STUDIA MATHEMATICA 223 (1) (2014)

A note on multilinear Muckenhoupt classes for multiple weights

by

SONGQING CHEN (Shanghai), HUOXIONG WU (Xiamen) and QINGYING XUE (Beijing)

Abstract. This paper is devoted to investigating the properties of multilinear $A_{\vec{P}}$ conditions and $A_{(\vec{P},q)}$ conditions, which are suitable for the study of multilinear operators on Lebesgue spaces. Some monotonicity properties of $A_{\vec{P}}$ and $A_{(\vec{P},q)}$ classes with respect to $\vec{P}$ and q are given, although these classes are not in general monotone with respect to the natural partial order. Equivalent characterizations of multilinear $A_{(\vec{P},q)}$ classes in terms of the linear A_p classes are established. These results essentially improve and extend the previous results.

1. Introduction. In the study of the weighted theory of multilinear Calderón–Zygmund operators, Lerner et al. [LOPTT] introduced multilinear $A_{\vec{P}}$ conditions for multiple weights, which are the natural extension to the multilinear setting of Muckenhoupt's classes and are the largest classes of weights for which all m-linear Calderón–Zygmund operators are bounded on weighted Lebesgue spaces. As the natural generalization of the classical linear $A_{(p,q)}$ classes, multilinear $A_{(\vec{P},q)}$ conditions were introduced by Moen [Mo] and Chen–Xue [CX], in studying the weighted theory of multilinear fractional type integral operators. The properties of the multilinear $A_{\vec{P}}$ and $A_{(\vec{P},q)}$ conditions, and their relations to the classical linear A_p conditions played key roles in establishing multiple weighted norm inequalities for multilinear Calderón–Zygmund operators, multilinear fractional integral operators and their commutators (see [LOPTT, Mo, CX, PPTT] etc.).

In this paper, we continue the investigation of the properties of $A_{\vec{P}}$ conditions and $A_{(\vec{P},q)}$ conditions. Unlike linear A_p classes, multilinear $A_{\vec{P}}$ classes are not increasing with respect to the natural partial order. In Section 2, we will show, however, that $A_{\vec{P}}$ classes do have certain monotonicity properties in terms of $\vec{P}$. In Section 3, we will establish some equivalent characteriza-

2010 *Mathematics Subject Classification*: Primary 42B35; Secondary 42B99.
Key words and phrases: multiple weights, $A_{\vec{P}}$ conditions, $A_{(\vec{P},q)}$ conditions.

DOI: 10.4064/sm223-1-1

tions of multilinear $A_{(\vec{P},q)}$ conditions in terms of classical linear A_p classes, which improve and extend the corresponding results in [CX, Mo]. As an application of these properties, some monotonicity properties of $A_{(\vec{P},q)}$ classes are also established.

2. Multilinear $A_{\vec{P}}$ conditions. Following the notation in [LOPTT], for m exponents $p_1, \ldots, p_m$, we will often write p for the number given by $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{P}$ for the vector $\vec{P} = (p_1, \ldots, p_m)$.

DEFINITION 2.1. Let $1 \le p_1, \ldots, p_m < \infty$, and let p and $\vec{P}$ be as above. A multiple weight $\vec{w} = (w_1, \ldots, w_m)$ is said to satisfy the *multilinear $A_{\vec{P}}$ condition* if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}(x)\, dx \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j(x)^{1-p_j'}\, dx \right)^{1/p_j'} < \infty, \tag{2.1}$$

where $\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$. When $p_j = 1$, $(|Q|^{-1} \int_Q w_j(x)^{1-p_j'}\, dx)^{1/p_j'}$ is understood as $(\inf_Q w_j(x))^{-1}$.

Obviously, for $m = 1$, $A_{\vec{P}}$ is the classical Muckenhoupt A_p condition. For $m > 1$, Lerner et al. [LOPTT] showed that

$$\prod_{j=1}^m A_{p_j} \subsetneq A_{\vec{P}}, \tag{2.2}$$

which implies that something more general happens for the $A_{\vec{P}}$ classes. In [LOPTT], the authors also established the following interesting characterization in term of the classical A_p condition.

THEOREM 2.2 (cf. [LOPTT, Theorem 3.6]). *Let $\vec{w} = (w_1, \ldots, w_m)$ and $1 \le p_1, \ldots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

$$\nu_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j} \in A_{mp} \quad \textit{and} \quad w_j^{1-p_j'} \in A_{mp_j'}, \quad j = 1, \ldots, m, \tag{2.3}$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.

It should be pointed out that when $m = 1$, both conditions in (2.3) represent the same A_p condition, but when $m \ge 2$, none of the two conditions in (2.3) implies the other (see [LOPTT, Remark 7.1]). This theorem also shows that as the index m increases, the $A_{\vec{P}}$ condition gets weaker.

On the other hand, it is well-known that the classical A_p classes have the natural partial order, that is,

$$A_q \subsetneq A_p \quad \text{for } 1 \le q < p \le \infty, \quad \text{and} \quad A_p = \bigcup_{1 \le q < p} A_q. \tag{2.4}$$

The classes $A_{\vec{P}}$ are not increasing under the natural partial order (see [LOPTT, Remark 7.3]), although it is easy to check that $A_{(1,\dots,1)} \subseteq A_{\vec{P}}$ for each $\vec{P}$. However, applying the above Theorem 2.2, Lerner et al. [LOPTT] proved the following result.

THEOREM 2.3 (cf. [LOPTT, Lemma 6.1]). *Assume that $\vec{w}=(w_1,\dots,w_m)$ satisfies the $A_{\vec{P}}$ condition. Then there exists a finite constant $r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$.*

In this section, we will continue the study of the properties of $A_{\vec{P}}$ classes. We will establish the following results:

THEOREM 2.4. *Let $1 \le p_1, \dots, p_m < \infty$ and $p_0 = \min_{1 \le i \le m} p_i$. Then the classes $A_{r\vec{P}}$ are strictly increasing as r increases with $r \ge 1/p_0$. More precisely, for $1/p_0 \le r_1 < r_2 < \infty$ we have*

$$A_{r_1\vec{P}} \subsetneq A_{r_2\vec{P}}. \tag{2.5}$$

THEOREM 2.5. *Let $1 < p_1, \dots, p_m < \infty$ and $p_0 = \min_{1 \le i \le m} p_i$. Then*

$$A_{\vec{P}} = \bigcup_{1/p_0 \le r < 1} A_{r\vec{P}}. \tag{2.6}$$

We will prove only Theorem 2.4, since Theorem 2.5 is an immediate consequence of Theorems 2.3 and 2.4.

Proof of Theorem 2.4. We consider the following two cases:

CASE 1: $r_1 p_0 > 1$. We first show that $A_{r_1\vec{P}} \subset A_{r_2\vec{P}}$. Suppose $\vec{w} \in A_{r_1\vec{P}}$. Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{r_2 p/(r_2 p_j)}\,dx\right)^{1/(r_2 p)} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j(x)^{1-(r_2 p_j)'}\,dx\right)^{1/(r_2 p_j)'} \\
&= \left(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\right)^{1/(r_2 p)} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_2 p_j)}\,dx\right)^{(r_2 p_j - 1)/(r_2 p_j)} \\
&= \left[\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_2 p_j)}\,dx\right)^{(r_2 p_j - 1)/p_j}\right]^{1/r_2}
\end{aligned}$$

$$\leq\left[\left(\frac{1}{|Q|}\int_Q\prod_{j=1}^m w_j(x)^{p/p_j}\,dx\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_1p_j)}\,dx\right)^{(r_1p_j-1)/p_j}\right]^{1/r_2}$$

$$\leq[\vec{w}]_{A_{r_1\vec{P}}}^{r_1/r_2}<\infty.$$

Next we prove that $A_{r_1\vec{P}}\neq A_{r_2\vec{P}}$. Set $a_j=nr_1p_j-n$ for $j=1,\dots,m$, and $\vec{w}=(|x|^{a_1},\dots,|x|^{a_m})$. Then

$$\begin{cases} n(-mr_2p_j-1+r_2p_j)<0<a_j<-n+nr_2p_j, \quad j=1,\dots,m,\\ \dfrac{-n}{p}<\displaystyle\sum_{j=1}^m\frac{a_j}{p_j}<nmr_2-\frac{n}{p}. \end{cases}$$

Hence,

$$\begin{cases} \dfrac{a_j}{1-r_2p_j}\in\left(-n,n\left(\dfrac{mr_2p_j}{r_2p_j-1}-1\right)\right), \quad j=1,\dots,m,\\ \dfrac{a_j}{1-r_1p_j}\notin\left(-n,n\left(\dfrac{mr_1p_j}{r_1p_j-1}-1\right)\right), \quad j=1,\dots,m,\\ p\displaystyle\sum_{j=1}^m\frac{a_j}{p_j}\in(-n,n(mr_2p-1)). \end{cases}$$

Consequently,

$$\begin{cases} |x|^{a_j/(1-r_2p_j)}\in A_{\frac{mr_2p_j}{r_2p_j-1}}, \quad j=1,\dots,m,\\ |x|^{a_j/(1-r_1p_j)}\notin A_{\frac{mr_1p_j}{r_1p_j-1}}, \quad j=1,\dots,m,\\ \displaystyle\prod_{j=1}^m|x|^{a_jp/p_j}\in A_{mr_2p}. \end{cases}$$

That is,

$$\begin{cases} |x|^{a_j[1-(r_2p_j)']}\in A_{m(r_2p_j)'}, \quad j=1,\dots,m,\\ |x|^{a_j[1-(r_1p_j)']}\notin A_{m(r_1p_j)'}, \quad j=1,\dots,m,\\ \displaystyle\prod_{j=1}^m|x|^{a_jp/p_j}\in A_{mr_2p}. \end{cases}$$

By Theorem 2.2, we have $\vec{w}\in A_{r_2\vec{P}}$, but $\vec{w}\notin A_{r_1\vec{P}}$. This implies that $A_{r_1\vec{P}}\subsetneq A_{r_2\vec{P}}$.

Case 2: $r_1p_0=1$. Set $r_3=(r_1+r_2)/2$; then $1/p_0=r_1<r_3<r_2$. We first show that $A_{r_1\vec{P}}\subset A_{r_3\vec{P}}$. Suppose that $\vec{w}\in A_{r_1\vec{P}}$. Since $r_1p_0=1$, without loss of generality we may assume that $r_1p_1=\cdots=r_1p_l=1$ for $1\leq l\leq m$, and $r_1p_j>1$ for $j=l+1,\dots,m$. Then

$$\begin{aligned}
&\bigg(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{r_3p/(r_3p_j)}\,dx\bigg)^{1/(r_3p)} \prod_{j=1}^m\bigg(\frac{1}{|Q|}\int_Q w_j(x)^{1-(r_3p_j)'}\,dx\bigg)^{1/(r_3p_j)'}\\
&=\bigg(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\bigg)^{1/(r_3p)} \prod_{j=1}^m\bigg(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_3p_j)}\,dx\bigg)^{(r_3p_j-1)/(r_3p_j)}\\
&=\bigg[\bigg(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\bigg)^{1/p} \prod_{j=1}^m\bigg(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_3p_j)}\,dx\bigg)^{(r_3p_j-1)/p_j}\bigg]^{1/r_3}\\
&\le\bigg[\bigg(\frac{1}{|Q|}\int_Q \prod_{j=1}^m w_j(x)^{p/p_j}\,dx\bigg)^{1/p} \prod_{j=1}^l\Big(\inf_Q w_j(x)\Big)^{-r_1}\\
&\qquad\qquad\times\prod_{j=l+1}^m\bigg(\frac{1}{|Q|}\int_Q w_j(x)^{1/(1-r_1p_j)}\,dx\bigg)^{(r_1p_j-1)/p_j}\bigg]^{1/r_3}\\
&\le [\vec{w}]_{A_{r_1\vec{P}}}^{r_1/r_3}<\infty,
\end{aligned}$$

where in the last inequality we have used the Hölder inequality and the fact that $r_1 = 1/p_j$ for $j = 1,\dots,l$.

On the other hand, since $r_3p_0 > 1$ and $r_3 < r_2$, by the result in Case 1, we know that $A_{r_3\vec{P}} \subsetneq A_{r_2\vec{P}}$. Hence, $A_{r_1\vec{P}} \subset A_{r_3\vec{P}} \subsetneq A_{r_2\vec{P}}$. Theorem 2.4 is proved. ∎

3. Multilinear $A_{(\vec{P},q)}$ conditions

3.1. Definition and main results

Definition 3.1. Let $1 \le p_1,\dots,p_m < \infty$, $q > 0$, and let p and $\vec{P}$ be as before. Suppose that $\vec{w} = (w_1,\dots,w_m)$ and each w_i $(i = 1,\dots,m)$ is a nonnegative function on $\mathbb{R}^n$. We say that $\vec{w} \in A_{(\vec{P},q)}$ if

$$\sup_Q\bigg(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\bigg)^{1/q}\prod_{i=1}^m\bigg(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\bigg)^{1/p_i'}<\infty,$$

where $v_{\vec{w}} = \prod_{i=1}^m w_i$. If $p_i = 1$, $(|Q|^{-1}\int_Q w_i^{-p_i'}(x)\,dx)^{1/p_i'}$ is understood as $(\inf_Q w_i(x))^{-1}$.

Clearly, when $m = 1$, $A_{(\vec{P},q)}$ is the classical $A_{(p,q)}$ class (see [MW]). For m arbitrary, Moen [Mo] showed that

$$\bigcup_{q_1,\dots,q_m}\prod_{i=1}^m A_{(p_i,q_i)} \subsetneq A_{(\vec{P},q)},$$

where the union is over all $q_i \ge p_i$ that satisfy $1/q = 1/q_1 + \dots + 1/q_m$. Furthermore, Moen [Mo] and Chen–Xue [CX] gave the following relations between $A_{(\vec{P},q)}$ and the classical A_p weights.

THEOREM 3.2 (cf. [Mo, Theorem 3.4]). *Let* $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, *and* q *with* $1/m < p \le q < \infty$. *Suppose that* $\vec{w} \in A_{(\vec{P},q)}$. *Then*

(i) $v_{\vec{w}}^q \in A_{mq}$;

(ii) $w_j^{-p_j'} \in A_{mp_j'}$ $(i = 1, 2, \dots, m)$.

THEOREM 3.3 (cf. [CX, Theorem 2.2]). *Let* $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $1/q = 1/p - \alpha/n$. *Suppose that* $\vec{w} \in A_{(\vec{P},q)}$. *Then*

(i) $v_{\vec{w}}^q \in A_{q(m-\alpha/n)}$;

(ii) $w_i^{-p_i} \in A_{p_i'(m-\alpha/n)}$ $(i = 1, \dots, m)$ *if* $\alpha/n < (m-2) + 1/p_i + 1/p_j$ *for any* $1 \le i, j \le m$.

When $p_i = 1$, $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}$ *is understood as* $w_i^{n/(mn-\alpha)} \in A_1$.

As is well-known, for the classical $A_{(p,q)}$ weights and A_p weights we have the following relations:

$$w \in A_{(p,q)} \ \Leftrightarrow\ w^q \in A_{q(1-\alpha/n)} \ \Leftrightarrow\ w^{-p'} \in A_{p'(1-\alpha/n)} \ \Leftrightarrow\ w^q \in A_s, \tag{3.1}$$

$$w \in A_{(p,q)} \ \Rightarrow\ w^q \in A_q \text{ and } w^p \in A_p \ \Leftrightarrow\ w^q \in A_q \text{ and } w^{-p'} \in A_{p'}, \tag{3.2}$$

where $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $s = 1 + q/p'$. For multilinear $A_{(\vec{P},q)}$ classes, Pradolini [Pr] gave the following equivalence.

THEOREM 3.4 (cf. [Pr, Remark 2.11]). *Let* $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < mn/\alpha$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $1/q = 1/p - \alpha/n$. *Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if* $\vec{w}^{\vec{q}} \in A_{\vec{S}}$, *where* $\vec{w}^{\vec{q}} = (w_1^{q_1}, \dots, w_m^{q_m})$, $1/q_i = 1/p_i - \alpha/mn > 0$, $s_i = (1 - \alpha/mn)q_i \ge 1$ *and* $\vec{S} = (s_1, \dots, s_m)$.

Obviously, when $m = 1$, Theorem 3.4 is the last equivalence in (3.1). In this paper, we will generalize the other equivalence properties in (3.1) and (3.2) to the multilinear setting in the next theorems.

THEOREM 3.5. *Let* $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* q *with* $1/m \le p \le q < \infty$. *Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if*

$$v_{\vec{w}}^q \in A_{mq} \quad \text{and} \quad w_j^{-p_j'} \in A_{mp_j'}, \quad j = 1, \dots, m, \tag{3.3}$$

where the condition $w_j^{-p_j'} \in A_{mp_j'}$ *in the case* $p_j = 1$ *is understood as* $w_j^{1/m} \in A_1$.

REMARK 3.6. Clearly, Theorem 3.5 generalizes Theorem 3.1 in the following two aspects: (i) the ranges of p_i are enlarged from $(1, \infty)$ to $[1, \infty)$; (ii) the conditions in (3.3) are shown to be sufficient and necessary. Also, compared with (3.2), the result of Theorem 3.5 is new, even for $m = 1$.

THEOREM 3.7. *Let $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < mn/\alpha$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. Then $\vec{w} \in A_{(\vec{P},q)}$ if and only if*

$$v_{\vec{w}}^q \in A_{q(m-\alpha/n)} \quad \text{and} \quad w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}, \quad i = 1, \dots, m. \tag{3.4}$$

When $p_i = 1$, the condition $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}$ is understood as $w_i^{n/(mn-\alpha)} \in A_1$.

REMARK 3.8. Compared with Theorem 3.3, Theorem 3.7 not only shows that the conditions in (3.4) are sufficient and necessary, but also removes the restrictions of condition (ii) in Theorem 3.3. We also remark that when $m = 1$, both conditions in (3.4) are the same $A_{(p,q)}$ condition; however, when $m \ge 2$, none of the two conditions in (3.4) implies the other. For example, let $n = 1$, $m = 2$, $p_1 = p_2 = 2$, $\alpha = 1/2$; then $p = 1$, $q = 2$ and $q(m - \alpha/2) = p_i'(m - \alpha/n) = 3$ ($i = 1, 2$). Take $w_1(x) = w_2(x) = |x|^{-3/4}$. Then $v_{\vec{w}}(x)^q = |x|^{-3} \notin L^1_{\mathrm{loc}}(\mathbb{R})$, so $v_{\vec{w}}^q \notin A_3$, but $w_i(x)^{-p_i'} = |x|^{3/2} \in A_3$ ($i = 1, 2$).

As a consequence of Theorems 3.5 and 3.7, we have the following surprising result.

THEOREM 3.9. *Let $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < mn/\alpha$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. Then*

$$v_{\vec{w}}^q \in A_{mq} \quad \text{and} \quad w_i^{-p_i'} \in A_{mp_i'},\ i = 1, \dots, m,$$

where the condition $w_i^{-p_i'} \in A_{mp_i'}$ in the case $p_i = 1$ is understood as $w_i^{1/m} \in A_1$, if and only if

$$v_{\vec{w}}^q \in A_{q(m-\alpha/n)} \quad \text{and} \quad w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}, \quad i = 1, \dots, m,$$

where the condition $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}$ in the case $p_i = 1$ is understood as $w_i^{n/(mn-\alpha)} \in A_1$.

In addition, applying Theorem 3.5, we will establish the following results.

THEOREM 3.10. *Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $q \ge p$ and $p_0 = \min_{1 \le i \le m} p_i$. Then*

$$A_{(\vec{P},q)} = \bigcup_{1<r<p_0} A_{(\vec{P},q,r)}.$$

More generally, for any $1 \le r_1 < p_0$,

$$A_{(\vec{P},q,r_1)} = \bigcup_{r_1<r<p_0} A_{(\vec{P},q,r)}.$$

Here and below, we denote

$$A_{(\vec{P},\,q,\,s)} := \{\vec{w} = (w_1, \dots, w_m) : \vec{w}^s = (w_1^s, \dots, w_m^s) \in A_{(\vec{P}/s,\, q/s)}\},\ \forall s \ge 1.$$

THEOREM 3.11. *Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $1/m < p \le q < \infty$ and $p_0 = \min_{1\le i\le m} p_i$. Then $A_{(\vec P,q,r)}$ is strictly decreasing as r increases, more precisely,*

$$A_{(\vec P,q,r_1)} \supsetneq A_{(\vec P,q,r_2)}, \qquad \forall\, 1 \le r_1 < r_2 < p_0.$$

In particular,

$$A_{(\vec P,q)} \supsetneq A_{(\vec P,q,r)}, \quad \forall\, 1 < r < p_0.$$

3.2. Proofs of main theorems

Proof of Theorem 3.5. Employing some techniques from [LOPTT, Mo], we first consider the case when there exists at least one $p_j > 1$. Without loss of generality we may assume that $p_1 = \cdots = p_l = 1$, $0 \le l < m$, and $p_j > 1$ for $j = l+1, \dots, m$. Then we have $mq \ge mp > 1$, $0 < q/(mq-1) \le p/(mp-1)$ and $\sum_{i=l+1}^m 1/p_i' = (mp-1)/p$.

(i) *The proof that $\vec w \in A_{(\vec P,q)} \Rightarrow$* (3.3): Suppose that $\vec w \in A_{(\vec P,q)}$. If $l = 0$, then (3.3) follows from Theorem 3.3. Now we suppose that $l > 0$. We first prove that $w_j^{-p_j'} \in A_{mp_j'}$ for $j \ge l+1$. Since $\vec w \in A_{(\vec P,q)}$, we have

$$\left(\frac{1}{|Q|}\int_Q v_{\vec w}(x)^q\,dx\right)^{1/q} \prod_{i=1}^{l}\Big(\inf_Q w_i(x)\Big)^{-1} \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/p_i'} \le C.$$

Then

$$\left(\frac{1}{|Q|}\int_Q \prod_{i=l+1}^{m} w_i(x)^q\,dx\right)^{1/q} \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/p_i'} \le C.$$

Since $p \le q$, we get

$$\left(\frac{1}{|Q|}\int_Q \prod_{i=l+1}^{m} w_i(x)^p\,dx\right)^{p_j'/p} \prod_{i=l+1,\, i\ne j}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{p_j'/p_i'} \times \left(\frac{1}{|Q|}\int_Q w_j(x)^{-p_j'}\,dx\right) \le C. \tag{3.5}$$

Notice that

$$\sum_{i=l+1, i\ne j}^{m} \frac{p_j'}{p_i'} + \frac{p_j'}{p} = mp_j' - 1, \quad j = l+1, \dots, m.$$

Applying Hölder's inequality, for $j = l+1, \dots, m$ we have

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q w_j(x)^{p_j'/(mp_j'-1)}\,dx\Big)^{mp_j'-1}\\ &= \Big[\frac{1}{|Q|}\int_Q\Big(\prod_{i=l+1}^m w_i(x)\Big)^{p_j'/(mp_j'-1)}\prod_{i=l+1,i\neq j}^m w_i(x)^{-p_j'/(mp_j'-1)}\,dx\Big]^{mp_j'-1}\\ &\leq \Big(\frac{1}{|Q|}\int_Q\prod_{i=l+1}^m w_i(x)^p\,dx\Big)^{p_j'/p}\prod_{i=l+1,i\neq j}^m\Big(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\Big)^{p_j'/p_i'}.\end{aligned}$$

This together with (3.5) implies that $w_j^{-p_j'}\in A_{mp_j'}$ for $j\geq l+1$.

Next we show that $v_{\vec{w}}^q\in A_{mq}$. Since $0<q/(mq-1)\leq p/(mp-1)$, we have

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q\prod_{i=l+1}^m w_i(x)^{-q/(mq-1)}\,dx\Big)^{(mq-1)/q}\\ &\leq\Big(\frac{1}{|Q|}\int_Q\prod_{i=l+1}^m w_i(x)^{-p/(mp-1)}\,dx\Big)^{(mp-1)/p}.\end{aligned}$$

From $\sum_{i=l+1}^m 1/p_i'=(mp-1)/p$ and Hölder's inequality,

$$\Big(\frac{1}{|Q|}\int_Q\prod_{i=l+1}^m w_i(x)^{-p/(mp-1)}\,dx\Big)^{(mp-1)/p}\leq\prod_{i=l+1}^m\Big(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\Big)^{1/p_i'}.$$

Hence,

(3.6)
$$\Big(\frac{1}{|Q|}\int_Q\prod_{i=l+1}^m w_i(x)^{-q/(mq-1)}\,dx\Big)^{mq-1}\leq\prod_{i=l+1}^m\Big(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\Big)^{q/p_i'}.$$

Then,

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{-q/(mq-1)}\,dx\Big)^{mq-1}\\ &\leq\prod_{i=1}^l\Big(\inf_Q w_i(x)\Big)^{-q}\prod_{i=l+1}^m\Big(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\Big)^{q/p_i'}.\end{aligned}\tag{3.7}$$

Since $\vec{w}\in A_{(\vec{P},q)}$, we have

(3.8)
$$\Big(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\Big)\prod_{i=1}^l\Big(\inf_Q w_i(x)\Big)^{-q}\prod_{i=l+1}^m\Big(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\Big)^{q/p_i'}\leq C.$$

Combining (3.7) with (3.8), we conclude that $v_{\vec{w}}^q\in A_{mq}$.

In what follows, we will show that $w_j^{1/m} \in A_1$, $j = 1, \dots, l$. By Hölder's inequality with qm and $(qm)' = qm/(qm-1)$, we have

$$\left(\frac{1}{|Q|}\int_Q w_j(x)^{1/m}\,dx\right) \le \left(\frac{1}{|Q|}\int_Q \prod_{i=l+1}^{m} w_i(x)^{-q/(qm-1)}\,dx\right)^{(qm-1)/(qm)} \times \left(\frac{1}{|Q|}\int_Q w_j(x)^q \prod_{i=l+1}^{m} w_i(x)^q\,dx\right)^{1/(qm)}.$$

This together with (3.6) yields

$$\left(\frac{1}{|Q|}\int_Q w_j(x)^{1/m}\,dx\right) \le \left(\frac{1}{|Q|}\int_Q w_j(x)^q \prod_{i=l+1}^{m} w_i(x)^q\,dx\right)^{1/(qm)} \times \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/(p_i'm)}. \tag{3.9}$$

On the other hand, since $\vec{w} \in A_{(\vec{P},q)}$, we have

$$\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)^{1/q} \prod_{i=1}^{l}\Big(\inf_Q w_i(x)\Big)^{-1} \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/p_i'} \le C.$$

Then,

$$\left(\frac{1}{|Q|}\int_Q w_j(x)^q \prod_{i=l+1}^{m} w_i(x)^q\,dx\right)^{1/q} \Big(\inf_Q w_j(x)\Big)^{-1} \times \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/p_i'} \le C.$$

Hence,

$$\left(\frac{1}{|Q|}\int_Q w_j(x)^q \prod_{i=l+1}^{m} w_i(x)^q\,dx\right)^{1/(qm)} \Big(\inf_Q w_j(x)^{1/m}\Big)^{-1} \times \prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/(p_i'm)} \le C. \tag{3.10}$$

Using (3.9) and (3.10), we obtain $w_j^{1/m} \in A_1$ for $j = 1, \dots, l$. This completes the proof that $\vec{w} \in A_{(\vec{P},q)} \Rightarrow$ (3.3).

(ii) *The proof that* (3.3) *is sufficient for* $\vec{w} \in A_{(\vec{P},q)}$: Suppose that (3.3) holds, i.e., $v_{\vec{w}}^q \in A_{mq}$, $w_i^{1/m} \in A_1$ for $i = 1, \dots, l$, and $w_i^{-p_i'} \in A_{mp_i'}$ for

$i = l+1, \dots, m$. Then,

$$\begin{cases} \left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{1/m})\, dx\right)\left(\inf\limits_Q w_i(x)^{1/m}\right)^{-1} \leq C, \quad i = 1, \dots, l, \\ \left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{-p_i'}\, dx\right)\left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{p_i'/(mp_i'-1)}\, dx\right)^{mp_i'-1} \leq C, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = l+1, \dots, m, \\ \left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^q\, dx\right)\left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^{-q/(mq-1)}\, dx\right)^{mq-1} \leq C. \end{cases}$$

Hence,

$$\begin{cases} \left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{1/m}\, dx\right)^m\left(\inf\limits_Q w_i(x)\right)^{-1} \leq C, \quad i = 1, \dots, l, \\ \left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{-p_i'}\, dx\right)^{1/p_i'}\left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{p_i'/(mp_i'-1)}\, dx\right)^{(mp_i'-1)/p_i'} \leq C, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = l+1, \dots, m, \\ \left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^q\, dx\right)^{1/q}\left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^{-q/(mq-1)}\, dx\right)^{(mq-1)/q} \leq C. \end{cases}$$

Consequently,

$$\begin{aligned} \left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^q\, dx\right)^{1/q} &\prod_{i=1}^{l}\left(\inf\limits_Q w_i(x)\right)^{-1}\prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{-p_i'}\, dx\right)^{1/p_i'} \\ &\times\left(\frac{1}{|Q|}\int\limits_Q v_{\vec{w}}(x)^{-q/(mq-1)}\, dx\right)^{(mq-1)/q}\prod_{i=1}^{l}\left(\frac{1}{|Q|}\int\limits_Q w_i(x)^{1/m}\, dx\right)^m \\ &\times\prod_{i=l+1}^{m}\left(\frac{1}{|Q|}\int\limits_Q w_j(x)^{p_i'/(mp_i'-1)}\, dx\right)^{(mp_i'-1)/p_i'} \leq C. \end{aligned} \tag{3.11}$$

Notice that $(mq-1)/q > 0$, $m > 0$ and $(mp_i'-1)/p_i' > 0$ for $i = l+1, \dots, m$. Set

$$\frac{1}{t} = \frac{mq-1}{q} + ml + \sum_{i=l+1}^{m}\frac{mp_i'-1}{p_i'}.$$

By Hölder's inequality,

$$
\begin{aligned}
1 &= \Big(\frac{1}{|Q|}\int_Q \Big(v_{\vec w}(x)^{-1}\prod_{i=1}^m w_i(x)\Big)^t\,dx\Big)^{1/t}\\
&\le \Big(\frac{1}{|Q|}\int_Q v_{\vec w}(x)^{-q/(mq-1)}\,dx\Big)^{(mq-1)/q}\\
&\quad\times \prod_{i=1}^l\Big(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\Big)^m \prod_{i=l+1}^m\Big(\frac{1}{|Q|}\int_Q w_j(x)^{p_i'/(mp_i'-1)}\,dx\Big)^{(mp_i'-1)/p_i'}
\end{aligned}
$$

This together with (3.11) implies that $\vec w\in A_{(\vec P,q)}$.

It remains to consider the case $p_j=1$ for all $j=1,\dots,m$. Notice that $q\ge p=1/m$. Assume that $\vec w\in A_{((1,\dots,1),q)}$, i.e.,

$$
\Big(\frac{1}{|Q|}\int_Q v_{\vec w}(x)^q\,dx\Big)^{1/q}\prod_{i=1}^m\Big(\inf_Q w_i(x)\Big)^{-1}\le C. \tag{3.12}
$$

We first prove that $w_j^{1/m}\in A_1$, $j=1,\dots,m$. By (3.12), we have

$$
\Big(\frac{1}{|Q|}\int_Q w_j(x)^q\,dx\Big)^{1/q}\Big(\inf_Q w_j(x)\Big)^{-1}\le C.
$$

Since $1/m\le q$, by Hölder's inequality,

$$
\Big(\frac{1}{|Q|}\int_Q w_j(x)^{1/m}\,dx\Big)^m\Big(\inf_Q w_j(x)\Big)^{-1}\le C.
$$

This implies $w_j^{1/m}\in A_1$.

Next we show that $v_{\vec w}^q\in A_{mq}$. It follows from (3.12) that

$$
\Big(\frac{1}{|Q|}\int_Q v_{\vec w}(x)^q\,dx\Big)^{1/q}\le C\prod_{i=1}^m\inf_Q w_i(x)\le C\inf_Q\Big(\prod_{i=1}^m w_i(x)\Big).
$$

Hence,

$$
\frac{1}{|Q|}\int_Q v_{\vec w}(x)^q\,dx\le C\inf_Q v_{\vec w}(x)^q.
$$

Then $v_{\vec w}^q\in A_1$. Since $mq\ge 1$, we have $v_{\vec w}^q\in A_{mq}$.

Now we prove the converse. Suppose that

$$
v_{\vec w}^q\in A_{mq}\quad\text{and}\quad w_i^{1/m}\in A_1,\ i=1,\dots,m. \tag{3.13}
$$

Consider first the case when $q>p=1/m$. By (3.13), we have

$$\begin{cases} \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)\left(\inf_Q w_i(x)^{1/m}\right)^{-1} \le C, \quad i=1,\dots,m, \\ \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{-q/(mq-1)}\,dx\right)^{mq-1} \le C. \end{cases}$$

Hence,

$$\begin{cases} \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m \left(\inf_Q w_i(x)\right)^{-1} \le C, \quad i=1,\dots,m, \\ \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{-q/(mq-1)}\,dx\right)^{(mq-1)/q} \le C. \end{cases}$$

Then,

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)^{1/q} &\prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{-q/(mq-1)}\,dx\right)^{(mq-1)/q} \\ &\times \left(\inf_Q w_i(x)\right)^{-1}\prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m \le C. \end{aligned} \tag{3.14}$$

On the other hand, notice that $q/(mq-1)>0$, $1/m>0$, and set

$$\frac{1}{s} = \frac{mq-1}{q} + m^2.$$

By Hölder's inequality,

$$\begin{aligned} 1 &= \left(\frac{1}{|Q|}\int_Q \left(v_{\vec{w}}(x)^{-1}\prod_{i=1}^m w_i(x)\right)^s dx\right)^{1/s} \\ &\le \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{-q/(mq-1)}\,dx\right)^{(mq-1)/q}\prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m \end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15) shows that $\vec{w} \in A_{(\vec{P},q)}$.

Now let us consider the case when $q=p=1/m$. It follows from (3.13) that

$$\begin{cases} \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)\left(\inf_Q w_i(x)^{1/m}\right)^{-1} \le C, \quad i=1,\dots,m, \\ \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{1/m}\,dx\right)\left(\inf_Q v_{\vec{w}}(x)^{1/m}\right)^{-1} \le C. \end{cases}$$

Then,

$$\begin{cases} \left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m\left(\inf_Q w_i(x)\right)^{-1} \le C, \quad i=1,\dots,m, \\ \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{1/m}\,dx\right)^m\left(\inf_Q v_{\vec{w}}(x)\right)^{-1} \le C. \end{cases}$$

Consequently,

$$\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^{1/m}\,dx\right)^m \prod_{i=1}^m\left(\inf_Q w_i(x)\right)^{-1}\left(\inf_Q v_{\vec{w}}(x)\right)^{-1} \times \prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m \le C. \tag{3.16}$$

On the other hand, by Hölder's inequality,

$$\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q w_i(x)^{1/m}\,dx\right)^m \ge \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^{1/m^2}\,dx\right)^{m^2} \ge \inf_Q v_{\vec{w}}(x).$$

Together with (3.16) this implies that $\vec{w} \in A_{(\vec{P},q)}$ and completes the proof of Theorem 3.5. ■

Proof of Theorem 3.7. By Theorem 3.4, $\vec{w} \in A_{(\vec{P},q)}$ if and only if $\vec{w}^{\vec{q}} \in A_{\vec{S}}$, where $\vec{q}$ and $\vec{S}$ are defined as in Theorem 3.3. By Theorem 2.1, $\vec{w}^{\vec{q}} \in A_{\vec{S}}$ is equivalent to

$$\prod_{i=1}^m w_i^{q_i s/s_i} \in A_{ms} \quad \text{and} \quad w_i^{q_i(1-s_i')} \in A_{ms_i'}, \quad i=1,\dots,m, \tag{3.17}$$

where the condition $w_i^{q_i(1-s_i')} \in A_{ms_i'}$ in the case $s_i = 1$ is understood as $w_i^{q_i/m} \in A_1$ $(i=1,\dots,m)$. Notice that

$$q_i(1-s_i') = -p_i', \quad ms_i' = p_i'(m-\alpha/n), \quad q_i s/s_i = q, \quad ms = q(m-\alpha/n),$$

and $s_i = 1$ if and only if $p_i = 1$, while

$$\frac{q_i}{m} = \frac{np_i}{mn-\alpha p_i} = \frac{n}{mn-\alpha} \quad \text{if } s_i = p_i = 1,\ i=1,\dots,m.$$

Thus (3.17) is equivalent to $v_{\vec{w}}^q \in A_{q(m-\alpha/n)}$ and $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}$ for $i=1,\dots,m$. When $p_i = 1$, the condition $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n)}$ is understood as $w_i^{n/(mn-\alpha)} \in A_1$. Theorem 3.7 is proved. ■

Proof of Theorem 3.10. We first prove that $A_{(\vec{P},q)} \subset \bigcup_{1<r<p_0} A_{(\vec{P},q,r)}$. Suppose that $\vec{w} \in A_{(\vec{P},q)}$, By Theorem 3.2, each $w_i^{-p_i'}$ is in A_∞, and hence there are constants $c_i, t_i > 1$ (t_i sufficiently close to 1) such that for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i' t_i}\,dx\right)^{1/t_i} \le \frac{c_i}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx.$$

Let $r_i > 1$ be selected so that

$$\frac{r_i}{p_i - r_i} = \frac{t_i}{p_i - 1}.$$

Let $r = \min\{r_1, \ldots, r_m\}$. Then r sufficiently close to 1, with $1 < r < p_0$. We have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m w_i(x)^r\,dx\right)^{q/r}\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i r/(p_i - r)}\,dx\right)^{(p_i - r)/p_i}\\
&\quad = \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^q\,dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i r/(p_i - r)}\,dx\right)^{(p_i - r)r/(p_i r)}\\
&\quad \le \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^q\,dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i r_i/(p_i - r_i)}\,dx\right)^{(p_i - r_i)r/(p_i r_i)}\\
&\quad = \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i' t_i}\,dx\right)^{r/(p_i' t_i)}\\
&\quad \le C\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}(x)^q\,dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{r/p_i'}\\
&\quad \le C[\vec{w}]^r_{A_{(\vec{P},q)}}.
\end{aligned}$$

Next we will show that $A_{(\vec{P},q,r)} \subset A_{(\vec{P},q)}$ for any $1 < r < p_0$. Let $\vec{w} \in A_{(\vec{P},q,r)}$, i.e., $\vec{w}^r \in A_{(\vec{P}/r,q/r)}$, which implies that

$$\left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m w_i(x)^r\right)^{q/r} dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-r(p_i/r)'}\,dx\right)^{1/(p_i/r)'} \le C.$$

Then,

$$\left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^q\,dx\right)^{r/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-rp_i/(p_i - r)}\,dx\right)^{(p_i - r)/p_i} \le C.$$

Hence,

$$\left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^q\,dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-rp_i/(p_i - r)}\,dx\right)^{(p_i - r)/(rp_i)} \le C.$$

Since $p_i' \le rp_i/(p_i - r)$, by Hölder's inequality,

$$\left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m w_i(x)^q\,dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i(x)^{-p_i'}\,dx\right)^{1/p_i'} \le C,$$

which implies that $\vec{w} \in A_{(\vec{P},q)}$. This completes the proof of the equality $A_{(\vec{P},q)} = \bigcup_{1<r<p_0} A_{(\vec{P},q,r)}$.

It remains to show that for any $1 < r_1 < p_0$,

$$A_{(\vec{P},q,r_1)} = \bigcup_{r_1<r<p_0} A_{(\vec{P},q,r)}.$$

Since $p_0/r_1 = \min_{1\le i\le m}\ p_i/r_1$, we have

$$A_{(\vec{P}/r_1,q/r_1)} = \bigcup_{1<s<p_0/r_1} A_{(\vec{P}/r_1,q/r_1,s)}. \tag{3.18}$$

Suppose that $\vec{w} \in A_{(\vec{P},q,r_1)}$, i.e.,

$$\vec{w}^{r_1} \in A_{(\vec{P}/r_1,q/r_1)} = \bigcup_{1<s<p_0/r_1} A_{(\vec{P}/r_1,q/r_1,s)}.$$

Then $\vec{w}^{r_1} \in A_{(\vec{P}/r_1,q/r_1,s_0)}$ for some $1 < s_0 < p_0/r_1$, which implies that $\vec{w}^{r_1 s_0} \in A_{(\vec{P}/(r_1 s_0),\ q/(r_1 s_0))}$. Hence,

$$\vec{w} \in A_{(\vec{P},\,q,\,r_1 s_0)}.$$

Since $r_1 < r_1 s_0 < p_0$, we get

$$\vec{w} \in \bigcup_{r_1<r<p_0} A_{(\vec{P},\,q,\,r)}.$$

Conversely, suppose that $\vec{w} \in A_{(\vec{P},q,r)}$ for some $r_1 < r < p_0$, so $\vec{w}^r \in A_{(\vec{P}/r,q/r)}$. Hence,

$$(\vec{w}^{r_1})^{r/r_1} \in A_{((\vec{P}/r_1)/(r/r_1),(q/r_1)/(r/r_1))}.$$

Equivalently, we have $\vec{w}^{r_1} \in A_{(\vec{P}/r_1,q/r_1,\,r/r_1)}$. Note that $1 < r/r_1 < p_0/r_1$, combining the previous condition with (3.18), we have $\vec{w}^{r_1} \in A_{(\vec{P}/r_1,\,q/r_1)}$, i.e., $\vec{w} \in A_{(\vec{P},q,r_1)}$. This proves Theorem 3.10. ∎

Proof of Theorem 3.11. By Theorem 3.10, $A_{(\vec{P},q,r)}$ is decreasing as r increases, i.e., for $1 \le r_1 < r_2 < p_0$,

$$A_{(\vec{P},q,r_1)} \supseteq A_{(\vec{P},q,r_2)}.$$

It remains to show that $A_{(\vec{P},q,r_1)} \ne A_{(\vec{P},q,r_2)}$ for $1 \le r_1 < r_2 < p_0$. We first prove that $A_{(\vec{P},q)} \neq A_{(\vec{P},q,r)}$ for $1 < r < p_0$. Set $\varepsilon_i = \min\{n - n/r, n - n/p_i\}$, $a_i = n - n/p_i - \varepsilon_i$ and $w_i = |x|^{a_i}$, $i = 1, \ldots, m$. Then

$$\begin{cases} -nm + n - n/p_i < 0 \le a_i < n - n/p_i, & i = 1, \ldots, m, \\ a_i \ge n/r - n/p_i, & i = 1, \ldots, m, \\ -n < 0 \le q \displaystyle\sum_{i=1}^{m} a_i < qmn - qn/p \le qmn - n. \end{cases}$$

Hence,

$$\begin{cases} -a_i p_i' \in (-n,\, n(mp_i' - 1)), & i = 1, \ldots, m, \\ -a_i r (p_i/r)' \notin (-n,\, n(m(p_i/r)' - 1)), & i = 1, \ldots, m, \\ q \displaystyle\sum_{i=1}^{m} a_i \in (-n,\, n(mq - 1)). \end{cases}$$

This leads to $v_{\vec{w}}^q \in A_{mq}$, $w_i^{-p_i'} \in A_{mp_i'}$ and $(w_i^r)^{-(p_i/r)'} \notin A_{m(p_i/r)'}$ for $i = 1, \ldots, m$. By Theorem 3.4, we get $\vec{w} \in A_{(\vec{P},q)}$ and $\vec{w} \notin A_{(\vec{P},q,r)}$. This proves that $A_{(\vec{P},q)} \supsetneq A_{(\vec{P},q,r)}$ for $1 < r < p_0$.

Next we show that $A_{(\vec{P},q,r_1)} \neq A_{(\vec{P},q,r_2)}$ for $1 < r_1 < r_2 < p_0$. Since $A_{(\vec{P},q)} \supsetneq A_{(\vec{P},q,r)}$ for $1 < r < p_0$ and $1 < r_2/r_1 < p_0/r_1 = \min_{i=1}^{m} p_i/r_1$, we have $A_{(\vec{P}/r_1,\, q/r_1)} \supsetneq A_{(\vec{P}/r_1,\, q/r_1,\, r_2/r_1)}$. Thus there exists a $\vec{w} \in A_{(\vec{P}/r_1,\, q/r_1)}$ such that $\vec{w} \notin A_{(\vec{P}/r_1,\, q/r_1,\, r_2/r_1)}$. Hence, $\vec{w}^{1/r_1} \in A_{(\vec{P},\, q,\, r_1)}$, but $\vec{w}^{1/r_1} \notin A_{(\vec{P},\, q,\, r_2)}$. This implies that $A_{(\vec{P},\, q,\, r_1)} \neq A_{(\vec{P},\, q,\, r_2)}$. Theorem 3.11 is proved. ∎

Acknowledgements. This research was partly supported by NNSF of China (grants no. 11371295 and 11471041).

References

[CX] X. Chen and Q. Xue, *Weighted estimates for a class of multilinear fractional type operators*, J. Math. Anal. Appl. 362 (2010), 355–373.

[LOPTT] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory*, Adv. Math. 220 (2009), 1222–1264.

[Mo] K. Moen, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math. 60 (2009), 213–238.

[MW] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. 192 (1974), 261–274.

[PPTT] C. Pérez, G. Pradolini, R. H. Torres and R. Trujillo-González, *End-point estimates for iterated commutators of multilinear singular integrals*, Bull. London Math. Soc. 46 (2014), 26–42.

[Pr] G. Pradolini, *Weighted inequalities and point-wise estimates for the multilinear fractional integral and maximal operators*, J. Math. Anal. Appl. 367 (2010), 640–656.

Songqing Chen
School of Mathematical Sciences
Xiamen University
Xiamen, Fujian 361005, China
and
School of Economics
Shanghai University of Finance and Economics
Shanghai 240033, China
E-mail: chensongqing0@126.com

Huoxiong Wu (corresponding author)
School of Mathematical Sciences
Xiamen University
Xiamen, Fujian 361005, China
E-mail: huoxwu@xmu.edu.cn

Qingying Xue
School of Mathematical Sciences
Beijing Normal University
Beijing 100875, China
E-mail: qyxue@bnu.edu.cn

Received August 15, 2013
Revised version July 27, 2014 (7521)