by assuming that the $(k-1)$th derivative in the classical sense satisfies $f^{(k-1)}\in \mathrm{A.C.}_{\mathrm{loc}}$. This is not possible for $0<p<1$ as $f\in L_p$ does not necessarily imply that $f$ is a distribution. Moreover, even if $f'\in L_p$ $(p<1)$, it does not imply that $f\in \mathrm{A.C.}_{\mathrm{loc}}$. In Section 2 we deal with $L_p(T)$ where $0<p<1$ and prove a result that will be useful for the proof of the inverse inequality. The sharp inverse inequality (1.3) is proved in Section 3. Analogous results to those in Section 2 are proved for $L_p[-1,1]$, $0<p<1$, in Section 4. The sharp converse (1.10) is proved in Section 5.

**2 Some positive and negative results for $L_p(T)$, $0<p<1$.** For $f\in L_p(T)$, $0<p\le\infty$, we define the derivative of $f$ as a function $g$ satisfying

$$\Big\|\frac{1}{h}\,(f(\cdot+h)-f(\cdot))-g(\cdot)\Big\|_{L_p(T)} \to 0 \quad \text{as } h\to 0, \tag{2.1}$$

in which case we write $g=f'$. (For $p\ge 1$, (2.1) is the commonly used strong derivative of $f$.) The $k$th derivative is given as usual as the $k$th iterate of the first derivative. When $f$ is locally absolutely continuous $(f\in \mathrm{A.C.}_{\mathrm{loc}})$ the definition in (2.1) coincides with the classical definition of a derivative. For $0<p<1$ the derivative in $L_p$ is problematic or, as Peetre described it, "pathological" (see [Pe]) even when it is the derivative of a function satisfying $f\in \mathrm{A.C.}_{\mathrm{loc}}$.

Some aspects of the behaviour of derivatives were described earlier (see for instance, [Pe], [Di-Hr-Iv], [Pe-Po] and [Di,95]). Here another aspect of this anomaly is described. This may serve as a warning to ourselves and others against using a certain type of argument which is absolutely acceptable when $1\le p\le\infty$. In the following example when we say $f'$ is a derivative of $f$, it will be in the most elementary sense $(f\in \mathrm{A.C.}_{\mathrm{loc}})$. We will prove our result for $[0,1]$ but similar outcomes occur on $[a,b]$ or $T$.

THEOREM 2.1. *When $0<p<1$ it is possible for $\varphi_n$ to converge to $f$ in $L_p[0,1]$, for $\varphi_n'$ to converge to $g$ in $L_p[0,1]$, and for $f'$ to exist and belong to $L_p[0,1]$, but $f'(x)\ne g(x)$.*

Remark 2.2. Other versions of Theorem 2.1 can be:

(I) When $0 < p < 1$ it is possible that $\varphi_n$ and $\psi_n$ converge to $f$ in $L_p[0,1]$, that $\varphi_n'$ converges to $g_1$ in $L_p[0,1]$, and that $\psi_n'$ converges to $g_2$ in $L_p[0,1]$, but $g_1 \neq g_2$ in $L_p[0,1]$.

(II) When $0 < p < 1$ it is possible that $\varphi_n$ and $\varphi_n'$ are Cauchy sequences in $L_p[0,1]$, and hence $\varphi_n \to f$, $\varphi_n' \to g$ but $g$ is not the derivative of $f$.

*Proof of Theorem 2.1.* We choose $f(x) = x$ and $\varphi_n(x)$ given by

$$\varphi_n(x) = \begin{cases} \dfrac{k}{n}, & \dfrac{k}{n} \le x < \dfrac{k+1}{n} - \dfrac{1}{n^2}, \\ \dfrac{k}{n} + \Big(x - \dfrac{k+1}{n} + \dfrac{1}{n^2}\Big)n, & \dfrac{k+1}{n} - \dfrac{1}{n^2} \le x < \dfrac{k+1}{n}, \end{cases}$$

for $k = 0, 1, \ldots, n-1$, and

$$\varphi_n'(x) = \begin{cases} n, & \dfrac{k+1}{2} - \dfrac{1}{n^2} \le x < \dfrac{k+1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

We clearly have $\|\varphi_n - f\|_p \le 1/n$ (where $\|F\|_p^p = \int_0^1 |F(x)|^p\,dx$) and

$$\|\varphi_n' - 0\|_p \le \Big(n \frac{1}{n^2} n^p\Big)^{1/p} = n^{(p-1)/p} \to 0.$$

We note that this example covers Remark 2.2 as well where we choose $\psi_n(x) = x$ and $\psi_n'(x) = 1$. ∎

For $1 \le p \le \infty$ the situation is different from what is described in Theorem 2.1, and for that reason some are inclined to believe in the opposite of that theorem. Under some restrictions on $\varphi_n$ and the rate of convergence we have $f' = g$ for $0 < p < 1$ as well, and Theorem 2.1 was given mainly to show that we need to prove the following result which will be useful in Section 3.

Theorem 2.3. *For $f \in L_p(T)$ and $T_n$ a sequence of trigonometric polynomials of degree $n$ satisfying*

$$(2.2) \qquad \|f - T_n\|_{L_p(T)} = o\Big(\frac{1}{n}\Big) \quad \text{and} \quad \|g - T_n'\|_{L_p(T)} = o(1), \qquad n \to \infty,$$

*we have $f' = g$, that is, $g$ satisfies* (2.1).

*Proof.* For $1 \le p \le \infty$ the theorem is a special case of known results and we prove it here only for $0 < p < 1$. For any $\varepsilon > 0$ we choose $n_0 = n_0(\varepsilon)$ such that for $n \ge n_0$,

$$\|f - T_n\|_{L_p(T)} \le \varepsilon \frac{1}{n} \quad \text{and} \quad \|g - T_n'\|_{L_p(T)} \le \varepsilon.$$

For $h$ satisfying $\sqrt{\varepsilon}/n \le h \le 1/n$ we have

$$\left\| \frac{f(\cdot+h)-f(\cdot)}{h} - \frac{T_n(\cdot+h)-T_n(\cdot)}{h} \right\|_{L_p(T)}^p \le 2\,\frac{\varepsilon^p}{\varepsilon^{p/2}} = 2\,\varepsilon^{p/2}. \tag{2.3}$$

Following [Di-Hr-Iv] (proof of Theorem 3.1 there), we have, for $\sqrt{\varepsilon}/n \le h \le 2\sqrt{\varepsilon}/n$,

$$\begin{aligned} \left\| \frac{T_n(\cdot+h)-T_n(\cdot)}{h} - T_n'(\cdot) \right\|_{L_p(T)}^p &\le \sum_{k=2}^{\infty} \left( \frac{h^{k-1}}{k!} \right)^p \|T_n^{(k)}\|_{L_p(T)}^p \\ &\le \sum_{k=2}^{\infty} (hn)^{(k-1)p} \|T_n\|_{L_p(T)}^p \\ &\le 4\varepsilon^p \, \frac{1}{1-2^p\varepsilon^{p/2}} \, \|T_n\|_{L_p(T)}^p \\ &\le C\varepsilon^p \|T_n\|_{L_p(T)}^p. \end{aligned} \tag{2.4}$$

We note that in (2.4) as well as in [Di-Hr-Iv] we utilized the important inequality by Arestov [Ar] who established that

$$\|T_n'\|_{L_p(T)} \le n\|T_n\|_{L_p(T)} \tag{2.5}$$

for $0 < p < 1$. Therefore, for $\sqrt{\varepsilon}/n \le h < 2\sqrt{\varepsilon}/n$ we have

$$\left\| \frac{f(\cdot+h)-f(\cdot)}{h} - g \right\|_{L_p(T)}^p \le C(\varepsilon^{p/2} + \varepsilon^p \|f\|_{L_p(T)} + \varepsilon^p), \tag{2.6}$$

and as the right-hand side does not depend on $n$ or $T_n$, we have $g = f'$. ■

Repeating the process in Theorem 2.3, we obtain the following corollary.

COROLLARY 2.4. *Suppose $f, g_1, \dots, g_k \in L_p(T)$ and $T_n$ is a sequence of trigonometric polynomials satisfying*

$$\begin{aligned} \|f - T_n\|_p &= o\left(\frac{1}{n^k}\right), \qquad n \to \infty, \\ \|g_i - T_n^{(i)}\|_p &= o\left(\frac{1}{n^{k-i}}\right), \qquad n \to \infty, \ \text{for } i = 1, \dots, k. \end{aligned} \tag{2.7}$$

*Then $g_i = g_{i-1}'$ $(f = g_0)$ in the sense of* (2.1).

**3. Functions in $L_p(T)$.** For $L_p(T)$ our estimate of $\omega^{r-k}(f^{(k)}, t)_p$ is given in the following theorem.

THEOREM 3.5. *For $f \in L_p(T)$, $0 < p < \infty$, and integers $k, r$ satisfying $k < r$ we have*

$$\omega^{r-k}(f^{(k)}, t)_p \le C \left\{ \int_0^t \frac{\omega^r(f,u)_p^q}{u^{qk+1}}\, du \right\}^{1/q} \tag{3.1}$$

*where $q = \min(p, 2)$.*

REMARK 3.6. (I) In Theorem 3.5 inequality (3.1) means that if its right-hand side converges, then $f^{(k)}$ exists in the sense of (2.1) (or for $p \ge 1$ as a distribution) and satisfies both $f^{(k)} \in L_p(T)$ and inequality (3.1).

(II) For $p = \infty$ (and $p = 1$) we have $q = 1$, which is the classical result (1.2).

(III) For $1 < p < 2$, $r = 2$ and $k = 1$, (3.1) is essentially proved by Marcinkiewicz in [Ma]. For $1 < p < \infty$, (3.1) is related to a result of Besov on the rate of best approximation by trigonometric polynomials (see [Be]).

(IV) For $1 < p < \infty$, (3.1) is actually stronger than (1.2). This is illustrated by examining cases for which $\omega^r(f,t)_p \le Mt^k/|\log t|^\alpha$ for $t < 1/2$. In such a situation we need $\alpha > 1$ for (1.2) to converge but only $\alpha > 1/q$ for (3.1) to converge. Moreover, in this case using (3.1) ($\alpha q > 1$), we obtain

$$\omega^{r-k}(f^{(k)},t)_p \le M_1 \frac{1}{|\log t|^{\alpha-1/q}} \quad \text{for } t < \frac{1}{2},$$

but using (1.2) (for $\alpha > 1$), we have only

$$\omega^{r-k}(f^{(k)},t)_p \le M_1 \frac{1}{|\log t|^{\alpha-1}} \quad \text{for } t < \frac{1}{2}.$$

We note that if $\omega^r(f,t)_p = O(t^{k+\beta})$ for some $\beta > 0$, then (3.1) does not have an advantage over (1.2) for $1 < p < \infty$. While proving (3.1) for $1 \le p < \infty$, we show that it implies (1.2) for $1 \le p < \infty$.

(V) For $0 < p < 1$ no inverse inequality was proved earlier, and in fact to call it an inverse result is a misnomer since the direct result (1.1) is not valid when $0 < p < 1$ (see [Pe-Po, p. 188]).

(VI) As an example of the use of (3.1) for $0 < p < 1$ we set $f(x) = x^{r-1}\operatorname{sgn} x$ for $|x| < \pi$ and define $f(x)$ by $f(x+2\pi) = f(x)$ elsewhere. We have $\omega^r(f,t)_p \approx t^{r-1+1/p}$ (see [Pe-Po, p. 188]). For $k < r$, $f^{(k)}(x) = (\Gamma(r)/\Gamma(r-k))x^{r-k-1}\operatorname{sgn} x$ when $|x| < \pi$ and hence $\omega^{r-k}(f^{(k)},t)_p \approx t^{r-k-1+1/p}$ as expected by (3.1). For instance, if $p = 1/2$, $r = 2$ and $k = 1$, we have $\omega^2(f,t)_p \approx t^3$ and $\omega(f',t)_p \approx t^2$.

*Proof of Theorem 3.5.* Since $\omega^m(F,t)_p$ is nondecreasing and

$$\omega^m(F,2t)_p \le 2^m \omega^m(F,t)_p \quad \text{for } 1 \le p \le \infty \tag{3.2}$$

while (see [Pe-Po, p. 187])

$$\omega^m(F,2t)_p \le C(m,p)\omega^m(F,t)_p \quad \text{for } 0 < p < 1, \tag{3.3}$$

it is sufficient to prove (2.1) for $t = 2^{-n}$. Using (3.2) and (3.3), we also have

$$\Big\{ \int_0^{2^{-n}} \frac{\omega^r(f,u)_p^q}{u^{qk+1}}\,du \Big\}^{1/q} \approx \Big\{ \sum_{l=n}^{\infty} 2^{lqk}\omega^r(f,2^{-l})_p^q \Big\}^{1/q}, \tag{3.4}$$

and hence it is sufficient to prove that for all $n$,

$$\omega^{r-k}(f^{(k)},2^{-n})_p \le C\Big\{\sum_{l=n}^{\infty} 2^{lqk}\omega^r(f,2^{-l})_p^q\Big\}^{1/q}. \tag{3.5}$$

Inequality (3.5) demonstrates that (3.1) for $1<p<\infty$ is stronger than (1.2) as the $l_1$ norm of the sequence $\{2^l\omega^r(f,2^{-l})_p\}_{l=n}^{\infty}$ is bigger than the $l_q$ norm of that sequence. (That (3.1) is actually stronger in some cases was shown in Remark 3.2(IV).)

Since for any trigonometric polynomial $Q_n$ of degree $cn$ we have

$$\omega^r(Q_n,u)_p \le C(r,L,p)u^r\|Q_n^{(r)}\|_p, \qquad u\le L/n,\ p>0$$

(see [St-Kr-Os] for $r=1$ and [Di-Hr-Iv]), we have

$$\omega^m(F,1/n)_p \le M(\|F-Q_n\|_p + n^{-m}\|Q_n^{(m)}\|_p) \tag{3.6}$$

for $0<p<\infty$, and hence our task is to find $Q_{2^n}$ of degree $c2^n$ (not necessarily the best or near best $c2^n$ trigonometric approximant to $f^{(k)}$) such that both $\|f^{(k)}-Q_{2^n}\|_p$ and $2^{-n(r-k)}\|Q_{2^n}^{(r-k)}\|_p$ are bounded by the right-hand side of (3.5).

We deal first with $0<p<1$. Let $T_n$ be the best $n$th degree trigonometric polynomial approximant to $f$ in $L_p(T)$, that is,

$$\|f-T_n\|_p = \inf\Big(\|f-T\|_p : T = a_0 + \sum_{l=1}^{n}(a_l\cos lx + b_l\sin lx)\Big) \equiv E_n(f)_p. \tag{3.7}$$

As trigonometric polynomials are dense in $L_p$, we have $\|f-T_{2^l}\|_p\to 0$. Clearly,

$$T_{2^l}-T_{2^n} = \sum_{m=n}^{l-1}(T_{2^{m+1}}-T_{2^m}),$$

and if $\sum_{m=n}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_p^p$ converges for $0<p\le 1$, then

$$f-T_{2^n} = \sum_{m=n}^{\infty}(T_{2^{m+1}}-T_{2^m}) \quad \text{in } L_p(T) \text{ for } 0<p\le 1.$$

Following [Di-Hr-Iv], for $0<p\le\infty$, $T_n$ of (3.7) and any integer $r$ we have

$$\|f-T_{2^m}\|_p + 2^{-mr}\,\|T_{2^m}^{(r)}\|_p \approx \omega^r(f,\,2^{-m})_p \tag{3.8}$$

and hence

$$\sum_{m=n}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_p^p \le C\sum_{m=n}^{\infty}\omega^r(f,2^{-m})_p^p \le C2^{-nkp}\sum_{m=n}^{\infty}2^{mkp}\omega^r(f,2^{-m})^p,$$

which converges assuming (3.1) and hence (3.5). In addition, the series $\sum_{m=n}^{\infty}(T_{2^{m+1}}-T_{2^m})$ has $k$ derivatives in $L_p$ as the Bernstein inequality (2.5) implies

$$\sum_{m=n}^{\infty} \|T_{2^{m+1}}^{(k)} - T_{2^m}^{(k)}\|_p^p \le \sum_{m=n}^{\infty} (2^{m+1})^{kp} \|T_{2^{m+1}} - T_{2^m}\|_p^p$$
$$\le C_1 \sum_{m=n}^{\infty} 2^{mkp} \omega^r(f, 2^{-m})_p^p,$$

which converges using (3.5). Therefore, there exists a function $g \in L_p(T)$ for which

$$\|g - T_{2^n}^{(k)}\|_p = \lim_{l\to\infty} \|T_{2^l}^{(k)} - T_{2^n}^{(k)}\|_p \le C_1^{1/p}\Big(\sum_{m=n}^{\infty} 2^{mkp}\omega^r(f,\, 2^{-m})_p^p\Big)^{1/p}.$$

Using (3.8), we also have

$$2^{-n(r-k)} \|(T_{2^n}^{(k)})^{(r-k)}\|_p = 2^{-n(r-k)} \|T_n^{(r)}\|_p \le C_2 2^{nk} \omega^r(f,\, 2^{-n})_p$$
$$\le C_2 \Big(\sum_{m=n}^{\infty} 2^{mkp}\omega^r(f,\, 2^{-m})_p^p\Big)^{1/p}.$$

If we show $g = f^{(k)}$, the above would imply the result of our theorem for $0 < p \le 1$ via $Q_{2^n} = T_{2^n}^{(k)}$ and (3.6). Following Theorem 2.3 and Corollary 2.4 (its iterate), we in fact have $g = f^{(k)}$.

We now turn to the case $1 \le p < \infty$. For a function $f \in L_p(T)$ with Fourier expansion

$$f(x) \sim a_0 + \sum_{l=1}^{\infty}(a_l \cos lx + b_l \sin lx) = \sum_{l=0}^{\infty} P_l(f)$$

the trigonometric polynomial $\eta_N f$ is given by

$$\eta_N f = \sum_{l=0}^{\infty} \eta\Big(\frac{l}{N}\Big) P_l(f) \tag{3.9}$$

where $\eta \in C^\infty[0,\infty)$, $\eta(x) = 1$ for $x \le 1/2$ and $\eta(x) = 0$ for $x \ge 1$. We now have: (I) $\eta_N f$ is a trigonometric polynomial of degree smaller than $N$; (II) $\eta_N \varphi = \varphi$, where $\varphi$ is a trigonometric polynomial of degree $[N/2]$; (III) $\|\eta_N f\|_{L_p(T)} \le C\|f\|_{L_p(T)}$ for $1 \le p \le \infty$. Therefore, $\eta_N f$ is a de la Vallée Poussin-type operator and $\|\eta_N f - f\|_{L_p(T)} \le (C+1)E_{N/2}(f)_p$ for $1 \le p \le \infty$ where $E_l(f)_p$ is the best rate of approximation of $f$ by a trigonometric polynomial of degree $l$ in $L_p(T)$ (see (3.7)). We now choose the $Q_n$ of (3.6) for $F = f^{(k)}$ to be $(\eta_n f)^{(k)}$. Clearly, $\|f - \eta_n f\|_p = o(1)$ as $n \to \infty$. We estimate $\eta_{2^l} f - \eta_{2^n} f$ using

$$\eta_{2^l} f - \eta_{2^n} f = \sum_{m=n}^{l-1} (\eta_{2^{m+1}} f - \eta_{2^m} f) \equiv \sum_{m=n}^{l-1} \theta_m f.$$

We now write

$$(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)} = \sum_{m=n}^{l-1} (\theta_m f)^{(k)}.$$

Following [Da-Di, Theorem 2.1], we have the Littlewood–Paley inequality

$$\begin{aligned}(3.10)\qquad B_p \|(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\|_{L_p(T)} &\le \Big\| \Big( \sum_{m=n}^{l-1} \{(\theta_m f)^{(k)}\}^2 \Big)^{1/2} \Big\|_{L_p(T)} \\ &\le A_p \|(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\|_{L_p(T)}\end{aligned}$$

with $A_p$ and $B_p$ independent of $l$, $n$, $f$ or $k$. For $1 \le p < \infty$, using [Da-Di, Corollary 2.2], for any integer $k$ we have

$$\begin{aligned}(3.11)\qquad & \Big\| \Big( \sum_{m=n}^{l-1} \{(\theta_m f)^{(k)}\}^2 \Big)^{1/2} \Big\|_{L_p(T)} \\ &\le \Big( \sum_{m=n}^{l-1} \|(\theta_m f)^{(k)}\|^q_{L_p(T)} \Big)^{1/q}, \qquad q = \min(p, 2).\end{aligned}$$

The equivalence

$$(3.12)\qquad \omega^r(f, 1/n)_p \approx \|f - \eta_n f\|_{L_p(T)} + n^{-r} \|(\eta_n f)^{(r)}\|_{L_p(T)} \quad \text{for } 1 \le p \le \infty$$

follows from the realization result in [Di-Hr-Iv] using the fact that $\eta_n f$ is a de la Vallée Poussin operator and hence near best approximant to $f$ in $L_p(T)$.

Using (3.10)–(3.12) and the Bernstein inequality, we have

$$\begin{aligned}\|(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\|_{L_p(T)} &\le C \Big( \sum_{m=n}^{l-1} 2^{mkq} \|\theta_m(f)\|^q_{L_p(T)} \Big)^{1/q} \\ &\le C_1 \Big( \sum_{m=n}^{l-1} 2^{mkq} \omega^r(f, 2^{-m})^q_p \Big)^{1/q}\end{aligned}$$

with $C_1$ independent of $m$, $l$ and $f$ (but it may depend on $r$, $p$ and $q$). The version of the right-hand side of (3.1) given in (3.4) establishes now the convergence of $(\eta_{2^l} f)^{(k)}$ to $f^{(k)}$ and hence $f^{(k)} \in L_p$. (Here, for $1 \le p \le \infty$ the difficulty described in Section 2 does not exist.) We now use $Q_{2^n} = \eta_{2^n} f$ and (3.12) to obtain

$$\begin{aligned}2^{-n(r-k)} \|((\eta_{2^n} f)^{(k)})^{(r-k)}\|_{L_p(T)} &= 2^{-n(r-k)} \|(\eta_{2^n} f)^{(r)}\|_{L_p(T)} \\ &\le C_2 2^{nk} \omega^r(f, 2^{-n})_p \\ &\le C_2 \Big( \sum_{m=n}^{\infty} 2^{mkq} \omega^r(f, 2^{-m})^q_p \Big)^{1/q},\end{aligned}$$

and thus complete the proof. ■

Remark 3.7. We can combine Theorem 3.5 (using the weaker $q = \min(p,1)$) with our earlier theorem [Di-Ti, Section 2] ($d = 1$) to obtain a result with different norms. This will yield the inequality

$$\omega^{r-k}(f^{(k)},t)_{L_{p_1}(T)} \le C\left\{\int_0^t [u^{-k-1/p+1/p_1}\omega^r(f,u)_{L_p(T)}]^{q_1}\,\frac{du}{u}\right\}^{1/q_1} \tag{3.13}$$

for $k + 1/p - 1/p_1 < r$, $0 < p < p_1 \le \infty$ and

$$q_1 = \begin{cases} p_1, & p_1 < \infty,\\ 1, & p_1 = \infty.\end{cases}$$

Corollary 3.8. *For $f \in L_p(T)$, $1 \le p < \infty$, $0 < k < r$ and $q = \min(p,2)$ we have*

$$\|f^{(k)}\|_p \le C\left\{\int_0^1 \frac{\omega^r(f,u)_p^q}{u^{qk+1}}\,du\right\}^{1/q}. \tag{3.14}$$

Remark 3.9. For $k = 1$, $r = 2$ and $1 \le p \le 2$ Corollary 3.8 is the theorem of Marcinkiewicz given in [Ma]. For $p = \infty$, (3.14) holds with $q = 1$.

*Proof of Corollary 3.8.* We note that if $f^{(k)} \in L_p(T)$ for some $f \in L_p(T)$ $(1 \le p \le \infty)$, one has

$$\frac{1}{2\pi}\int_0^{2\pi} f^{(k)}(x+y)\,dy = 0. \tag{3.15}$$

Therefore

$$\begin{aligned}\|f^{(k)}\|_p &= \left\|f^{(k)}(\cdot) - \frac{1}{2\pi}\int_0^{2\pi} f^{(k)}(\cdot+y)\,dy\right\|_p\\ &\le \omega(f^{(k)},2\pi)_p \le (2\pi+1)\omega(f^{(k)},1)_p.\end{aligned}$$

Using Theorem 3.5, we now have

$$\|f^{(k)}\|_p \le (2\pi+1)C_1\left\{\int_0^1 \frac{\omega^{k+1}(f,u)_p^q}{u^{kq+1}}\,du\right\}^{1/q},$$

which establishes (3.14) for $k = r-1$. If $r - 1 > k$, we use [Di,83], which establishes $\|f^{(k)}\|_p \le C_2\|f^{(k+1)}\|_p$ for any $f \in L_p(T)$, and $k > 0$ satisfying (3.15). ∎

**4. Convergence of polynomials and their derivatives in $L_p$, $0 < p < 1$.** As explained in Section 2 (see also [Di,95]), one cannot expect automatically that $P_n \to f$ and $P_n' \to g$ in $L_p$ $(0 < p < 1)$ imply $g = f'$. However, if some additional conditions are satisfied, that is in fact the case.

THEOREM 4.10. *Suppose* $-1 < a_1 < a < b < b_1 < 1$ *and* $P_n$ *is a sequence of polynomials of degree* $n$ *satisfying*

$$\|f - P_n\|_{L_p[a_1,b_1]} = o(1/n), \qquad \|g - P_n'\|_{L_p[a_1,b_1]} = o(1), \qquad n \to \infty. \tag{4.1}$$

*Then*

$$\left\| \frac{f(x+h) - f(x)}{h} - g(x) \right\|_{L_p[a,b]} = o(1), \qquad h \to 0, \tag{4.2}$$

*or, in other words,* $g$ *is the derivative of* $f$ *in* $L_p[a,b]$.

*Proof.* We follow the proof of Theorem 2.3 (with the appropriate modifications). For $h$ satisfying $h < \min(b_1 - b, a - a_1) \equiv d$ and $\sqrt{\varepsilon d}/n \le h \le 1/n$, and for $f, g$ and $P_n$ satisfying

$$\|f - P_n\|_{L_p[a_1,b_1]} \le \varepsilon/n \quad \text{and} \quad \|g - P_n'\|_{L_p[a_1,b_1]} \le \varepsilon \tag{4.3}$$

we have

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{P_n(\cdot + h) - P_n(\cdot)}{h} \right\|_{L_p[a,b]}^p \le 2\left(\frac{\varepsilon}{d}\right)^{p/2}. \tag{4.4}$$

We now follow [Di-Hr-Iv, Section 6] to obtain, for $\sqrt{\varepsilon d}/n < h \le 2\sqrt{\varepsilon d}/n$,

$$\left\| \frac{P_n(\cdot + h) - P_n(\cdot)}{h} - P_n'(\cdot) \right\|_{L_p[a,b]} \le \sum_{k=2}^{n+1} \left(\frac{h^{k-1}}{k!}\right)^p \|P_n^{(k)}\|_{L_p[a,b]}^p \equiv S.$$

Defining

$$\widetilde{d}(x) = (x - a_1)(b_1 - x) \qquad \text{for } x \in [a_1, b_1], \tag{4.5}$$

we have

$$S \le C \sum_{k=2}^{\infty} \left(\frac{h^{k-1}}{k!}\right)^p d^{-pk/2} \|\widetilde{d}(x)^{k/2} P_n^{(k)}\|_{L_p[a_1,b_1]}^p \equiv CS_1.$$

Using the Bernstein inequality

$$\|\varphi(x)^k P_n^{(k)}\|_{L_p[-1,1]} \le Cn^k \|P_n\|_{L_p[-1,1]} \quad \text{with} \quad \varphi(x)^2 = 1 - x^2,$$

we have by a change of variable

$$\|\widetilde{d}(x)^{k/2} P_n^{(k)}\|_{L_p[a_1,b_1]} \le Cn^k \|P_n\|_{L_p[a_1,b_1]}.$$

Therefore,

$$\begin{aligned} S_1 &\le C \sum_{k=2}^{\infty} \left(\frac{h^{k-1}}{k!}\right)^p d^{-pk/2} n^{kp} \|P_n\|_{L_p[a_1,b_1]}^p \\ &= C(\|f\|_{L_p[a_1,b_1]}^p + \varepsilon^p) \sum_{k=2}^{\infty} \left(\frac{hn}{\sqrt{d}}\right)^{(k-1)p} \\ &\le C_1(\|f\|_{L_p[a_1,b_1]}^p + \varepsilon^p)\varepsilon^p. \end{aligned}$$

Hence,

$$\left\|\frac{f(\cdot+h)-f(\cdot)}{h}-g(\cdot)\right\|^p_{L_p[a,b]} \le C_2(\varepsilon^p(\|f\|_{L_p[a_1,b_1]}+1)+\varepsilon^{p/2}+\varepsilon^p), \tag{4.6}$$

and as both sides of (4.6) do not depend on $P_n$ or $n$, (4.6) implies (4.2). ∎

We may iterate the result in Theorem 4.10 to obtain

Corollary 4.11. *Suppose $-1 < a_1 < a < b < b_1 < 1$ and $P_n$ is a sequence of polynomials of degree $n$ satisfying*

$$\begin{aligned} \|f-P_n\|_{L_p[a_1,b_1]} &= o\left(\frac{1}{n^k}\right), \\ \|g_i-P_n^{(i)}\|_{L_p[a_1,b_1]} &= o\left(\frac{1}{n^{k-i}}\right) \quad \text{for } i=1,\dots,k. \end{aligned} \tag{4.7}$$

*Then $g_i$ is the derivative of $g_{i-1}$ in $L_p[a,b]$ in the sense of* (4.2) (*with $g_0 = f$*).

For the proof of Corollary 4.11 we use a finite sequence of nested intervals and the proof of Theorem 4.10.

We also have the following corollary of the above.

Corollary 4.12. *Suppose $P_n$ is a sequence of polynomials of degree $n$ satisfying*

$$\|f-P_n\|_{L_p[-1,1]} = o\left(\frac{1}{n^k}\right), \tag{4.8}$$

$$\|\varphi^i(g_i-P_n^{(i)})\|_{L_p[-1,1]} = o\left(\frac{1}{n^{k-i}}\right) \quad \text{for } i=1,\dots,k.$$

*Then in any interval $[a,b]$, $-1<a<b<1$, $g_i$ is the derivative of $g_{i-1}$ and $g_1$ is the derivative of $f$ in the sense of* (4.2).

For the proof we just confirm that the conditions of Corollary 4.11 are satisfied.

**5. The estimate of $\Omega^{r-k}_\varphi(f^{(k)},t)_p$ by $\omega^r_\varphi(f,t)_p$.** For a function $f \in L_p[-1,1]$ the inverse result of our paper is given in the following theorem.

Theorem 5.13. *For $f \in L_p[-1,1]$, $0<p<\infty$, and integers $k,r$ satisfying $k<r$, we have*

$$\Omega^{r-k}(f^{(k)},t)_{p,\varphi^k} \le C\left\{\int_0^t \frac{\omega^r_\varphi(f,u)^q_p}{u^{qk+1}}\,du\right\}^{1/q} \tag{5.1}$$

*where $q=\min(p,2)$.*

Remark 5.14. For $1 \le p < \infty$, (5.1) implies the inequality with $q=1$ for that range but with $q=1$ Theorem 5.13 is included in Theorem 6.3.1

of [Di-To]. For $q = 1$ and $p = \infty$ we have the result in [Di-To, Theorem 6.3.1(a)].

For the proof of Theorem 5.13 we need the following lemma.

LEMMA 5.15. *For $0 < p \le \infty$, integer $m$ and $g \in L_p[a,b]$, for any $-1 < a < b < 1$ and $Q_n$ a polynomial of degree $n$ we have*

$$\Omega_\varphi^m(g,1/n)_{p,\varphi^k} \le C(\|\varphi^k(g-Q_n)\|_{L_p[-1,1]} + n^{-m}\|\varphi^{k+m}Q_n^{(m)}\|_{L_p[-1,1]}). \tag{5.2}$$

*Proof.* To prove (5.2) we observe that

$$\Omega_\varphi^m(g,1/n)_{p,\varphi^k} \le C_1\{\Omega_\varphi^m(g-Q_n,1/n)_{p,\varphi^k} + \Omega_\varphi^m(Q_n,1/n)_{p,\varphi^k}\}$$

where $C_1 = 1$ for $1 \le p \le \infty$ and $C_1 = 2^{1/p}$ for $0 < p < 1$. Following [Di-Hr-Iv] with minor changes, we obtain

$$\Omega_\varphi^m(Q_n,1/n)_{p,\varphi^k} \le C_2 n^{-m}\|\varphi^{k+m}Q_n^{(m)}\|_{L_p[-1,1]}.$$

To complete the proof of (5.2) we need to show that

$$\Omega_\varphi^m(g-Q_n,1/n)_{p,\varphi^k} \le C_3\|\varphi^k(g-Q_n)\|_{L_p[-1,1]}.$$

To prove the last inequality, we note that all we need to show is that for $-1+2m^2h^2 \le x \le 1-2m^2h^2$ and $|\alpha| \le m/2$,

$$A^{-1} \le \left(\frac{\varphi(x)}{\varphi(x+\alpha h\varphi(x))}\right)^l \le A \tag{5.3}$$

with $A$ independent of $x$ and $h$. Without loss of generality it is sufficient to prove (5.3) for $l = 2$, $h \ge 0$ and $x \le 0$. With the restriction on $x$, i.e. $-1+2m^2h^2 \le x \le 0$, we have $h < 1/\sqrt{2}\,m$ as otherwise (5.3) is vacuous. For $-m/2 \le \alpha \le 0$ (recall $-1+2m^2h^2 \le x$ and hence $1+x-mh\varphi(x) \ge 0$) we have

$$\begin{aligned}\frac{2}{3} &\le \frac{1}{1+\frac{m}{2}h} \le \frac{1}{1+\frac{m}{2}h\sqrt{\frac{1+x}{1-x}}} \le \frac{1-x}{1-x+\frac{m}{2}h\varphi(x)}\\ &\le \frac{(1-x)(1+x)}{(1-x-\alpha h\varphi(x))(1+x+\alpha h\varphi(x))} \equiv \frac{\varphi^2(x)}{\varphi^2(x+\alpha h\varphi(x))}\\ &\le \frac{1+x}{1+x-\frac{m}{2}h\varphi(x)} \le 2.\end{aligned}$$

For $0 \le \alpha \le m/2$ (the simpler case when $x < 0$ and $h > 0$) we have

$$1 \le \frac{\varphi^2(x)}{\varphi^2(x+\alpha h\varphi(x))} \le \frac{1-x}{1-x-\frac{m}{2}h\varphi(x)} \le 2. \ \blacksquare$$

*Proof of Theorem 5.13.* The function $\omega_\varphi^l(F,t)_p$ is nondecreasing. We also have

$$\omega_\varphi^l(F,2t)_p \le C\omega_\varphi^l(F,t)_p \quad \text{for } 0 < p \le \infty, \tag{5.4}$$

which follows for $1 \le p \le \infty$ from the equivalence of $\omega_\varphi^l(F,t)_p$ with the appropriate $K$-functional (see [Di-To]) and for $0 < p < 1$ from [Dr-Hr-Iv, Corollary 5.13, (5.13)].

Therefore,

$$(5.5)\qquad \Big\{\int_0^{2^{-n}} \frac{\omega_\varphi^r(f,u)_p^q}{u^{qk+1}}\,du\Big\}^{1/q} \approx \Big\{\sum_{l=n}^{\infty} 2^{lkq}\omega_\varphi^r(f,2^{-l})_p^q\Big\}^{1/q},$$

and as $\Omega^{r-k}(F,t)_{p,\varphi^k}$ and $\{\int_0^t \frac{\omega_\varphi^r(f,u)_p^q}{u^{qk+1}}\,du\}^{1/q}$ are monotonic in $t$, it is sufficient to prove

$$(5.6)\qquad \Omega^{r-k}(f^{(k)},2^{-n})_{p,\varphi^k} \le C\Big\{\sum_{l=n}^{\infty} 2^{lkq}\omega_\varphi^r(f,2^{-l})_p^q\Big\}^{1/q}.$$

We note that monotonicity in $t$ of $\Omega^m(g,t)_{p,\varphi^k}$ and of $\int_0^t\cdots$ together with (5.4)–(5.6) implies (5.1).

We first proceed with the proof for $0<p<1$. We choose $P_{2^k}$ to be the best $2^k$th degree polynomial approximant to $f$ in $L_p[-1,1]$. As polynomials are dense in $L_p[-1,1]$ for $0<p<1$ (as well as for $1 \le p < \infty$), we have $\|f-P_{2^k}\|_{L_p[-1,1]} \to 0$. If

$$\sum_{l=n}^{\infty} \|P_{2^{l+1}} - P_{2^l}\|_{L_p[-1,1]}^p < \infty,$$

we have

$$\|f-P_{2^n}\|_{L_p[-1,1]}^p \le \sum_{l=n}^{\infty} \|P_{2^{l+1}} - P_{2^l}\|_{L_p[-1,1]}^p, \qquad 0<p<1,$$

and, in other words,

$$f - P_{2^n} = \sum_{l=n}^{\infty}(P_{2^{l+1}} - P_{2^l}) \quad \text{in } L_p[-1,1] \text{ for } 0<p<1.$$

Following [Di-Hr-Iv, Sections 5 and 6], we have

$$(5.7)\qquad \|f-P_{2^l}\|_{L_p[-1,1]} + 2^{-lr}\,\|\varphi^r P_{2^l}^{(r)}\|_{L_p[-1,1]} \approx \omega_\varphi^r(f,2^{-l})_p.$$

Hence with $\|\cdot\|_{L_p[-1,1]} \equiv \|\cdot\|_p$ we write

$$\begin{aligned}\sum_{l=n}^{\infty} \|P_{2^{l+1}} - P_{2^l}\|_p^p &\le C\sum_{l=n}^{\infty}\omega_\varphi^r(f,2^{-l})_p^p \\ &\le C2^{-nkp}\sum_{l=n}^{\infty} 2^{klp}\omega_\varphi^r(f,2^{-l})_p^p,\end{aligned}$$

and as the sum on the right-hand side converges following (5.5), we have

$\|f - P_{2^n}\|_p = o(2^{-nk})$ as $n \to \infty$. We now need the Bernstein inequality

$$\|\varphi^{j+1} Q_n'\|_p \le C(p,j) n \|\varphi^j Q_n\|_p \tag{5.8}$$

for $Q_n$ a polynomial of degree $n$ (for $0 < p < 1$ see for example [Di-Ji-Le, (2.3)]). We use (5.8) to obtain

$$\begin{aligned}\sum_{l=n}^{\infty} \|\varphi^i (P^{(i)}_{2^{l+1}} - P^{(i)}_{2^l})\|_p^p &\le C_i \sum_{l=n}^{\infty} 2^{lip} \omega_\varphi^r(f, 2^{-l})_p^p \\ &\le C_i 2^{-n(k-i)p} \sum_{l=n}^{\infty} 2^{lkp} \omega_\varphi^r(f, 2^{-l})_p^p.\end{aligned} \tag{5.9}$$

Hence $\varphi^i g_i = \varphi^i P^{(i)}_{2^n} + \varphi^i \sum_{l=n}^{\infty} (P^{(i)}_{2^{l+1}} - P^{(i)}_{2^l})$ converges in $L_p$ (for $g_0 = f$ it was shown earlier) and

$$\|\varphi^i (g_i - P^{(i)}_{2^n})\|_p = o(2^{-n(k-i)}), \qquad n \to \infty, \text{ for } i = 0, 1, \dots, k, \tag{5.10}$$

which implies the condition of Corollary 4.12, and therefore $g_i$ is locally the $i$th derivative of $f$ in $L_p$. To complete the proof (for $0 < p < 1$) we apply Lemma 5.15 with $g = g_k$, $m = r - k$, the integer $2^n$, and $P^{(k)}_{2^n}$ for the polynomial $Q_{2^n}$ of degree $2^n$. We now use (5.9) to obtain

$$\|\varphi^k (g_k - P^{(k)}_{2^n})\|_p^p \le C_k \sum_{l=n}^{\infty} 2^{lkp} \omega_\varphi^r(f, 2^{-l})_p^p.$$

The equivalence (5.7) with $l = n$ implies

$$2^{-n(r-k)} \, \|\varphi(x)^{r-k+k} (P^{(k)}_{2^n})^{(r-k)}\|_{L_p[-1,1]} \le C \, 2^{nk} \omega_\varphi^r(f, 2^{-n})_p.$$

The last two estimates yield (5.6) and our result is proved for $0 < p < 1$.

Let us now proceed with the case $1 \le p < \infty$. The function $f$ has the expansion

$$f \sim \sum_{m=0}^{\infty} a_m \psi_m$$

where $\psi_m$ is the Legendre polynomial of degree $n$ normalized to satisfy $\|\psi_m\|_{L_2[-1,1]} = 1$, and where

$$a_m = \int_{-1}^{1} f(x) \psi_m(x) \, dx.$$

We choose $P_n(f) = \eta_n(f)$ to be given by

$$\eta_n(f) = \sum_{m=0}^{\infty} \eta\left(\frac{m}{n}\right) a_m \psi_m \tag{5.11}$$

where $\eta \in C^\infty$, $\eta(x) = 1$ for $x \le 1/2$ and $\eta(x) = 0$ for $x \ge 1$.

It is well-known that $\eta_n(f)$ is a de la Vallée Poussin-type operator on $L_p[-1,1]$, $1 \le p \le \infty$, that is,

(I) $\|\eta_n f\|_p \le C\|f\|_p$,
(II) $\|\eta_n f - f\|_p \le C E_{n/2}(f)_p \equiv C \inf\Big\{\|f-\Psi_n\|_p : \Psi_n = \sum_{m\le n/2} b_m \psi_m\Big\}$,
(III) $\eta_n f \in \operatorname{span}\{\psi_0,\dots,\psi_n\}$.

We choose $Q_{2^n}$ of (5.2) to be $(\eta_n f)^{(k)}$. Using (II) and the density of polynomials in $L_p[-1,1]$, $1 \le p < \infty$, we have $\|f-\eta_n f\|_{L_p[-1,1]} = o(1)$ as $n \to \infty$.

Following [Da-Di], we write

$$\eta_{2^l} f - \eta_{2^n} f = \sum_{m=n}^{l-1} (\eta_{2^{m+1}} f - \eta_{2^m} f) \equiv \sum_{m=n}^{l-1} \theta_m f.$$

We now write

$$\varphi^k\{(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\} = \sum_{m=n}^{l-1} \varphi^k (\theta_m f)^{(k)}.$$

Following [Da-Di, Theorem 2.1], we have the Littlewood–Paley inequality

$$\begin{aligned}(5.12)\qquad B_p\|\varphi^k\{(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\}\|_{L_p[-1,1]} &\le \Big\|\Big(\sum_{m=n}^{l-1}\{\varphi^k(\theta_m f)^{(k)}\}^2\Big)^{1/2}\Big\|_{L_p[-1,1]}\\ &\le A_p\Big\|\varphi^k\{(\eta_{2^l} f)^{(k)} - (\eta_{2^n} f)^{(k)}\}\Big\|_{L_p[-1,1]}\end{aligned}$$

with $A_p$ and $B_p$ independent of $l$, $n$, $f$ or $k$. Using [Da-Di, Corollary 2.2], for $1 < p < \infty$ and $q = \min(p,2)$ we have

$$\begin{aligned}\Big\|\Big(\sum_{m=n}^{l-1}\{\varphi^k(\theta_m f)^{(k)}\}^2\Big)^{1/2}\Big\|_{L_p[-1,1]} &\le \Big(\sum_{m=n}^{l-1}\|\varphi^k(\theta_m f)^{(k)}\|^q_{L_p[-1,1]}\Big)^{1/q}\\ &\le C\Big(\sum_{m=n}^{l-1} 2^{mkq}\|\theta_m f\|^q_{L_p[-1,1]}\Big)^{1/q}\\ &\qquad \text{(by the Bernstein inequality [Di-To, Chapter 7])}\\ &\le C_1\Big(\sum_{m=n}^{l-1} 2^{mkq}\omega^r_\varphi(f,2^{-m})^q_p\Big)^{1/q}\\ &\qquad \text{(by the Jackson inequality [Di-To, Chapter 7]).}\end{aligned}$$

In view of (5.5), the last sum converges as $l \to \infty$, and hence $f^{(k)}$ exists and satisfies

$$\|\varphi^k(f^{(k)} - \eta_{2^n} f^{(k)})\|_p = C_1\Big(\sum_{m=n}^{\infty} 2^{mkq}\omega_\varphi^r(f, 2^{-m})_p^q\Big)^{1/q}.$$

Using Lemma 5.15, we will complete the proof when we show

$$2^{-(r-k)n}\|\varphi^r(\eta_{2^n} f)^{(r)}\|_p \le C_2\, 2^{kn}\omega_\varphi^r(f, 2^{-n})_p \le C_3\Big(\sum_{m=n}^{\infty} 2^{kmq}\omega_\varphi^r(f, 2^{-n})_p^q\Big)^{1/q}.$$

The second inequality is clear, and the first follows from the realization result in [Di-Hr-Iv] which holds for $\eta_n f$, as well as from

$$\omega_\varphi^r(f, 2^{-n})_p \approx \|f - \eta_n f\|_p + \frac{1}{2^{nr}}\,\|\varphi^r(\eta_n f)^{(r)}\|_p. \ \blacksquare$$

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Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
E-mail: zditzian@math.ualberta.ca

Scuola Normale Superiore
Piazza dei Cavalieri, 7
Pisa 56126, Italy
E-mail: s.tikhonov@sns.it

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