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A local Landau type inequality for semigroup orbits

by

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Abstract. Given a strongly continuous semigroup $(S(t))_{t\ge 0}$ on a Banach space X with generator A and an element $f \in D(A^2)$ satisfying $\|S(t)f\| \le e^{-\omega t}\|f\|$ and $\|S(t)A^2 f\| \le e^{-\omega t}\|A^2 f\|$ for all $t \ge 0$ and some $\omega > 0$, we derive a Landau type inequality for $\|Af\|$ in terms of $\|f\|$ and $\|A^2 f\|$. This inequality improves on the usual Landau inequality that holds in the case $\omega = 0$.

1. Introduction. For C^2-functions $f : [0,\infty) \to \mathbb{R}$, Edmund Landau proved in 1913 that $\|f'\|_\infty^2 \le 4\|f''\|_\infty \|f\|_\infty$. Hardy, Littlewood, and Pólya proved in 1934 that $\|f'\|_2^2 \le 2\|f''\|_2\|f\|_2$, and Hardy, Landau, and Littlewood proved in 1935 that $\|f'\|_p^2 \le 4\|f''\|_p\|f\|_p$ for $1 \le p \le \infty$ (see also, e.g., [4], where much more results can be found, and the references therein).

Later on, these inequalities were recognized as special cases of more general inequalities that hold in Banach spaces X for generators A of strongly continuous semigroups $(S(t))_{t\ge 0} = (e^{tA})_{t\ge 0}$ which are *contractive*, i.e. which satisfy $\|S(t)\| \le 1$ for all $t \ge 0$. If $f \in D(A^2)$, then

$$\|Af\|^2 \le 4\|A^2 f\|\|f\| \tag{1.1}$$

(cf. [2, 1]), and if X is in addition a Hilbert space, then

$$\|Af\|^2 \le 2\|A^2 f\|\,\|f\|, \qquad f \in D(A^2) \tag{1.2}$$

(cf. [3]). The constants 4 and 2 are known to be optimal.

In this paper we study estimates of $\|Af\|$ for $f \in D(A^2)$ in terms of $\|f\|$ and $\|A^2 f\|$ under the assumption that

$$\|S(t)f\| \le e^{-\omega t}\|f\| \quad \text{and} \quad \|S(t)A^2 f\| \le e^{-\omega t}\|A^2 f\|, \qquad t \ge 0, \tag{1.3}$$

where $\omega > 0$. Compared to the contractivity assumption we assume an exponential *decay*, but only for the two elements f and $A^2 f$. It turns out that our estimate is best formulated in terms of the three quantities

$$a = \|f\|, \qquad b = \|Af\|/\omega, \qquad c = \|A^2 f\|/\omega^2.$$

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This corresponds to the fact that changing the time $s = t/\omega$, or in other words, replacing A by A/ω in (1.3), we can resort to the case $\omega = 1$.

Remark 1.1. If $\omega > 0$ is such that

$$\|S(t)\| \le e^{-\omega t}, \qquad t \ge 0, \tag{1.4}$$

then $0 \in \rho(A)$ and

$$-A^{-1} f = \int_0^\infty S(t) f \, dt, \qquad f \in X,$$

which implies $\|A^{-1}\| \le 1/\omega$. In this case we clearly have

$$\|f\| = \|A^{-1} A f\| \le \|Af\|/\omega = \|A^{-1} A^2 f\|/\omega \le \|A^2 f\|/\omega^2$$

for any $f \in D(A^2)$, i.e. $a \le b \le c$.

In our main result, we relax the assumption (1.4) on the semigroup considerably. We only assume that $f \in D(A^2)$ is such that the estimate (1.3) holds. In contrast to the situation in Remark 1.1, the inequality $a \le b \le c$ is then no longer immediate. Nevertheless, this inequality still holds true, and much more can be said.

2. Main result

Theorem 2.1. *Let $(S(t))_{t \ge 0}$ be a strongly continuous semigroup in a Banach space with generator A. Suppose that $f \in D(A^2) \setminus \{0\}$ and $\omega > 0$ are such that*

$$\|S(t) f\| \le e^{-\omega t} \|f\|, \quad \|S(t) A^2 f\| \le e^{-\omega t} \|A^2 f\|, \qquad t \ge 0.$$

Writing $a = \|f\|$, $b = \|Af\|/\omega$ and $c = \|A^2 f\|/\omega^2$, we have

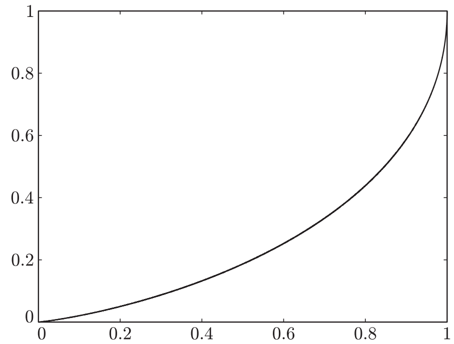
$$a \le b \le c \quad \text{and} \quad b = c \Rightarrow a = c.$$

Moreover,

$$b \le c - (a + c) h\left(\frac{c - a}{c + a}\right) \tag{2.1}$$

where $h : [0,1] \to [0,1]$ is the inverse function of the continuous, bijective, and strictly increasing function

$$g : [0,1] \to [0,1], \qquad \beta \mapsto \begin{cases} \beta(1 - \log \beta), & \beta \in (0,1], \\ 0, & \beta = 0. \end{cases}$$



Graph of the function $h : [0,1] \to [0,1]$

We start the proof with the following

LEMMA 2.2. *Under the assumptions of Theorem* 2.1 *we have* $a \le c$, *i.e.* $\|f\| \le \|A^2 f\|/\omega^2$.

Proof. This follows by a standard estimate, once we have shown

$$f = \int_0^\infty tS(t)A^2 f\,dt.$$

Observe that the integral on the right hand side converges absolutely by assumption. Let $T > 0$. Integration by parts gives

$$\int_0^T tS(t)A^2 f\,dt = TS(T)Af - \int_0^T S(t)Af\,dt = f - S(T)f + TS(T)Af.$$

Letting $T \to \infty$ we obtain

$$\lim_{T\to\infty} TS(T)Af = \int_0^\infty tS(t)A^2 f\,dt - f =: \tilde{f}.$$

We shall show $\tilde{f} = 0$. To this end we take $\varepsilon > 0$. First we find $\gamma_0 > 0$ such that

$$\|tS(t)Af - \tilde{f}\| < \varepsilon \quad \text{for } t \ge \gamma_0.$$

For $\gamma \ge \gamma_0$ we then have

$$\begin{aligned}\|\tilde{f}\| &= \Big\| \int_\gamma^{e\gamma} \frac{\tilde{f}}{t}\,dt \Big\| \le \Big\| \int_\gamma^{e\gamma} S(t)Af\,dt \Big\| + \int_\gamma^{e\gamma} \frac{\|\tilde{f} - tS(t)Af\|}{t}\,dt \\ &\le \Big\| \int_\gamma^{e\gamma} S(t)Af\,dt \Big\| + \varepsilon \int_\gamma^{e\gamma} \frac{dt}{t} = \Big\| \int_\gamma^{e\gamma} S(t)Af\,dt \Big\| + \varepsilon.\end{aligned}$$

Taking γ large we see that the first term is $\le \varepsilon$, since

$$\int_0^T S(t)Af\,dt = S(T)f - f$$

shows that the integral $\int_0^\infty S(t)Af\,dt$ converges as an improper Riemann integral. We thus have $\|\tilde f\| \le 2\varepsilon$, and $\tilde f = 0$ follows. ∎

Proof of Theorem 2.1. For any $f \in D(A^2)$ we have

$$S(t)f - f = \int_0^t S(\tau)Af\,d\tau, \qquad t \ge 0,$$

$$S(\tau)Af - Af = \int_0^\tau S(s)A^2f\,ds, \qquad \tau \ge 0,$$

which implies

$$S(t)f = f + tAf + \int_0^t (t-\tau)S(\tau)A^2f\,d\tau, \qquad t \ge 0.$$

For $f \in D(A^2)$ satisfying the assumptions we hence obtain, for any $t \ge 0$,

$$\begin{aligned} e^{-\omega t}\|f\| \ge \|S(t)f\| &\ge \|f + tAf\| - \int_0^t (t-\tau)e^{-\omega\tau}\,d\tau\,\|A^2f\| \\ &\ge t\|Af\| - \|f\| - \|A^2f\|\Big(\frac{e^{-\omega t}}{\omega^2} - \frac{1-\omega t}{\omega^2}\Big). \end{aligned}$$

We rewrite this as

$$e^{-\omega t}\Big(\|f\| + \frac{\|A^2f\|}{\omega^2}\Big) + \omega t\Big(\frac{\|A^2f\|}{\omega^2} - \frac{\|Af\|}{\omega}\Big) + \Big(\|f\| - \frac{\|A^2f\|}{\omega^2}\Big) \ge 0.$$

Using $a = \|f\|$, $b = \|Af\|/\omega$ and $c = \|A^2f\|/\omega^2$ and writing $s = \omega t$, we arrive at

$$e^{-s}(a+c) + s(c-b) + (a-c) \ge 0 \quad \text{for any } s \ge 0. \tag{2.2}$$

By Lemma 2.2 we have $a \le c$. Moreover, we can rewrite (2.2) as

$$e^{-s}(a+c) \ge s(b-c) + (c-a), \qquad s \ge 0,$$

and letting $s \to \infty$ we see that $b \le c$ and that $b = c$ implies $c = a$ via Lemma 2.2.

We postpone the proof of $a \le b$ and discuss the properties of g. We have $g(0) = 0$, $g(1) = 1$, and g is continuous. On $(0,1]$, g is differentiable with $g'(\beta) = -\log\beta$ for $\beta \in (0,1]$. We conclude that g is strictly increasing and maps $[0,1]$ onto $[0,1]$. The inverse function $h : [0,1] \to [0,1]$ of g is thus bijective, strictly increasing and differentiable on $[0,1)$.

For the proof of $a \le b$ we modify the estimate above and use $\|f+tAf\| \ge \|f\| - t\|Af\|$ instead of $\|f+tAf\| \ge t\|Af\| - \|f\|$. This yields

$$e^{-s}(a+c) + s(c+b) - (a+c) \ge 0, \qquad s \ge 0. \tag{2.3}$$

For $s = 0$ the left hand side of (2.3) equals 0. Hence its derivative at $s = 0$ has to be ≥ 0, i.e. $-(a+c) + (c+b) \ge 0$, which means $a \le b$.

For (2.1) we may assume $b < c$. We minimize the left hand side of (2.2), i.e. we take $s = s_0$ given by $e^{-s_0}(a+c) = c - b$ and let $\beta_0 = e^{-s_0} = \frac{c-b}{a+c}$. This yields

$$e^{-s_0}(a+c)(s_0+1) \ge c - a, \quad \text{or} \quad g(\beta_0) = \beta_0(1 - \log \beta_0) \ge \frac{c-a}{a+c},$$

and finally

$$\frac{c-b}{a+c} = \beta_0 \ge h\left(\frac{c-a}{a+c}\right), \quad \text{or} \quad b \le c - (a+c)h\left(\frac{c-a}{a+c}\right),$$

as asserted. ∎

Remark 2.3. We discuss the quality of the estimate (2.1). First observe that (2.1) is invariant under the scaling $f \mapsto \lambda f$. Hence we may assume $a = 1$. Taking, for the moment, $c = \alpha b^2$ where $\alpha > 0$, we study the quotient

$$\frac{c - (a+c)h\left(\frac{c-a}{c+a}\right)}{b} = \alpha b - \frac{1+\alpha b^2}{b} h\left(\frac{\alpha b^2 - 1}{1 + \alpha b^2}\right)$$

for $b \to \infty$. Since $h(1) = 1$ we have

$$\frac{1}{b} h\left(\frac{\alpha b^2 - 1}{1+\alpha b^2}\right) \to 0 \qquad (b \to \infty),$$

and we have to study

$$\psi(b) := \alpha b\left(1 - h\left(\frac{\alpha b^2 - 1}{1+\alpha b^2}\right)\right).$$

To this end we let

$$\gamma = \frac{2}{1+\alpha b^2} \quad \text{so that} \quad \frac{\alpha b^2 - 1}{1+\alpha b^2} = 1 - \gamma.$$

Letting $s = h(1-\gamma)$ we have $\gamma = 1 - g(s)$ and

$$\begin{aligned} \lim_{\gamma \to 0+} \frac{(1 - h(1-\gamma))^2}{\gamma} &= \lim_{s \to 1-} \frac{(1-s)^2}{1 - g(s)} = \lim_{s \to 1-} \frac{-2+2s}{-g'(s)} \\ &= 2 \lim_{s \to 1} \frac{s-1}{\log s - 1} = 2. \end{aligned}$$

Hence $\frac{1-h(1-\gamma)}{\sqrt{\gamma}} \to \sqrt{2}$ as $\gamma \to 0+$, and recalling the definition of γ we obtain

$$\lim_{b\to\infty} \psi(b) = \lim_{b\to\infty} \left(\alpha b \frac{\sqrt{2}}{\sqrt{1+\alpha b^2}}\right) \cdot \lim_{\gamma\to 0+} \frac{1 - h(1-\gamma)}{\sqrt{\gamma}} = 2\sqrt{\alpha}.$$

In particular, we can take $f \in D(A^2)$ with $a = \|f\| = 1$ satisfying the assumptions of Theorem 2.1 for some $\omega_0 > 0$, and then also for all $\omega \in (0, \omega_0)$. We have $b = \|Af\|/\omega$ and $c = \|A^2 f\|/\omega^2 = \alpha b^2$ for $\alpha = \|A^2 f\|/\|Af\|^2$. Now $\omega \to 0+$ means $b \to \infty$, and the limit inequality reads $1 \le 2\sqrt{\alpha}$, i.e.

$$\|Af\| \le 2\|A^2 f\|^{1/2},$$

which is exactly (1.1).

This means that (2.1) can be understood as an interpolation between the classical Kallmann–Landau–Rota inequality (1.1), which in our notation reads $b \le 2\sqrt{ca}$, and the case $a = b = c$, which happens, e.g., if f is an eigenvector for the eigenvalue ω and satisfies the assumptions of Theorem 2.1. We study this in a classical example and show in particular that (1.1) is also optimal for $f \in D(A^2)$ satisfying the assumptions of Theorem 2.1 for some $\omega > 0$.

Example 2.4. Let $X = C_0[0, \infty)$, the space of all continuous functions $f : [0, \infty) \to \mathbb{R}$ satisfying $\lim_{x\to\infty} f(x) = 0$ equipped with the sup-norm. The operator $A = \frac{d}{dx}$ with domain

$$D(A) = \{f \in C^1[0, \infty) : f, f' \in X\}$$

is the generator of the left shift semigroup $(S(t))_{t\ge 0}$ given by

$$(S(t)f)(x) = f(x + t), \qquad x \ge 0,\ t \ge 0,$$

which is clearly contractive: $\|S(t)\| \le 1$ for all $t \ge 0$. Moreover, we see that $A^2 = \frac{d^2}{dx^2}$ with

$$D(A^2) = \{f \in C^2[0, \infty) : f, f', f'' \in X\}.$$

We take the extremal for (1.1) given in [1], i.e. we let

$$f(x) := \begin{cases} 1 - 6(x - \xi)^2, & x \in [0, \xi], \\ 1 - 6(x - \xi)^2 + 8(x - \xi)^3 - 3(x - \xi)^4, & x \in (\xi, \xi + 1], \\ 0, & x > \xi + 1, \end{cases}$$

where $\xi = 1/\sqrt{3}$. It is easy to check that f is a C^2-function, hence $f \in D(A^2)$. Moreover, f is increasing on $[0, \xi]$ with $f(0) = -1$, $f(\xi) = 1$, f' is decreasing on $[0, \xi]$ with $f'(0) = 4\sqrt{3}$, $f'(\xi) = 0$, and $f'' = -12$ on $[0, \xi]$. Concerning the interval $[\xi, \xi + 1]$ we have $f(\xi + 1) = f'(\xi + 1) = f''(\xi + 1) = 0$, f'' has an additional zero at $\xi + 1/3$ with $f'(\xi + 1/3) = -16/9$, and f''' has a zero at $\xi + 2/3$ with $f''(\xi + 2/3) = 4$. We conclude

$$\|f\|_\infty = 1, \quad \|f'\|_\infty = 4\sqrt{3}, \quad \|f''\|_\infty = 12,$$

so that equality holds in (1.1). Now let $\omega > 0$ and define $f_\omega := e^{-\omega(\cdot)} f$. Then $f_\omega \in D(A^2)$, $\|f_\omega\|_\infty = 1$ (since $|f|$ attains its maximal value at $x = 0$), and

$$\forall t \ge 0 : \|S(t) f_\omega\|_\infty \le e^{-\omega t} \|f_\omega\|_\infty.$$

We have to check the assumption for

$$A^2 f_\omega = f_\omega'' = e^{-\omega(\cdot)}(f'' - 2\omega f' + \omega^2 f).$$

We clearly have

$$\begin{aligned}\|f_\omega''\|_\infty &\le \|f''\|_\infty + 2\omega\|f'\|_\infty + \omega^2\|f\|_\infty \\ &= 12 + 8\sqrt{3}\omega + \omega^2 = -(f'' - 2\omega f' + \omega^2 f)(0) = -f_\omega''(0),\end{aligned}$$

and this implies

$$\|f_\omega''\|_\infty = 12 + 8\sqrt{3}\,\omega + \omega^2 \quad \text{and} \quad \forall t \ge 0 : \|S(t)f_\omega''\|_\infty \le e^{-\omega t}\|f_\omega''\|_\infty.$$

We also have

$$\|f_\omega'\|_\infty \le \|f'\|_\infty + \omega\|f\|_\infty \le 4\sqrt{3} + \omega = (f' - \omega f)(0) = f_\omega'(0),$$

which implies $\|f_\omega'\|_\infty = 4\sqrt{3} + \omega$. Hence we can apply Theorem 2.1 to

$$a = 1, \qquad b_\omega = 1 + \frac{4\sqrt{3}}{\omega}, \qquad c_\omega = 1 + \frac{8\sqrt{3}}{\omega} + \frac{12}{\omega^2}.$$

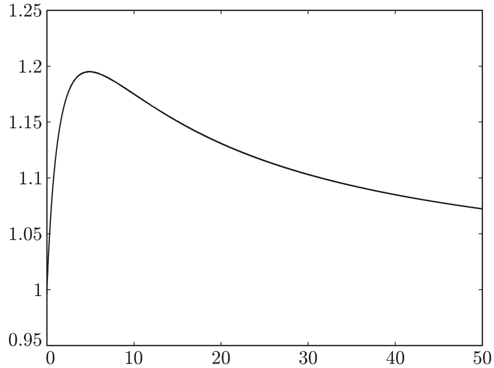
Arguments similar to those in Remark 2.3 show what happens for $\omega \to 0+$: since $c_\omega \sim b_\omega^2/4$ we have $\alpha = 1/4$ here and

$$\frac{c_\omega - (a + c_\omega)h\left(\frac{c_\omega - b_\omega}{a + c_\omega}\right)}{b_\omega} \;\to\; 1 \qquad (\omega \to 0+).$$

For $\omega \to \infty$ we observe that $b_\omega \to 1$ and $c_\omega \to 1$, which implies

$$\frac{c_\omega - (a + c_\omega)h\left(\frac{c_\omega - b_\omega}{a + c_\omega}\right)}{b_\omega} \;\to\; 1 \qquad (\omega \to \infty).$$

We illustrate the behaviour of the quotient for intermediate values by a picture:



Numerical plot of the quotient $\frac{c_\omega - (a+c_\omega)h\left(\frac{c_\omega - b_\omega}{a+c_\omega}\right)}{b_\omega}$ for $\omega \in [0, 50]$

This should be compared with (1.1), which reads here $b_\omega \le 2\sqrt{c_\omega}$. For $\omega \to 0+$ we also have $2\sqrt{c_\omega}/b_\omega \to 1$ (since $c_\omega \sim b_\omega^2/4$), but for $\omega \to \infty$ we have $2\sqrt{c_\omega}/b_\omega \to 2$. In particular, we see that, in contrast to (1.1), the inequality (2.1) is asymptotically optimal in this example and, numerically, the maximum value of the plotted quotient is 1.1952.

Open problem. Is it possible to give an analog of Theorem 2.1 in Hilbert spaces that is related to Kato's inequality (1.2) in the way Theorem 2.1 is related to (1.1)?

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