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Representations of Polish groups and continuity

by

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Abstract. In the first part of the paper, some criteria of continuity of representations of a Polish group in a Banach algebra are given. The second part uses the result of the first part to deduce automatic continuity results of Baire morphisms from Polish groups to locally compact groups or unitary groups. In the final part, the spectrum of an element in the range of a strongly but not norm continuous representation is described.

1. Introduction. In [8], [7], [3], [24], [19] various sufficient conditions of continuity for representations of locally compact groups on Banach spaces, or more generally in Banach algebras, are given. In [8], [7], [3] these conditions are of spectral nature. To be more precise, in [8], it is proved that if G is a locally compact abelian group, A a unital Banach algebra and $\theta : G \to A$ a locally bounded (norm bounded on compact subsets of G) representation, then the continuity of θ is equivalent to the a priori weaker condition $\rho(\theta(g) - I) \to 0$ as $g \to e$ where ρ denotes the spectral radius in A and e is the unit of G. (This condition is often called *spectral continuity* for θ.) In [7] this result is generalized to some representations of nonabelian or non-locally compact groups.

In [3], [4] and [24], the results above are used to obtain continuity criteria of the form:

- If for any $\omega \in \Omega$, where Ω is a subset of the topological dual A' of A, $\omega \circ \theta$ is continuous then θ is continuous.

In the locally compact abelian case [3], [4], we can take Ω to be the Gelfand spectrum $\hat{A}$ of the (abelian) algebra A. In the case $G = \mathbb{R}$, such theorems can already be found in [26] and in the classical treatise [14].

If G is nonabelian (but again locally compact) and θ is a unitary representation of G in a unital C^*-algebra A, one can take Ω to be the set of states of A [24].

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In [19], J. Kuznetsova gave another type of theorem, namely:

- Let G be a locally compact group and $\theta : G \to \mathcal{L}(H)$ a unitary representation of G on a Hilbert space. Moreover, suppose that θ is Haar measurable when $\mathcal{L}(H)$ is endowed with the weak operator topology. Then θ is strongly continuous.

Here, we have a criterion of strong continuity and not of norm continuity as in the other cases.

Except for some results in [7], all the theorems in the cited works need actually the local compactness of the group. In this work, we wish to extend them, whenever possible, to Polish and not necessarily locally compact groups. The aim of Section 2 is to generalize to any Polish group the continuity criteria of [3] and [24].

In Section 3, the most important part is a rewriting of [19] where locally compact groups and Haar measurability are replaced by Polish groups and the Baire property. We use the theorem obtained in this way to prove automatic continuity of Baire morphisms from a Polish group to a locally compact group. In the final part, the aim is to generalize and unify the description of spectra of elements in the range of a strongly continuous but not norm continuous representation, given in [3], [24] and [28], thus showing interesting applications of the continuity criteria provided in Section 2.

2. Continuity through linear forms

2.1. Preliminaries. In this subsection, we collect some facts used in the proofs of Theorems 2.7 and 2.8. The first one concerns sequences of positive definite functions (see [13] for definitions and classical properties of positive definite functions).

Proposition 2.1 (Banaszczyk, [1, Prop. 3.4]). *Let G be a Baire group and $(\chi_n)_{n\in\mathbb{N}}$ a pointwise convergent sequence of positive definite continuous functions such that the limit function is continuous. Then $(\chi_n)_{n\in\mathbb{N}}$ is equicontinuous.*

We will also need a result on compactness and sequential compactness in some nonmetrizable spaces. Recall that a topological Hausdorff space X is called *angelic* if:

- Every relatively countably compact subset of X is relatively compact in X.
- The closure of a relatively countably compact subset of X is the sequential closure of A.

We have the following property:

Proposition 2.2. *Let X be an angelic space and A a subset of X. The following assertions are equivalent:*

(1) *$\overline{A}$ is compact.*
(2) *Every sequence in A admits a subsequence converging in $\overline{A}$.*

The following theorem (see [9]) gives a large class of angelic spaces.

Theorem 2.3. *Let X be a topological space such that there is a sequence $(X_n)_{n\in\mathbb{N}}$ of relatively countably compact subspaces with $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$, and let Z be a metric space. Then the space $(\mathcal{C}(X,Z))_p$ of continuous functions from X to Z is angelic when endowed with the pointwise convergence topology.*

As a consequence, the space $(\mathcal{C}(X))_p$ of continuous complex functions on a separable space X, with its pointwise convergence topology, is angelic.

We will now give a result linking spectral continuity and continuity for group representations (one of the few where local compactness is not needed).

Theorem 2.4 ([7, Th. 3]). *Let G be a topological group and $\theta : G \to A$ a locally bounded representation of G on a unital algebra A. If θ is spectrally continuous (i.e. $\lim_{g\to e} \rho(\theta(g) - I) = 0$) and if there exists a neighborhood V of the unit e in G such that*

$$\sup_{g\in V} \|(\theta(g) - I)^n\| \le 2^n$$

for some positive integer n, then θ is continuous (i.e. $\lim_{g\to e} \|\theta(g) - I\| = 0$).

From this, one can show

Corollary 2.5. *Let G be a topological group and $\theta : G \to A$ a norm bounded representation of G in a unital Banach algebra A. If θ is spectrally continuous, then it is continuous.*

Proof. First, we consider the particular case where $\|\theta(g)\| = 1$ for every g in G. Clearly,

$$\forall g \in G, \forall n \in \mathbb{N}, \qquad \|(\theta(g) - I)^n\| \le \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

and so Theorem 2.4 applies to θ. If θ is a norm bounded representation, one can renorm A so as to be in the case above. Indeed, set $|||a||| = \sup_{g\in G} \|\theta(g)a\|$ for a in A. Clearly $|||\cdot|||$ is a Banach algebra norm on A equivalent to $\|\cdot\|$ and

$$\forall a \in A, \forall g \in G, \qquad |||\theta(g)a||| = |||a|||.$$

Set now $\|a\|_G = \sup_{|||a'|||\le 1} |||aa'|||$; then $\|\cdot\|_G$ is also a Banach algebra norm equivalent to $|||\cdot|||$ and $\|\cdot\|$, satisfying $\|\theta(g)\|_G = 1$ for any g in G, so the result can be reduced to the particular case above. ∎

2.2. Continuity through characters in the abelian case. We will prove that if $\theta : G \to A$ is a norm bounded representation of a Polish abelian group G in a unital abelian Banach algebra A, then the continuity of θ is equivalent to the continuity of $\chi \circ \theta$ for all characters χ of A.

We will begin with a lemma allowing us to introduce positive definite functions in our setting:

LEMMA 2.6. *If $\theta : G \to A$ is a locally bounded (bounded on some neighborhood of the unit e in G) representation of a topological group G on a Banach algebra A, then, for each positive ϵ, there is a neighborhood V_ϵ of e such that for all $g \in V_\epsilon$,*

$$\sigma(\theta(g)) \subset \{z \in \mathbb{C} \mid 1 - \epsilon \le |z| \le 1 + \epsilon\}.$$

Proof. By the local boundedness of θ, there are $M > 1$ and a neighborhood V of e such that $\|\theta(g)\| \le M$ for all $g \in V$. By continuity of the group operation, for each positive integer n, there exists a neighborhood V_n of e such that for all $g \in V_n$, $\|\theta(g^n)\| \le M$ and $\|\theta(g^{-n})\| \le M$. Since $\sigma(\theta(g^{-n})) = \{1/\lambda \mid \lambda \in \sigma(\theta(g^{-n}))\}$, we have $\sigma(g^n) \subset \{z \in \mathbb{C} \mid 1/M \le |z| \le M\}$ and since $\sigma(\theta(g^n)) = (\sigma(\theta(g)))^n$, we have

$$[\forall g \in V_n,\, z \in \sigma(\theta(g))] \;\Rightarrow\; \frac{1}{M^{1/n}} \le |z| \le M^{1/n}.$$

Hence, we conclude that, for each $\epsilon > 0$, there is a neighborhood V_ϵ of e such that

$$g \in V_\epsilon \;\Rightarrow\; \sigma(\theta(g)) \subset \{1 - \epsilon \le |z| \le 1 + \epsilon\}. \;\blacksquare$$

THEOREM 2.7. *Let G be an abelian Polish group, A an abelian unital Banach algebra and $\theta : G \to A$ a locally bounded representation. Then the following conditions are equivalent:*

(i) *For any character χ of A, $\chi \circ \theta$ is continuous.*
(ii) *θ is spectrally continuous.*

If, moreover, θ is norm bounded, then any of these conditions is equivalent to the continuity of θ.

Proof. Clearly (ii)⇒(i) since $\lim_{g\to e} \rho(\theta(g) - I) = 0$ implies that for all χ in the Gelfand spectrum $\hat{A}$ of A, we have $\lim_{g\to e}((\chi \circ \theta)(g) - 1) = 0$, which expresses the continuity of $\chi \circ \theta$ at e and hence, by the fact that $\chi \circ \theta$ is a morphism from G to the multiplicative group of complex numbers, the continuity of $\chi \circ \theta$.

The point is to prove (i)⇒(ii). Suppose that $\chi \circ \theta$ is continuous for all χ in $\hat{A}$ but θ is not spectrally continuous. Then we can find a strictly positive number ϵ and a sequence $(g_n)_{n\in\mathbb{N}}$ in G converging to e such that

$$\forall n \in \mathbb{N}, \quad \rho(\theta(g_n) - I) > \epsilon.$$

So, there is a complex sequence (λ_n) with $\lambda_n \in \sigma(\theta(g_n))$ and $|\lambda_n - 1| > \epsilon$, for all $n \in \mathbb{N}$. As λ_n is in the spectrum of $\theta(g_n)$, one can find, for every n, a character χ_n of A such that $\lambda_n = \chi_n(\theta(g_n))$.

The spectrum $\hat{A}$ being compact (in the restricted w^*-topology of A'), the set $\{\chi \circ \theta \mid \chi \in \hat{A}\}$ is compact in $(\mathcal{C}(G))_p$ (the continuity of $\chi \mapsto \chi \circ \theta$ from the w^*-topology of A' to the pointwise convergence topology is immediate). But, since G is Polish, $(\mathcal{C}(G))_p$ is angelic (Th. 2.3) and so $\{\chi \circ \theta \mid \chi \in \hat{A}\}$ is sequentially compact. By extracting a subsequence if necessary, we can suppose that $(\chi_n \circ \theta)_{n\in\mathbb{N}}$ is pointwise convergent to $\chi \circ \theta$ for some $\chi \in \hat{A}$. Lemma 2.6 implies that $\lim_{n\to\infty} |\lambda_n| = 1$. So we can assume without loss of generality that, for $n \geq 1$,

$$\left|\frac{\lambda_n}{|\lambda_n|} - 1\right| > \frac{\epsilon}{2}.$$

Setting

$$\varphi_n(g) = \frac{(\chi_n \circ \theta)(g)}{|\chi_n \circ \theta(g)|}, \qquad \varphi(g) = \frac{(\chi \circ \theta)(g)}{|\chi \circ \theta(g)|}, \qquad \mu_n = \frac{\lambda_n}{|\lambda_n|},$$

one finds that $\varphi_n(g_n) = \mu_n$ for all $n \in \mathbb{N}$; and φ and φ_n are morphisms from G to the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and hence positive definite. The maps φ and φ_n are, by hypothesis, continuous and $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to φ. By Proposition 2.1, $\{\varphi_n \mid n \in \mathbb{N}\}$ is equicontinuous. Hence there is a neighborhood W of e such that

$$\forall n \in \mathbb{N}, \forall g \in W, \qquad |\varphi_n(g) - 1| \leq \epsilon/2$$

and for n so large that $g_n \in W$,

$$|\varphi_n(g_n) - 1| = |\mu_n - 1| \leq \epsilon/2,$$

which is a contradiction. ■

For a norm bounded representation, spectral continuity is equivalent to continuity (Cor. 2.5), so (i) and (ii) are equivalent to the continuity of θ.

2.3. Continuity through states in the unitary case. In this section, G is a Polish group (perhaps nonabelian) and A a unital C^*-algebra. We recall that u in A is called *unitary* if $u^*u = uu^* = 1$ (see e.g. [5], [23] for the properties of C^*-algebras and unitary, self-adjoint and positive elements). A *unitary representation* of G in A is a representation $\theta : G \to A$ such that $\theta(g)$ is unitary for all $g \in G$.

We recall that a *state* on A is an element ω of A' such that ω is positive on positive elements of A and $\|\omega\| = 1$ (equivalently, $\|\omega\| = \omega(1) = 1$). The set of states of A (denoted $S(A)$) is a convex, w^*-compact subset of A'. Its extremal points are called *pure states*; we denote by $PS(A)$ the set of pure states of A. Note that for every ω in $S(A)$ and every unitary representation of a group G in A, $\omega \circ \theta$ is positive definite (clear from definitions).

For unitary representations of Polish groups in C^*-algebras, we have the following conditions of continuity:

THEOREM 2.8. *Let G be a Polish group, A a unital C^*-algebra and $\theta : G \to A$ a unitary representation. Then the following assertions are equivalent:*

(i) *θ is continuous.*
(ii) *$\omega \circ \theta$ is continuous for all $\omega \in S(A)$.*
(iii) *$\omega \circ \theta$ is continuous for all $\omega \in PS(A)$.*

Proof. (i)⇒(ii)⇒(iii) is clear.

(iii)⇒(ii) is proved, as in [24] for locally compact Polish groups, using the Choquet–Bishop–De Leeuw integral representation theorem (see [25]). Let μ_ω be a probability Baire measure on $PS(A)$ representing ω (i.e. for any w^*-continuous linear functional u on A', we have $u(\omega) = \int_{PS(A)} u(\eta)\, d\mu_\omega(\eta)$). For $g \in G$, the linear functional $\widehat{\theta(g)}$ on A' defined by $\widehat{\theta(g)}(\omega) = \omega(\theta(g))$ is w^*-continuous, and so for all $\omega \in S(A)$,

$$\omega(\theta(g)) = \widehat{\theta(g)}(\omega) = \int_{PS(A)} \widehat{\theta(g)}(\eta)\, d\mu_\omega = \int_{PS(A)} (\eta \circ \theta)(g)\, d\mu_\omega(\eta).$$

Assuming (iii), we have to show that $\omega \circ \theta$ is also continuous. If g is in G and (g_n) is a sequence in G converging to g, then for all $\eta \in PS(A)$,

$$(\eta \circ \theta)(g_n) = \widehat{\theta(g_n)}(\eta) \to (\eta \circ \theta)(g) = \widehat{\theta(g)}(\eta),$$

i.e. $\widehat{\theta(g_n)}$ converges pointwise to $\widehat{\theta(g)}$ on $PS(A)$. Moreover, $|\widehat{\theta(h)}(\eta)| \le 1$ for all $\eta \in PS(A)$ and $h \in G$, so by dominated convergence,

$$(\omega \circ \theta)(g_n) = \int_{PS(A)} \widehat{\theta(g_n)}(\eta)\, d\mu_\omega \to \int_{PS(A)} \widehat{\theta(g)}(\eta)\, d\mu_\omega = (\omega \circ \theta)(g).$$

As G is metrizable, this proves the continuity of $\omega \circ \theta$.

We have now to prove that (ii)⇒(i). For a in A, set

$$\rho_S(a) = \sup_{\omega \in S(A)} |\omega(a)|$$

(the *numerical radius* of a). If a is normal (i.e. $a^*a = aa^*$) in A, we have $\rho_S(a) = \|a\|$. If θ is unitary, then, for any $g \in G$, $\theta(g) - I$ is normal and

$$\|\theta(g) - I\| = \rho_S(\theta(g) - I).$$

We assume (ii) ($\omega \circ \theta$ continuous for all $\omega \in S(A)$) and we have to show that $\lim_{g \to e} \rho_S(\theta(g) - I) = 0$. We proceed exactly as in the abelian case with states instead of characters. If $\rho_S(\theta(g) - I) \not\to 0$ as $g \to e$, one can find a positive ϵ, a sequence (g_n) in G converging to e and a sequence (ω_n) in the w^*-compact set $S(A)$ such that

$$\forall n \in \mathbb{N}, \quad |\omega_n(\theta(g_n)) - 1| \ge \epsilon$$

(recall that $\omega(1) = 1$ for any state). As in the proof of Theorem 2.7, $\{\omega \circ \theta \mid \omega \in S(A)\}$ is compact in the angelic space $(\mathcal{C}(G))_p$, and thus sequentially compact. So we can suppose that the sequence $(\omega_n \circ \theta)$ converges pointwise to $\omega \circ \theta$ for some state ω. The conclusion follows as in the proof of Theorem 2.7 by using Proposition 2.1 and equicontinuity in e. ■

We can also give a corollary of the theorem above where the continuity of unitary representations can be seen, not through continuity of composition with linear functionals but through the strong continuity of the composition with the hilbertian representations associated to states via the Gelfand–Naimark–Segal construction. This corollary will be used to prove an automatic continuity result for group morphisms later (for G.N.S. representations, see e.g. [5] or [23]).

Corollary 2.9. *Let G be a Polish group, A a unital C^*-algebra and $\theta : G \to A$ a unitary representation. The following are equivalent:*

(i) *θ is continuous.*
(ii) *For each ω in $S(A)$, $\pi_\omega \circ \theta$ is strongly continuous, where π_ω is the hilbertian unitary representation of A associated to ω by the G.N.S. construction.*

Proof. Since π_ω is continuous, clearly (i)⇒(ii). Conversely, assume that $\pi_\omega \circ \theta$ is strongly continuous for all ω in $S(A)$. If x_ω is a cyclic vector of H_ω (the representation space of π_ω) such that

$$\forall a \in A, \qquad \omega(a) = \langle \pi_\omega(a)x_\omega, x_\omega\rangle$$

(the existence of x_ω is an essential fact from the G.N.S. construction), then

$$\forall g \in G, \quad (\omega \circ \theta)(g) = \langle \pi_\omega(\theta(g))x_\omega, x_\omega\rangle,$$

so $\omega \circ \theta$ is continuous (for all ω in $S(A)$) and by Theorem 2.8, θ is continuous. ■

2.4. Remarks. Theorem 2.7 for locally bounded representations and Theorem 2.8 have been proved for Polish locally compact groups in [3], [24]. The proofs used classical arguments of abstract harmonic analysis (dual groups and Fourier transforms in the abelian case, Raikov's theorem and functions of positive type in the unitary case) that are not available without Haar measure. In [4], we obtain generalizations in the case of locally compact, but perhaps non-Polish groups, by applying Glicksberg–De Leeuw decomposition (used here in Section 3) and some facts on weak topology on spaces of continuous functions. Here, the crucial fact which enables us to bypass local compactness is the equicontinuity result 2.1 of Banaszczyk and the angelicity of $(\mathcal{C}(G))_p$ when G is separable.

3. Continuity and Baire property. J. Kuznetsova [19] has proved that, for a unitary representation $\theta : G \to \mathcal{L}(H)$ of a locally compact group G on a Hilbert space H, Haar measurability of θ (when $\mathcal{L}(H)$ is endowed with the weak operator topology) implies continuity (a result already known when H is separable). We wish to prove an analogous result if G is assumed to be Polish and Haar measurability is replaced by the Baire property.

Let X be a Polish topological space. We define a map of X into a topological space to be *Baire-measurable* (or to have the *Baire property*) if the inverse image of every open set in the range has the Baire property (see [15] for properties of sets having the Baire property, and of Baire (measurable) maps).

We will use the Glicksberg–De Leeuw decomposition of representations but the essential Theorem 3.6 is essentially a rewriting of the analog in [19] using the Baire property instead of Haar measurability. We have, nevertheless, chosen to give a detailed proof because it is used in the final part of the section to give an automatic continuity result.

We begin by recalling the Glicksberg–De Leeuw decomposition theorem. We state it only for unitary representations and not in its more general form irrelevant for our purpose.

THEOREM 3.1 ([6]). *Let G be a topological group, H a Hilbert space and θ a unitary representation of G on H. There is a hilbertian decomposition $H = H_c \oplus H_0$ which reduces θ (H_c and H_0 are $\theta(G)$-invariant) such that:*

- *$H_c = \{x \in H \mid g \mapsto \theta(g)x$ is continuous from G to $H\}$.*
- *H_0 is the subspace of all vectors $x \in H$ such that, for every neighborhood V of e in G, 0 is in the closed convex hull of $\theta(V)x$ in H (a representation such that each vector has this property is called* averaging to zero*).*

REMARKS 3.2. In the cited theorem of [6], we only have the weak continuity of the restriction of θ to H_c, but in its general form, this theorem deals with Banach spaces which are not necessarily Hilbert spaces. In the case of a Hilbert space, the strong and weak operator topology are the same when restricted to the subset of unitary operators [10], so, for our unitary representation, this distinction is irrelevant.

In the same vein, [6] shows that the sum $H_c \oplus H_0$ is a topological direct sum (which is sufficient for us). Nevertheless, for unitary representations, it is actually hilbertian. (This requires looking a little more at the proofs in [6].)

It follows from Theorem 3.1 that, in order to show that a unitary representation on a Hilbert space H is strongly continuous, it is sufficient to show that the component H_0 of the decomposition of H given by the theorem reduces to $\{0\}$.

Let x be a vector in H_0. If we set $\varphi_x(g) = \langle\theta(g)x, x\rangle$, then φ_x is a positive definite function and one can prove that:

- For every neighborhood V of e in G and any $\epsilon > 0$, there are an integer n, positive numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$ and $g_1, \ldots, g_n$ in V such that
$$\Big|\sum_{i=1}^n \alpha_i \varphi_x(gg_i)\Big| < \epsilon \quad \text{for any } g \in G.$$

It suffices to write that the vector 0 is in the convex hull of $\theta(V)x$ and apply Cauchy–Schwarz. (Such a positive definite function is also called *averaging to zero*).

Before proving the main result of this section, we need some facts on positive definite functions averaging to zero.

LEMMA 3.3 ([12]). *Let G be a topological group and B a nonmeager subset of G having the Baire property. There is a neighborhood W of the unit e of G such that, for any sequence $(g_n)_{n\in\mathbb{N}}$ in W, $B \cap \bigcap_{n\in\mathbb{N}} Bg_n$ is not a meager set.*

Proof. By assumption, we can find a nonempty open set U and a meager set M such that B is the symmetric difference $U \triangle M$. Pick g_0 in U; there exists an open neighborhood V of g_0 and a symmetric neighborhood W (i.e. $W = W^{-1}$) of e in G such that $VW \subset U$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in W and set $M_{(g_n)} = M \cup \bigcup_{n\in\mathbb{N}} Mg_n$. Then $M_{(g_n)}$ is a meager set. For g in $V \setminus M_{(g_n)}$, we have $g \in U \setminus M$ and for all $n \in \mathbb{N}$ we have $gg_n^{-1} \in U \setminus M$, so $V \setminus M_{(g_n)} \subset B \cap \bigcap_{n\in\mathbb{N}} Bg_n$, which is not a meager set. ∎

PROPOSITION 3.4. *Let G be a Polish group and φ a positive definite function on G averaging to zero.*

(i) *If B is a nonmeager subset of G having the Baire property and P an open half-plane in the complex plane $\mathbb{C}$ such that $0 \in P$, then $\varphi^{-1}(P) \cap B \neq \emptyset$ (and is not, actually, a meager set).*

(ii) *If φ has the Baire property, then $\varphi = 0$ on a comeager subset of G.*

Proof. (i) Set $\epsilon = d(0, \mathbb{C}\setminus P)$. Using Lemma 3.3, we can find a symmetric neighborhood W of e such that, for any sequence (g_k) in W, $B \cap \bigcap_{k\in\mathbb{N}} Bg_k$ is *nonmeager*. Because φ averages to zero, we can also find $g_1, \ldots, g_n$ in W and positive numbers $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 + \cdots + \alpha_n = 1$ such that, for g in G,
$$\Big|\sum_{k=1}^n \alpha_k \varphi(gg_k)\Big| < \epsilon.$$

So, $B \cap \bigcap_{k=1}^n Bg_k^{-1}$ is not meager. If $g \in G$ is such that $gg_k \notin \varphi^{-1}(P)$ (i.e. $\varphi(gg_k) \in \mathbb{C} \setminus P$) for all $k \in \{1, \ldots, n\}$, then, $\mathbb{C} \setminus P$ being convex,

$\sum_{k=1}^{n} \alpha_k \varphi(gg_k) \in \mathbb{C} \setminus P$ and $|\sum_{k=1}^{n} \alpha_k \varphi(gg_k)| \geq \epsilon$ $(= d(0, \mathbb{C} \setminus P))$, which contradicts the definitions of ϵ, $(g_k)_{1 \leq k \leq n}$ and $(\alpha_k)_{1 \leq k \leq n}$.

Thus, for all $g \in G$, there exists $k_g \in \{1, \ldots, n\}$ such that $gg_{k_g} \in \varphi^{-1}(P)$. Moreover, if $g \in B \cap \bigcap_{k=1}^{n} Bg_k^{-1}$, we have $gg_{k_g} \in B$. Therefore

$$B \cap \bigcap_{k=1}^{n} Bg_k^{-1} \subset \bigcup_{k=1}^{n} (\varphi^{-1}(P) \cap B) g_k^{-1},$$

which is nonmeager and hence so is $\varphi^{-1}(P) \cap B$ (and a fortiori $\varphi^{-1}(P) \cap B \neq \emptyset$).

(ii) If φ has the Baire property, then for every open half-plane P in $\mathbb{C}$ such that $0 \in P$, $\varphi^{-1}(P)$ has the Baire property and is comeager by (i) (if not, $\varphi^{-1}(P) \cap (G \setminus \varphi^{-1}(P)) \neq \emptyset$).

To conclude, it is sufficient to remark that one can find a sequence $(P_n)_{n \in \mathbb{N}}$ of open half-planes in $\mathbb{C}$ such that $\{0\} = \bigcap_{n \in \mathbb{N}} P_n$, and so $\varphi^{-1}(\{0\})$ is comeager in G. ∎

Before proving the "Baire" analog of the "Haar" result of Kuznetsova, we need the following theorem from set-theoretic topology.

Theorem 3.5 ([16, pp. 225–226]). *Let X be a Polish space and $\mathcal{A}$ a point finite family (i.e. for each x in X, $\{A \in \mathcal{A} \mid x \in A\}$ is finite) of meager sets with $\bigcup_{A \in \mathcal{A}} A = X$. Then there exists a subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcup_{A \in \mathcal{B}} A$ does not have the Baire property.*

From this, one can deduce that if $\mathcal{A}$ is a point finite family of meager subsets of X such that $\bigcup_{A \in \mathcal{A}} A$ is not meager, then one can find a subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcup_{A \in \mathcal{B}} A$ does not have the Baire property.

If $\bigcup_{A \in \mathcal{A}} A$ does not have the Baire property, one can choose $\mathcal{B} = \mathcal{A}$. If $\bigcup_{A \in \mathcal{A}} A$ does have the Baire property, there is an open subset U (nonempty) such that $\bigcup_{A \in \mathcal{A}} A$ is comeager in U, i.e. $U \setminus \bigcup_{A \in \mathcal{A}} A = A'$ where A' is meager (in X and in U).

The set U with its induced topology is Polish [15] and each $A \in \mathcal{A}$ is meager in U. We can thus apply the preceding theorem to the point finite family $\mathcal{A} \cup \{A'\}$ in U. Thus, there is $\mathcal{B} \subset \mathcal{A}$ such that $(\bigcup_{A \in \mathcal{B}} A) \cup A'$ does not have the Baire property. But A' being a meager set disjoint from $\bigcup_{A \in \mathcal{B}} A$, $\bigcup_{A \in \mathcal{B}} A$ does not have the Baire property.

We are now ready to prove:

Theorem 3.6. *Let G be a Polish group and θ a unitary representation of G on a Hilbert space H. If θ has the Baire property when $\mathcal{L}(H)$ is endowed with the weak operator topology, then θ is strongly continuous.*

Proof. By Theorem 3.1, we have a decomposition $H = H_0 \oplus H_c$ reducing H and such that the part of θ on H_c is strongly continuous. So, it suffices to show that if θ has the Baire property then $H_0 = \{0\}$. Towards a contradiction, pick x in $H_0 \setminus \{0\}$. Then $g \mapsto \langle \theta(g)x, x\rangle$ is, by assumption, a positive definite function with the Baire property and averaging to zero. By Proposition 3.4, $S = \{g \in G \mid \langle \theta(g)x, x\rangle \neq 0\}$ is a meager set with $e \in S$ since $\langle \theta(e)x, x\rangle = \|x\|^2 \neq 0$.

By an application of the Baire theorem, a countable Polish group is discrete, and an uncountable Polish group has cardinality c (the continuum) [15]. So we can assume that G has cardinality c. We can choose a well-ordering $\{g_\alpha \mid \alpha < c\}$ on G with $g_0 = e$. We define an increasing function ψ (for the chosen order) from an initial segment $\{g_\alpha \mid \alpha < m\}$ of G to G by induction. Set $\psi(e) = e$ and suppose that $\psi(g_\beta)$ has been chosen for $\beta < \alpha$. If $X_\alpha = \{\psi(g_\beta) \mid \beta < \alpha\}$ is such that $X_\alpha.S$ is not meager, we stop the procedure. If $X_\alpha.S$ is a meager subset of G, it is different from G and we take $\psi(g_\alpha) = \inf(G \setminus X_\alpha.S)$. Set for convenience $h_\alpha = \psi(g_\alpha)$. Then ψ is strictly increasing by construction and since $h_0 = \psi(e) = e = g_0$, we have $h_\alpha \geq g_\alpha$ for $\alpha < m$. We now consider the family $X_\alpha = \{h_\beta \mid \beta < \alpha\}$ for $\alpha < m$. If $m < c$, then $X_m.S$ is not meager. If $m = c$, then for $\alpha < c$ the initial segment of G associated to h_α satisfies $\{g < h_\alpha\} \subset X_\alpha.S$ since $h_\alpha = \inf(G \setminus X_\alpha.S)$. We have ($c$ being a limit ordinal)

$$G = \{g_\alpha \mid \alpha < c\} = \bigcup_{\alpha<c} \{g_\beta \mid \beta < \alpha\}$$
$$\subset \bigcup_{\alpha<c} \{g \mid g < h_\alpha\} \subset X_c.S.$$

So, $X_c.S = G$ and it is not meager.

We have thus constructed an ordered family $\{X_\alpha \mid \alpha < m\}$ for an $m \leq c$ such that $X_m.S$ is not a meager set (and all the $X_\alpha.S$ for $\alpha < m$ are). Moreover $\alpha > \beta \Rightarrow h_\beta^{-1}.h_\alpha \notin S$, because otherwise $h_\alpha \in h_\beta.S \subset X_\alpha.S$, which contradicts the definition of h_α.

So, $\langle \theta(h_\beta^{-1})\theta(h_\alpha)x, x\rangle = \langle \theta(h_\alpha)x, \theta(h_\beta)x\rangle = 0$ for $\alpha \neq \beta$ $(\alpha, \beta < m)$ using the definition of S, and so $(\theta(h_\alpha)x)_{\alpha<m}$ forms an orthogonal system. Set now, for all $\alpha < m$ and $n \in \mathbb{N}^*$,

$$A_{\alpha,n} = \{g \in G \mid |\langle \theta(g)x, \theta(h_\alpha)x\rangle| > 1/n\}.$$

We have

$$\bigcup_{n\in\mathbb{N}^*} A_{\alpha,n} = \{g \in G \mid \langle \theta(g)x, \theta(h_\alpha)x\rangle \neq 0\}$$
$$= \{g \in G \mid \langle \theta(h_\alpha^{-1}g)x, x\rangle \neq 0\} = h_\alpha.S,$$

and thus $\bigcup_{n\in\mathbb{N}^*} A_{\alpha,n}$ and each $A_{\alpha,n}$ are meager sets. Since

$$\bigcup_{n\in\mathbb{N}^*} \Big(\bigcup_{\alpha<m} A_{\alpha,n}\Big) = \bigcup_{\alpha<m} h_\alpha.S = X_m.S$$

is not a meager set, one can find $N \in \mathbb{N}^*$ such that $\bigcup_{\alpha<m} A_{\alpha,N}$ is not meager.

We will now show that the family $(A_{\alpha,N})_{\alpha<m}$ is point finite. We have

$$g \in A_{\alpha,N} \iff \exists \alpha < m \text{ such that } |\langle \theta(g)x, \theta(h_\alpha)x\rangle| > 1/N.$$

Since $(\theta(g_\alpha)x)_{\alpha<m}$ is an orthogonal system, we have

$$\|x\|^2 = \|\theta(g)x\|^2 \geq \sum_{\{\alpha \mid g\in A_{\alpha,N}\}} |\langle \theta(g)x, \theta(h_\alpha)x\rangle|^2 \geq \sum_{\{\alpha \mid g\in A_{\alpha,N}\}} \frac{1}{N^2},$$

and so, for any $g \in G$, the set $\{\alpha < m \mid g \in A_{\alpha,N}\}$ is finite. By Theorem 3.5, there is a subfamily $\mathcal{B}$ of $\{\alpha \mid \alpha < m\}$ such that $\bigcup_{\alpha\in\mathcal{B}} A_{\alpha,N}$ fails the Baire property in G. But

$$\bigcup_{\alpha\in\mathcal{B}} A_{\alpha,N} = \theta^{-1}\Big(\bigcup_{\alpha\in\mathcal{B}} \{T \in \mathcal{L}(T) \mid |\langle Tx, \theta(h_\alpha)x\rangle| > 1/N\}\Big),$$

which, by assumption, does have the Baire property in G. ∎

Remarks 3.7. (1) As already said (and as one can see), this proof owes much of its idea and structure to that of [19] (the latter being, moreover, more difficult because the Polish locally compact case is only the first step and other arguments are needed to treat general locally compact groups). We have, nevertheless, detailed it for the sake of completeness.

(2) Theorem 3.5 is actually a particular case ([16, Example, p. 226]) from which one can also deduce the analogous fact with a point finite family of Haar negligible sets, used in [19].

From Theorem 3.6, one can deduce an automatic continuity theorem for morphisms between topological groups.

Proposition 3.8. *Let G be a Polish group, H a Hilbert space and $\mathcal{U}(H)$ the unitary group of H. Suppose that $\varphi : G \to \mathcal{U}(H)$ is a group morphism having the Baire property (when $\mathcal{U}(H)$ is endowed with its usual topology induced by the norm topology of $\mathcal{L}(H)$). Then φ is continuous.*

Proof. By assumption, φ induces a representation of G on H having the Baire property for the norm topology on $\mathcal{L}(H)$. For any state ω of $\mathcal{L}(H)$, let $\pi_\omega : \mathcal{L}(H) \to \mathcal{L}(H_\omega)$ be the G.N.S. representation of $\mathcal{L}(H)$ associated to ω. Then $\pi_\omega \circ \varphi$ has the Baire property (since π_ω is norm continuous) for the norm and a fortiori for the weak operator topology on $\mathcal{L}(H_\omega)$. By the previous theorem, $\pi_\omega \circ \varphi$ is strongly continuous for each ω in $S(\mathcal{L}(H))$, and by Corollary 2.9, φ is continuous. ∎

From Theorem 3.6, we can also deduce the following:

Theorem 3.9. *Let G and K be topological groups, G Polish and K locally compact. If $\varphi : G \to K$ is a morphism with the Baire property, then φ is continuous.*

First, we must recall the following known lemma (see [17]):

LEMMA 3.10. *If K is a locally compact group and $\rho_K : K \to \mathcal{L}(\mathbb{L}^2(K))$ is the left regular representation of K, then ρ_K is a homeomorphism onto its range endowed with the topology induced by the strong operator topology on $\mathcal{L}(\mathbb{L}^2(K))$.*

Proof of Theorem 3.9. The composition $\rho_K \circ \varphi : G \to \mathcal{L}(\mathbb{L}^2(K))$ is a unitary representation with the Baire property when $\mathcal{L}(\mathbb{L}^2(K))$ has the strong (and a fortiori weak) operator topology. Thus, by Theorem 3.6, $\rho_K \circ \varphi$ is strongly continuous. If V is an open subset of K, we have

$$\varphi^{-1}(V) = (\rho_K \circ \varphi)^{-1}(\rho_K(V)).$$

By Lemma 3.10, $\rho_K(V)$ is open in $\rho_K(K)$, so there is a strongly open set U in $\mathcal{L}(\mathbb{L}^2(K))$ such that $U \cap \rho_K(K) = \rho_K(V)$. Hence

$$\varphi^{-1}(V) = (\rho_K \circ \varphi)^{-1}(\rho_K(V)) = (\rho_K \circ \varphi)^{-1}(U \cap \rho_K(K)) = (\rho_K \circ \varphi)^{-1}(U)$$

is an open set in G. ∎

REMARKS 3.11. (1) When G and K are both Polish groups, Theorem 3.9 is a well known result of Pettis.

When G is locally compact and K is Polish, a Haar measurable morphism is continuous (Steinhaus, Weil); the same is true when G and K are both locally compact (Kleppner [17], [18]).

(2) When one has the Baire property or Haar measurability, these automatic continuity theorems have an ambiguous status.

In [19], J. Kuznetsova has proved that a Haar measurable morphism from a locally compact group to any topological group is continuous assuming Martin's axiom. So, a much more general continuity theorem for Haar measurable morphisms than those above is consistent with ZFC. The consequence of Martin's axiom used in [19] is that, in a Polish locally compact group, the union of less than continuum many Haar negligible sets is negligible. Another consequence of Martin's axiom is the same in Polish spaces for meager sets.

So one can prove, exactly as in [19], that if S is a nonempty meager set in a Polish group G, then there is a set $A \subset G$ such that both $A.S$ and $G \setminus A.S$ intersect every perfect nonmeager set in G and so $A.S$ fails the Baire property, and, from this, that every Baire morphism from a Polish group into any topological group is continuous (assuming Martin's axiom). We do not develop this since the modifications needed in [19, proof of Lemma 6 and Theorem 8] are very minor.

4. Spectral properties of abelian group representations. In Section 5, combining the continuity criterion through characters (Th. 2.7), the regularity of the map associating the spectrum to an operator relative to the strong topology (see [20] or [29]) and the properties of discontinuous group morphisms with values in the torus, we are going to prove that, for "almost all" elements in a Polish group, the spectra in the range of the representation have particular geometric properties in the complex plane.

4.1. Preliminaries. We begin by recalling some definitions related to topological groups.

Let G be a topological group, $\mathcal{T}$ a translation invariant σ-algebra of subsets of G and $\mathcal{I}$ a σ-ideal in $\mathcal{T}$. Then $\mathcal{I}$ is said to have the *strong Steinhaus property* relative to $\mathcal{T}$ if for any set A in $\mathcal{T} \setminus \mathcal{I}$, $\{g \in A \mid gA \cap A \notin \mathcal{I}\}$ is a neighborhood of the unit e in G.

In locally compact groups, the sets of Haar measure zero form a σ-ideal with the strong Steinhaus property in the σ-algebra of Haar measurable subsets. The same is true for the σ-ideal of meager sets in the σ-algebra of subsets having the Baire property in Polish groups.

In abelian Polish groups, there is perhaps a less well known example of the Steinhaus property. Recall that a subset of a topological space X is called *universally measurable* if it is measurable for every complete Borel measure on X. If G is an abelian Polish group, a universally measurable subset A of G is called *Haar null* (in Christensen's sense) if one can find a Borel probability measure μ on G such that $\mu(gA) = 0$ for all $g \in G$.

One can show (see [2]) that, in abelian Polish groups, the Haar null sets form a σ-ideal with the Steinhaus property in the σ-algebra of universally measurable subsets.

4.2. Morphisms from an abelian Polish group to the torus. To study the spectra of elements in the range of a group representation, one can first study some properties of morphisms (in particular, discontinuous ones) from the group G under study to the multiplicative complex group $\mathbb{C}^*$ or to the torus $\mathbb{T}$.

Let G be a topological group and φ a morphism from G to $\mathbb{C}^*$. We set $\Gamma_\varphi = \bigcap_{V \in \mathcal{V}(e)} \overline{\varphi(V)}$ where $\mathcal{V}(e)$ is the set of all neighborhoods of e in G. One can see that Γ_φ consists of all complex numbers λ such that one can find a net (g_i) in G with $g_i \to e$ and $\varphi(g_i) \to \lambda$.

Since we are only interested in Polish groups, we can use sequences in place of nets, but most of the results in Proposition 4.1 can be extended to more general topological groups. A great part of the properties of locally bounded morphisms and of their associated sets Γ_φ are more or less explicitly stated in [14]. We collect them in the following proposition.

PROPOSITION 4.1. *Let G be a Polish group and $\varphi : G \to \mathbb{C}^*$ a locally bounded morphism from G to $\mathbb{C}^*$. Then:*

(i) *Γ_φ is a compact subgroup of the torus $\mathbb{T}$.*
(ii) *φ is continuous if and only if $\Gamma_\varphi = \{1\}$.*

(iii) $|\varphi|$ (*defined by* $|\varphi|(g) = |\varphi(g)|$) *is a continuous morphism from* G *to the multiplicative group* $\mathbb{R}_+^*$ *of positive real numbers.*

(iv) *If for each positive integer* n, G *admits local division by* n (*i.e. there are a neighborhood* V_n *of* e *and a continuous map* $\psi_n : V_n \to G$ *with* $\psi_n(e) = e$ *such that* $(\psi_n(g))^n = g$ *for all* $g \in V_n$), *then* $\Gamma_\varphi = \{1\}$ *or* $\mathbb{T}$.

(v) *Suppose that* φ *takes values in* $\mathbb{T}$ *and let* $\mathcal{K}(\mathbb{T})$ *be the Polish space of compact subsets of* $\mathbb{T}$ *endowed with the Vietoris topology. Then the map* $G \to \mathcal{K}(\mathbb{T})$ *defined by* $g \mapsto \varphi(g)\Gamma_\varphi$ *is continuous.*

Proof. (i) Clear from the definition of Γ_φ with sequences converging to e.

(ii) Clearly $\Gamma_\varphi = \{1\}$ if φ is continuous. Conversely, if $(g_n)_{n\in\mathbb{N}}$ is a sequence in G converging to e, then since φ is locally bounded, one can suppose that $(\varphi(g_n))_{n\in\mathbb{N}}$ is bounded and, actually, converges. The limit of $(\varphi(g_n))_{n\in\mathbb{N}}$ is in Γ_φ, so it is 1, and hence φ is continuous since it is continuous at e.

(iii) The local boundedness of φ (and $|\varphi|$) implies that $\Gamma_{|\varphi|}$ is a compact subgroup of $\mathbb{R}_+^*$, hence $\Gamma_{|\varphi|} = \{1\}$ and $|\varphi|$ is continuous by (ii).

(iv) Let ψ_n witness local division by n in G, and choose λ in Γ_φ. If (g_k) is a sequence converging to e such that $\varphi(g_k)$ converges to λ, by local boundedness, since $\psi(g_k) \to e$, $\varphi(\psi(g_k))$ is bounded, and hence can be supposed to converge to some λ_n in Γ_φ.

Since $(\varphi(\psi_n(g_k)))^n = \varphi(g_k) \to \lambda$, we have $\lambda_n^n = \lambda$, and thus Γ_φ is a closed divisible subgroup of $\mathbb{T}$, that is, $\Gamma_\varphi = \{1\}$ or $\Gamma_\varphi = \mathbb{T}$.

(v) We begin with the case of a morphism φ with values in $\mathbb{T}$. For $g \in G$ and a sequence $g_n \to g$, setting $A^\epsilon = \{z \in \mathbb{T} \mid d(z, A) < \epsilon\}$ for $A \subset \mathbb{T}$, we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \quad n \geq N \;\Rightarrow\; \varphi(g_n g^{-1}) \in \Gamma_\varphi^\epsilon.$$

Indeed, suppose $\Gamma_\varphi^\epsilon \neq \mathbb{T}$. If the assertion above is not true, one can extract a sequence g_{n_k} such that for all $k \in \mathbb{N}$, $\varphi(g_{n_k} g^{-1})$ is in the compact set $\mathbb{T} \setminus \Gamma_\varphi^\epsilon$, and such a sequence can be chosen so that $\varphi(g_{n_k} g^{-1})$ converges to some λ in $\mathbb{T} \setminus \Gamma_\varphi^\epsilon$. Since $g_{n_k} g^{-1} \to e$, λ must be in Γ_φ, a contradiction.

Thus

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \quad n \geq N \;\Rightarrow\; \varphi(g_n) \in \varphi(g)\Gamma_\varphi^\epsilon.$$

Since Γ_φ is a subgroup of $\mathbb{T}$ and, for $\lambda \in \mathbb{T}$, $z \mapsto \lambda z$ is an isometry, it follows that $\lambda\varphi(g_n) \in \varphi(g)\Gamma_\varphi^\epsilon$ for all $\lambda \in \Gamma_\varphi$ and $n \geq N$, and thus $\varphi(g_n)\Gamma_\varphi \subset \varphi(g)\Gamma_\varphi^\epsilon$. Moreover, $\varphi(g_n g^{-1}) \in \Gamma_\varphi^\epsilon \Rightarrow \varphi(g_n^{-1} g) \in \Gamma_\varphi^\epsilon$ (the metric on the torus is translation invariant and $z \mapsto z^{-1}$ is an isometry). So, $n \geq N \Rightarrow \varphi(g) \in \varphi(g_n)\Gamma_\varphi^\epsilon$, and, as above, $\varphi(g)\Gamma_\varphi \subset \varphi(g_n)\Gamma_\varphi^\epsilon$, which, by definition of the Hausdorff metric on $\mathcal{K}(\mathbb{T})$ defining the Vietoris topology (see [15]), proves the desired continuity. ■

REMARK 4.2. (1) In the case where $\Gamma_\varphi = \mathbb{T}$, the map $g \mapsto \varphi(g)\Gamma_\varphi$, being constant, is trivially continuous and (v) is irrelevant.

(2) By (iii), the continuity of a morphism φ from a Polish group G to $\mathbb{C}^*$ is equivalent to the continuity of the morphism $\tilde{\varphi}$ defined by $\tilde{\varphi}(g) = \varphi(g)/|\varphi(g)|$ with values in $\mathbb{T}$. In view of this remark, we only need to study morphisms (in particular discontinuous morphisms) into $\mathbb{T}$.

In the case $G = \mathbb{R}$, it is proved in [14] that if φ is a discontinuous morphism with values in the torus, then the inverse image $\varphi^{-1}(V)$ of any nonempty open subset V of $\mathbb{T}$ is dense in $\mathbb{R}$. This result is made more precise in [20] where it is proved that $\varphi^{-1}(V)$ meets every nonmeager subset of $\mathbb{R}$ having the Baire property. In this form, the result does not generalize to all abelian Polish groups in place of $\mathbb{R}$.

Consider the following example (see [8]). Set $G = \mathbb{U}_3^{\mathbb{N}}$ (where $\mathbb{U}_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$). Define $\varphi\big((z_n)_{n\in\mathbb{N}}\big)$ by taking the limit along a nontrivial ultrafilter on $\mathbb{N}$ (or a Banach limit) of the sequence (z_n). One can see that φ is a discontinuous morphism from G to $\mathbb{T}$ and $(\varphi(g))^3 = 1$ for g in G. So, if V is an open subset in $\mathbb{T}$ containing no cubic root of unity, $\varphi^{-1}(V)$ is empty. Nevertheless, if we restrict the classes of the open subsets of $\mathbb{T}$ and subsets of G considered (in a manner which depends on the morphism φ or, more precisely, on Γ_φ), we can obtain a satisfactory generalization of the result, valid when $G = \mathbb{R}$. Such theorems are proved in [3] and [28]. The approach below unifies the previous results and provides some details and generalizations.

We will first give a local version of the statement that an element of a σ-algebra is in an ideal.

Let X be a topological space, $\mathcal{T}$ a σ-algebra containing the Borel algebra of X, and $\mathcal{I}$ a σ-ideal in $\mathcal{T}$. For g in X we set

$$\mathcal{I}_g = \{A \in \mathcal{T} \mid \exists V \in \mathcal{V}(g) \text{ open}, V \cap A \in \mathcal{I}\}.$$

Clearly $\mathcal{I} \subset \mathcal{I}_g$ for all $g \in X$. If moreover X is a second countable topological space, one can show easily using the Lindelöf property that $A \in \mathcal{I} \Leftrightarrow \forall g \in A$, $A \in \mathcal{I}_g$.

Now, we consider a σ-algebra $\mathcal{T}$ of subsets of G which contains the Borel σ-algebra, and a σ-ideal $\mathcal{I}$ in $\mathcal{T}$ with the strong Steinhaus property relative to $\mathcal{T}$.

Lemma 4.3. *Let V be an open subset of $\mathbb{T}$ such that $V \cap \Gamma_\varphi \neq \emptyset$. Then $\varphi^{-1}(V) \cap A \neq \emptyset$ for each subset A of G with $A \in \mathcal{T} \setminus \mathcal{I}_e$.*

Proof. Let V_1 be an open subset of $\mathbb{T}$ such that $V_1 \cap \Gamma_\varphi \neq \emptyset$, and V_0 an open symmetric neighborhood of 1 such that $V_0 V_1 \subset V$.

If $\Gamma_\varphi = \mathbb{T}$, then $\mathbb{T} = \Gamma_\varphi V_1 = \bigcup_{\mu\in\Gamma_\varphi} \mu V_1$, and, by compactness, there exist an integer N and $\mu_1, \dots, \mu_n$ in $\mathbb{T}$ such that $\mathbb{T} = \bigcup_{i=1}^N \mu_i V_1$, and so $G = \bigcup_{i=1}^N \varphi^{-1}(\mu_i V_1)$ and $A \cap \bigcup_{i=1}^N \varphi^{-1}(\mu_i V_1) = A \in \mathcal{T} \setminus \mathcal{I}$.

If $\Gamma_\varphi = \{\mu_1, \ldots, \mu_N\}$ is a finite subgroup of $\mathbb{T}$, then $\Gamma_\varphi(\bigcup_{i=1}^N \mu_i V_1) = \bigcup_{i=1}^N \mu_i V_1$ and so

$$\varphi(g) \in \bigcup_{i=1}^{N} \mu_i V_1 \;\Leftrightarrow\; \varphi(g)\Gamma_\varphi \subset \bigcup_{i=1}^{N} \mu_i V_1.$$

Hence

$$\varphi^{-1}\Big(\bigcup_{i=1}^{N} \mu_i V_1\Big) = \Big\{g \in G \;\Big|\; \varphi(g)\Gamma_\varphi \subset \bigcup_{i=1}^{N} \mu_i V_1\Big\}$$

is an open neighborhood of e in G (by Prop. 4.1(v)) and $A \cap \bigcup_{i=1}^N \varphi^{-1}(\mu_i V_1)$ is in $\mathcal{T} \setminus \mathcal{I}$ since $A \notin \mathcal{I}_e$.

So, in any case, it suffices to show that if A is a subset of G such that $A \in \mathcal{T} \setminus \mathcal{I}$ and $A \subset \bigcup_{i=1}^N \varphi^{-1}(\mu_i V_1)$, then $\varphi^{-1}(V) \cap A \neq \emptyset$ (in the case $\Gamma_\varphi \neq \mathbb{T}$, one can consider $A \cap \bigcup_{i=1}^N \varphi^{-1}(\mu_i V_1)$ instead of A).

Let $J \subset \{1, \ldots, N\}$. We will say that J has *property* (P) if for each subset A of G such that $A \subset \bigcup_{i\in J} \varphi^{-1}(\mu_i V_1)$ and $A \in \mathcal{T} \setminus \mathcal{I}$, we have $\varphi^{-1}(V) \cap A \neq \emptyset$.

If $\{1, \ldots, N\}$ has property (P), our proposition is proved.

We proceed by induction on the cardinality $|J|$ of J. For each subset $A \in \mathcal{T} \setminus \mathcal{I}$, $U_A := \{g \in G \mid A \cap gA \notin \mathcal{I}\}$ is a neighborhood of e in G and $\Gamma_\varphi \subset \overline{\varphi(U_A)}$, thus for each $i \in \{1, \ldots, N\}$, there exists $g_{i,A} \in U_A$ such that $\mu_i(\varphi(g_{i,A}))^{-1} \in V_0$, hence $\mu_i \in \varphi(g_{i,A})V_0$ and $\mu_i V_1 \subset \varphi(g_{i,A})V_0V_1 \subset \varphi(g_{i,A})V$, so

$$\varphi^{-1}(\mu_i V_1) \subset \varphi^{-1}(\varphi(g_{i,A})V) = g_{i,A}\varphi^{-1}(V).$$

If for some i in $\{1, \ldots, N\}$, we have $A \cap g_{i,A}A \cap \varphi^{-1}(\mu_i V_1) \neq \emptyset$, then $\varphi^{-1}(V) \cap A \neq \emptyset$. Indeed, if $g \in A \cap g_{i,A}A \cap \varphi^{-1}(\mu_i V_1)$, then $g_{i,A}^{-1}g \in A$ and $g_{i,A}^{-1}g \in g_{i,A}^{-1}(g_{i,A}\varphi^{-1}(V)) = \varphi^{-1}(V)$.

Consider now $J \subset \{1, \ldots, N\}$. If $|J| = 1$, then $J = \{i_0\}$ with $1 \le i_0 \le N$, and if $A \in \mathcal{T} \setminus \mathcal{I}$ and $A \subset \bigcup_{i\in J} \varphi^{-1}(\mu_i V_1) = \varphi^{-1}(\mu_{i_0} V_1)$, we have

$$A \cap g_{i_0,A}A \cap \varphi^{-1}(\mu_{i_0} V_1) = A \cap g_{i_0,A}A \in \mathcal{T} \setminus \mathcal{I},$$

so $A \cap g_{i_0,A}A \cap \varphi^{-1}(\mu_{i_0} V_1) \neq \emptyset$ and by the preceding argument we have $\varphi^{-1}(V) \cap A \neq \emptyset$. Hence J has property (P).

Now, suppose that $|J| \in \{2, \ldots, N\}$ and that every subset of $\{1, \ldots, N\}$ of cardinality strictly less than $|J|$ has (P), for each $A \in \mathcal{T} \setminus \mathcal{I}$ such that $A \subset \bigcup_{i\in J} \varphi^{-1}(\mu_i V_1)$. There are two cases:

(i) There is some $i \in J$ such that $A \cap g_{i,A}A \cap \varphi^{-1}(\mu_i V_1) \neq \emptyset$. Then, as in the case of $|J| = 1$, we can conclude that $\varphi^{-1}(V) \cap A \neq \emptyset$.

(ii) $A \cap g_{i,A}A \cap \varphi^{-1}(\mu_i V_1) = \emptyset$ for all $i \in J$. Choosing $i_0 \in J$, we have $A \cap g_{i_0,A}A \cap \varphi^{-1}(\mu_{i_0} V_1) = \emptyset$ and so

$$A \cap g_{i_0,A}A \subset \bigcup_{i \in J \setminus \{i_0\}} \varphi^{-1}(\mu_i V_1)).$$

But $A \cap g_{i_0,A}A \in \mathcal{T} \setminus \mathcal{I}$ and, by induction hypothesis, $(A \cap g_{i_0,A}A) \cap \varphi^{-1}(V) \neq \emptyset$, hence $\varphi^{-1}(V) \cap A \neq \emptyset$. ∎

THEOREM 4.4. *If $g \in G$, V is an open subset of $\mathbb{T}$ such that $\varphi(g)\Gamma_\varphi \cap V \neq \emptyset$ and A is a subset of G with $A \in \mathcal{T} \setminus \mathcal{I}_g$, then $\varphi^{-1}(V) \cap A \neq \emptyset$.*

Proof. $\varphi(g^{-1})V$ is an open subset of $\mathbb{T}$ such that $\varphi(g^{-1})V \cap \Gamma_\varphi \neq \emptyset$, $g^{-1}A \notin \mathcal{I}_e$ since $A \notin \mathcal{I}_g$, and thus by the previous lemma, we have $\varphi^{-1}(\varphi(g^{-1})V) \cap g^{-1}A \neq \emptyset$, so $g^{-1}(\varphi^{-1}(V) \cap A) \neq \emptyset$ and $\varphi^{-1}(V) \cap A \neq \emptyset$. ∎

REMARK 4.5. (1) It is possible, for proper subsets J of $\{1, \ldots, N\}$, that $\bigcup_{i \in J} \varphi^{-1}(\mu_i V_1)$ contains no set in $\mathcal{T} \setminus \mathcal{I}$. In that case (P) would be trivially true for $J \neq \{1, \ldots, N\}$.

(2) If φ is continuous ($\Gamma_\varphi = \{1\}$), the result is a trivial consequence of the continuity.

(3) If $\Gamma_\varphi V = \mathbb{T}$, then $z\Gamma_\varphi \cap V \neq \emptyset$ for all $z \in \mathbb{T}$, so $\varphi(g)\Gamma_\varphi \cap V \neq \emptyset$ for each g in G, and if $A \in \mathcal{T}$ satisfies $\varphi^{-1}(V) \cap A = \emptyset$, then $A \in \mathcal{I}_g$ for all $g \in G$, so $A \in \mathcal{I}$. Hence: if $\Gamma_\varphi V = \mathbb{T}$, then $\varphi^{-1}(V) \cap A \neq \emptyset$ for all $A \in \mathcal{T} \setminus \mathcal{I}$. In particular, if $\Gamma_\varphi = \mathbb{T}$, then, for each nonempty open subset V of $\mathbb{T}$ and each $A \in \mathcal{T} \setminus \mathcal{I}$, we have $\varphi^{-1}(V) \cap A \neq \emptyset$.

COROLLARY 4.6. *Let G be an abelian Polish group, $\mathcal{T}$ a σ-algebra of G containing the Borel algebra, and $\mathcal{I}$ a σ-ideal in $\mathcal{T}$ having the strong Steinhaus property relative to $\mathcal{T}$. If $\omega : G \to \mathcal{K}(\mathbb{T})$ (where $\mathcal{K}(\mathbb{T})$ is the Polish space of compact subsets of $\mathbb{T}$ endowed with the Vietoris topology) is $\mathcal{T}$-measurable and if $\varphi : G \to \mathbb{T}$ is a group morphism such that $\varphi(g) \in \omega(g)$ for all $g \in G$, then, setting $\Omega_\varphi := \{g \in G \mid \varphi(g)\Gamma_\varphi \not\subset \omega(g)\}$, we have $\Omega_\varphi \in \mathcal{I}$.*

Proof. We have

$$\Omega_\varphi = \bigcup_{n \in \mathbb{N}} \{g \in G \mid V_n \cap \varphi(g)\Gamma_\varphi \neq \emptyset \text{ and } V_n \cap \omega(g) = \emptyset\},$$

where $(V_n)_{n \in \mathbb{N}}$ is a basis for the topology of $\mathbb{T}$.

If $\Omega_\varphi \notin \mathcal{I}$, there exists n_0 in $\mathbb{N}$ such that $A_{n_0} := \{g \in G \mid V_{n_0} \cap \varphi(g)\Gamma_\varphi \neq \emptyset$ and $V_{n_0} \cap \omega(g) = \emptyset\}$ is not in $\mathcal{I}$. By the definition of the Vietoris topology [15], $\{K \in \mathcal{K}(\mathbb{T}) \mid K \cap V_{n_0} \neq \emptyset\}$ is an open set in $\mathcal{K}(\mathbb{T})$, and by Proposition 4.1(v), $\{g \in G \mid \varphi(g)\Gamma_\varphi \cap V_{n_0} \neq \emptyset\}$ is an open subset of G. Analogously, the set $\{K \in \mathcal{K}(\mathbb{T}) \mid K \cap V_{n_0} = \emptyset\}$ is a closed subset of $\mathcal{K}(\mathbb{T})$, and by $\mathcal{T}$-measurability of ω, we have $\{g \in G \mid \omega(g) \cap V_{n_0} = \emptyset\} \in \mathcal{T}$, so

$$A_{n_0} = \{g \in G \mid V_{n_0} \cap \varphi(g)\Gamma_\varphi \neq \emptyset\} \cap \{g \in G \mid V_{n_0} \cap \omega(g) = \emptyset\}$$

is in $\mathcal{T} \setminus \mathcal{I}$.

If $g \in A_{n_0}$, then $\varphi(g) \in \omega(g)$ and so $\varphi(g) \notin V_{n_0}$, hence $\varphi^{-1}(V_{n_0}) \cap A_{n_0} = \emptyset$.

Let $g_0 \in A_{n_0}$ be such that $A_{n_0} \notin \mathcal{I}_{g_0}$ (such a g_0 exists since $A_{n_0} \notin \mathcal{I}$). Then we have $\varphi(g_0)\Gamma_\varphi \cap V_{n_0} \neq \emptyset$, and so "$\varphi^{-1}(V_{n_0}) \cap A_{n_0} = \emptyset$" contradicts Theorem 4.4. ∎

Corollary 4.7. *Let G, $\mathcal{T}$, $\mathcal{I}$ and $\omega : G \to \mathcal{K}(\mathbb{T})$ be as in the preceding corollary, and $(\varphi_i)_{i\in I}$ be a family of morphisms from G to $\mathbb{T}$ such that $\varphi_i(g)$ is in $\omega(g)$ for all $i \in I$ and $g \in G$. Setting $A_I = \bigcup_{i\in I} \Gamma_{\varphi_i}$, we have:*

(1) *If A_I is not finite, then*

$$\{g \in G \mid \omega(g) \neq \mathbb{T}\} = \Big\{g \in G \,\Big|\, \bigcup_{i\in I} \varphi_i(g)\Gamma_{\varphi_i} \not\subset \omega(g)\Big\} \in \mathcal{I}.$$

(2) *If A_I is a finite set, then there exist $N \in \mathbb{N}^*$ and $\{i_1, \ldots, i_N\} \subset I$ such that*

$$A_I = \bigcup_{k=1}^{N} \Gamma_{\varphi_{i_k}} \quad \text{and} \quad \Big\{g \in G \,\Big|\, \bigcup_{k=1}^{N} \varphi_{i_k}(g)\Gamma_{\varphi_{i_k}} \not\subset \omega(g)\Big\} \in \mathcal{I}.$$

Proof. We begin with the second case. Since A_I is finite, only a finite number of φ_i's have Γ_{φ_i} distinct, and all these sets Γ_{φ_i} are finite. Hence, $A_I = \bigcup_{k=1}^{N} \Gamma_{\varphi_{i_k}}$ for some integer N and $i_1, \ldots, i_N$ in I. By Corollary 4.6, we have $\{g \in G \mid \varphi_{i_k}(g)\Gamma_{\varphi_{i_k}} \not\subset \omega(g)\} \in \mathcal{I}$ for all $k \in \{1, \ldots, N\}$ and thus

$$\Big\{g \in G \,\Big|\, \bigcup_{k=1}^{N} \varphi_{i_k}(g)\Gamma_{\varphi_{i_k}} \not\subset \omega(g)\Big\} \in \mathcal{I}.$$

Now, we consider the first case where $\bigcup_{i\in I} \Gamma_{\varphi_i}$ is an infinite subset of $\mathbb{T}$. There is $i \in I$ such that $\Gamma_{\varphi_i} = \mathbb{T}$ or, for each $n \in \mathbb{N}$, Γ_{φ_i} is a group of roots of unit of order more than n for a certain $i \in I$. Hence, if U is a nonempty open subset of $\mathbb{T}$, there is $i \in I$ such that $\lambda\Gamma_{\varphi_i} \cap U \neq \emptyset$ for all $\lambda \in \mathbb{T}$ (i depends on U if there is no $i \in I$ such that $\Gamma_{\varphi_i} = \mathbb{T}$). Let $(V_n)_{n\in\mathbb{N}}$ be a basis of the topology in $\mathbb{T}$. Then $\{g \in G \mid \omega(g) \neq \mathbb{T}\} = \bigcup_{n\in\mathbb{N}} A_n$ with $A_n := \{g \in G \mid \omega(g) \cap V_n = \emptyset\}$. We have $A_n \in \mathcal{T}$ for all $n \in \mathbb{N}$, by the $\mathcal{T}$-measurability of ω, and $\{g \in G \mid \omega(g) \neq \mathbb{T}\}$ is also in $\mathcal{T}$. Since $\{g \in G \mid \omega(g) \neq \mathbb{T}\} \notin \mathcal{I}$, there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} \notin \mathcal{I}$, and so there exists $g_0 \in A_{n_0}$ such that $A_{n_0} \notin \mathcal{I}_{g_0}$. Since $\bigcup_{i\in I} \Gamma_{\varphi_i}$ is infinite, we see as above that there exists $i_0 \in I$ such that $\lambda V_{n_0} \cap \Gamma_{\varphi_{i_0}} \neq \emptyset$ for all $\lambda \in \mathbb{T}$; in particular, $\varphi_{i_0}(g_0)\Gamma_{\varphi_{i_0}} \cap V_{n_0} \neq \emptyset$. But, for $g \in A_{n_0}$, $\varphi_{i_0}(g) \in \omega(g)$, so $\varphi_{i_0}(g) \notin V_{n_0}$ and $\varphi_{i_0}^{-1}(V_{n_0}) \cap A_{n_0} = \emptyset$, which contradicts Theorem 4.4; hence $\{g \in G \mid \omega(g) \neq \mathbb{T}\} \in \mathcal{I}$. ∎

Remark 4.8. The heuristic meaning of Theorem 4.4 and its corollaries is that discontinuous morphisms from an abelian Polish group G to the torus have an extremely "oscillating" behavior. The inverse image of each open subset of $\mathbb{T}$, or, in some cases, of each well-distributed open subset of $\mathbb{T}$, is not only dense in G but it meets any "regular" and "big" subset of G (e.g. Haar measurable sets of positive measure if G is locally compact, nonmeager sets having the Baire property, or universally measurable and not Haar null sets in Christensen's sense in Polish groups).

If the morphism is constrained to take values in a compact subset of $\mathbb{T}$ varying in a somewhat regular manner (e.g. Baire or Haar measurability when $\mathcal{K}(\mathbb{T})$ is endowed with the Vietoris topology), then the regular variation of subsets cannot absorb the oscillation except if, at least on the complement of some "small" subset of G, the values of the set map are, in a sense depending on φ (or on Γ_φ), "big" subsets of $\mathbb{T}$.

5. Angular distribution of spectra in the ranges of strongly continuous representations. We will study the arguments of the elements of the spectrum $\sigma(\theta(g))$ when $g \mapsto \theta(g)$ is a strongly continuous representation of an abelian Polish group G on a Banach space. For that, we will fix some notation. Let U be a subset of $\mathbb{C}^*$; we set $U^1 = \{z/|z| \mid z \in U\}$ and, if T is an invertible linear operator on a Banach space, or more generally an invertible element of a Banach algebra, $\sigma^1(T) = (\sigma(T))^1$.

We denote by G an abelian Polish group, by X a Banach space, by $\mathcal{L}(X)$ the Banach algebra of bounded linear operators on X and by θ a representation of G on X.

If A is a closed subalgebra containing the range $\theta(G)$, we have $\sigma^1_A(\theta(g)) = \sigma^1(\theta(g))$, where σ_A is the spectrum relative to the algebra A. Indeed, for $g \in G$, $\theta(g)$ is invertible in A, thus $0 \notin \sigma_A(\theta(g))$. Clearly, $\sigma(\theta(g)) \subset \sigma_A(\theta(g))$ and it is known that $\partial\sigma_A(\theta(g)) \subset \sigma(\theta(g))$, so $\sigma^1(\theta(g)) \subset \sigma^1_A(\theta(g))$ and $(\partial\sigma_A(\theta(g)))^1 \subset \sigma^1(\theta(g))$. By a classical connectedness argument, each ray from 0 meeting $\sigma_A(\theta(g))$ meets $\partial\sigma_A(\theta(g))$, and $\sigma^1_A(\theta(g)) = (\partial\sigma_A(\theta(g)))^1$.

Since G is commutative, the remark above applies to the Banach subalgebra of $\mathcal{L}(X)$ generated by $\theta(G)$, which is commutative, or to any commutative subalgebra containing $\theta(G)$.

Let $\hat{A}$ be the character space of A, and let A^* be the group of invertible elements of A. For $\chi \in \hat{A}$, define $\chi^1 : A^* \to \mathbb{T}$ by $\chi^1(a) = \chi(a)/|\chi(a)|$. Clearly, $\sigma^1_A(a) = \{\chi^1(a) \mid \chi \in \hat{A}\}$ for all $a \in A^*$, and $\chi^1 \circ \theta : G \to \mathbb{T}$ is a group morphism for all $\chi \in \hat{A}$.

Now, we need the following regularity result on the variation of spectrum relative to the strong operator topology.

Theorem 5.1 (see [20], [29]). *Let X be a separable Banach space. The map $T \mapsto \sigma(T)$, from $\mathcal{L}(X)$ (endowed with the strong operator topology) to the topological space $\mathcal{K}(\mathbb{C})$ of compact subsets of $\mathbb{C}$, is a Borel map.*

From this, one can deduce easily that, if $\theta : G \to \mathcal{L}(X)$ is a strongly continuous representation of a Polish group G on a Banach space, then the map $g \mapsto \sigma^1(\theta(g))$ is Borel from G to $\mathcal{K}(\mathbb{T})$. As an application of this fact and our results on morphisms from G to $\mathbb{T}$, one can show:

Lemma 5.2. *Let $\theta : G \to \mathcal{L}(X)$ be a strongly continuous representation of an abelian Polish group G on a separable Banach space, and A be a commutative Banach subalgebra containing $\theta(G)$. Then*

$$\Big\{g \in G \;\Big|\; \sigma^1(\theta(g)) \neq \bigcup_{\chi \in \hat{A}} (\chi^1 \circ \theta)(g) \Gamma_{\chi^1 \circ \theta}\Big\}$$

is a meager and Haar null subset of G.

Proof. Since $\sigma^1(\theta(g)) = \{(\chi^1 \circ \theta)(g) \mid \chi \in \hat{A}\}$, we have, without any hypothesis,

$$\sigma^1(\theta(g)) \subset \bigcup_{\chi \in \hat{A}} (\chi^1 \circ \theta)(g) \Gamma_{\chi^1 \circ \theta} \quad \text{for all } g \in G.$$

Then it suffices to apply Corollary 4.7 to the family of morphisms $\{\chi^1 \circ \theta \mid \chi \in \hat{A}\}$ and to the Borel map $g \mapsto \sigma^1(\theta(g))$ from G to $\mathcal{K}(\mathbb{T})$ where the σ-ideal considered is the set of meager sets or of Haar null sets in G. ∎

Remark 5.3. If θ is norm continuous, then, for each $\chi \in \hat{A}$, $\chi^1 \circ \theta$ is continuous and $(\chi^1 \circ \theta)(g) \Gamma_{\chi^1 \circ \theta} = (\chi^1 \circ \theta)(g)$. Hence, the above lemma says nothing more than $\sigma^1(\theta(g)) = \{(\chi^1 \circ \theta)(g) \mid \chi \in \hat{A}\}$. The result is interesting only in the strongly but not norm continuous case.

We define a *regular polygon* to be the image under any rotation around 0 of a closed subgroup of $\mathbb{T}$ (hence $\mathbb{T}$ or any $\{z\}$, $z \in \mathbb{T}$, are polygons in our terminology); a polygon with more than one element is called *nontrivial.* We can now prove the main result of this section which gives some "angular scattering" properties of the spectra of "almost all" elements in the range of a strongly continuous but not norm continuous representation.

Theorem 5.4. *Let G be an abelian Polish group and let $\theta : G \to \mathcal{L}(X)$ be a strongly continuous representation of G on a Banach space X. Then we have the following dichotomies:*

(1) *If G is locally compact, then either θ is norm continuous, or the set of elements g in G such that $\sigma^1(\theta(g))$ does not contain any nontrivial polygon is meager and has Haar measure zero.*
(2) *If G is not assumed to be locally compact, but θ is norm bounded, then the first assertion remains true if we replace "has Haar measure zero" by "is Haar null (in Christensen's sense)".*

Proof. We first assume that X is separable. If θ is not norm continuous, by [28, Proposition 2.2] in case (1), and by Theorem 2.7 if G is not locally compact but θ is norm bounded (with A a commutative subalgebra of $\mathcal{L}(X)$ containing $\theta(G)$), there is a $\chi \in \hat{A}$ such that $\chi \circ \theta$ and clearly also $\chi^1 \circ \theta$ is discontinuous, and so $\Gamma_{\chi^1 \circ \theta} \neq \{1\}$. By the preceding lemma, we have $(\chi^1 \circ \theta)(g)\Gamma_{\chi^1 \circ \theta} \subset \sigma^1(\theta(g))$ for g in the complement of a meager and Haar null subset of G (and thus $\sigma^1(\theta(g))$ contains a nontrivial polygon).

If X is not assumed to be separable and θ is not norm continuous, we will show that one can find a separable $\theta(G)$-invariant subspace such that the restricted representation is already norm discontinuous.

Indeed, if θ is norm discontinuous, then there is $\delta > 0$ and a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $g_n \to e$ and $\|\theta(g_n) - I\| > \delta$; thus, there is a sequence (x_n) of unit vectors in X such that $\|\theta(g_n)x_n - x_n\| > \delta$ for all $n \in \mathbb{N}$. We denote by Y the closed subspace of X generated by the $\theta(G)$-orbits of the x_n's. As the group G is separable, Y is separable too and clearly $\theta(G)$-invariant. Writing $\theta_{|Y} : G \to \mathcal{L}(Y)$ for the restricted representation, we have $\|\theta_{|Y}(g_n)x_n - x_n\| > \delta$ for all $n \in \mathbb{N}$, and so $\|\theta_{|Y}(g) - I\| \nrightarrow 0$ as $g \to e$, and thus $\theta_{|Y}$ is norm discontinuous.

Hence, on the complement of a meager and Haar null subset of G, $\sigma^1(\theta_{|Y}(g)) = \sigma^1(\theta(g)_{|Y})$ contains a nontrivial polygon. It is known that if $T \in \mathcal{L}(X)$ and Y is a bounded T-invariant subspace, we have the inclusion $\sigma_{\mathrm{app}}(T_{|Y}) \subset \sigma_{\mathrm{app}}(T)$ of approximate spectra, and for each operator T, $\partial\sigma(T) \subset \sigma_{\mathrm{app}}(T)$.

By the remark at the beginning of the section, $\sigma^1(\theta(g)) = (\partial\sigma(\theta(g)))^1$. Hence, for all $g \in G$,

$$\sigma^1(\theta(g)) = (\partial\sigma(\theta(g)))^1 \subset (\sigma_{\mathrm{app}}(\theta(g)))^1 \subset \sigma^1(\theta(g)),$$

and $\sigma^1(\theta(g)) = (\sigma_{\mathrm{app}}(\theta(g)))^1$. In the same way, $\sigma^1(\theta_{|Y}(g)) = (\sigma_{\mathrm{app}}(\theta_{|Y}(g)))^1$. Thus, $\sigma^1(\theta_{|Y}(g)) \subset \sigma^1(\theta(g))$ and $\sigma^1(\theta(g))$ contains a nontrivial polygon on the complement of a meager and Haar null set.

Conversely, if θ is norm continuous, the upper semicontinuity of the map $T \mapsto \sigma(T)$ relative to the norm topology shows that, on a neighborhood of e, the spectra $\sigma(\theta(g))$ are all contained in the right half-plane, and thus cannot contain nontrivial polygons. ■

Example 5.5. For the regular representation of $\mathbb{R}$, the exceptional set reduces to $\{0\}$. If $\mathbb{R}$ acts by translations on $\mathbb{L}^2(\mathbb{T})$, one can see that it is $\mathbb{Q}\pi$.

Now consider the representation of $\mathbb{U}_3^{\mathbb{N}}$ on ℓ^2 defined by $\theta((z_n))(u_n) = (z_n u_n)$. Then θ is a strongly continuous representation which is norm discontinuous, and the spectrum of $\theta((z_n))$ is the set of cubic roots of 1 that appear in the sequence (z_n). Thus, if (z_n) contains all the elements of $\mathbb{U}_3$, then $\sigma(\theta((z_n))) = \sigma^1(\theta((z_n))) = \mathbb{U}_3$ and the exceptional set is the set of sequences (z_n) in $\mathbb{U}_3^{\mathbb{N}}$ where there is a missing cubic root; in particular, we can see that it cannot be countable.

Corollary 5.6. *If for every positive integer n, G admits local division by n, we have the following properties:*

(1) *If G is locally compact, then $\sigma^1(\theta(g)) = \mathbb{T}$ except on a meager and Haar measure zero subset of G.*
(2) *If θ is norm bounded and G is not assumed to be locally compact, we have the same as in* (1) *with "Haar measure zero" replaced by "Haar null".*

Proof. It is sufficient to recall that, in this case, for all $\chi \in \hat{A}$, $\Gamma_{\chi^1 \circ \theta} = \mathbb{T}$ (Proposition 4.1), and to apply the preceding theorem. ∎

6. Applications. The preceding theorem enables us to prove the non-existence of strongly but not norm continuous representations on some particular Banach spaces. We need to recall some definitions and properties.

Definition 6.1.

(1) An infinite-dimensional Banach space X is called *indecomposable* if there is no topological direct decomposition $X = X_1 \oplus X_2$ with X_1 and X_2 both infinite-dimensional.
(2) X is called *hereditarily indecomposable* (H.I.) if all its closed infinite-dimensional subspaces are indecomposable. These spaces are introduced in [11].

Definition 6.2. Let X and Y be Banach spaces. A bounded operator $T : X \to Y$ is said to be *strictly singular* if there is no closed infinite-dimensional subspace X_0 of X such that T is an isomorphism from X_0 onto $T(X_0)$.

Strictly singular operators have the same spectral theory as compact operators.

Proposition 6.3. *The spectrum of a strictly singular operator is at most countable with* 0 *as the only possible cluster point. Nonzero elements of the spectrum are eigenvalues of finite algebraic multiplicity.*

We have the following property of H.I. spaces:

Theorem 6.4 (see [22]). *Let X be a H.I. space and $T \in \mathcal{L}(X)$. Then $T = \lambda I + S$, where $\lambda \in \mathbb{C}$, I is the identity map on X, and S is a strictly singular operator.*

We will need the following definitions concerning Banach spaces.

Definition 6.5. Let X be a Banach space.

(1) X has the *Dunford–Pettis* (*D.P.*) *property* if for any couple of sequences (x_n) in X and (x_n^*) in the dual space X' such that (x_n) tends weakly to x and (x_n^*) tends weakly to x^*, we have $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

(This is equivalent to the statement that every weakly compact operator from X to a Banach space Y maps weakly compact subsets of X onto norm compact subsets of Y.)

(2) X has the *Grothendieck property* (G) if each sequence (x_n^*) in X' converging in the weak* topology converges also in the weak topology of X'.

A reference for these properties is e.g. [27].

Example 6.6. $\mathbb{L}^\infty(\Omega)$ (Ω a measure space) has both (D.P.) and (G).

We have the following result:

Theorem 6.7 (see [21, p. 211]). *Let X be a Banach space having* (D.P.) *and (T_n) a sequence of bounded operators on X such that:*

(1) *For all $x^* \in X'$, we have $\|T_n x^*\| \to 0$.*
(2) *For every norm bounded sequence (x_n^*) in X', $(T_n^* x_n^*)$ tends weakly to* 0.

Then $\lim_{n\to\infty} \|T_n^2\| = 0$.

We can now state the following result.

Theorem 6.8.

(1) *Let X be a H.I. space, let G be a locally compact abelian Polish group having local division by n for each integer n (e.g. a group of Lie type), and let $\theta : G \to \mathcal{L}(X)$ be a representation. Then the following assertions are equivalent:*

 (i) *θ is strongly continuous.*
 (ii) *θ is norm continuous.*

(2) *The same is true if X is a space with* (D.P.) *and* (G) *(e.g. $\mathbb{L}^\infty(\Omega)$) and G a locally compact Polish group.*

Proof. (1) By Theorem 6.4 and Proposition 6.3, $\sigma(\theta(g))$ is countable for each g, which contradicts Corollary 5.6(1).

(2) Assume that θ is strongly continuous and (g_n) is a sequence converging to e in G. Then $\theta(g_n)x \to x$ for all $x \in X$, and thus for all $x^* \in X'$ and $x \in X$, we have

$$\langle (\theta(g_n))^* x^*, x\rangle = \langle x^*, \theta(g_n)x\rangle \to \langle x^*, x\rangle,$$

so $(\theta(g_n))^* x^* \overset{w^*}{\to} x^*$ in X'; thus, the representation $g \mapsto (\theta(g))^*$ from G on X' is weakly continuous. It is known (see e.g. [6]) that for a locally compact group, a weakly continuous representation is also strongly continuous (this fact is much more classical for unitary representations but, nevertheless, remains true in the general case). The representation $g \mapsto (\theta(g))^*$ on X' is thus strongly continuous and for all $x^* \in X'$, $(\theta(g))^* x^* \to x^*$ in X' (endowed with its norm topology).

Moreover, if (x_n^*) is a bounded sequence in X', for all $x \in X$ we have

$$|\langle(\theta(g_n) - I)^* x_n^*, x\rangle| = |\langle x_n^*, (\theta(g_n) - I)x\rangle| \le \sup_{n\in\mathbb{N}} \|x_n^*\| \cdot \|(\theta(g_n) - I)x\| \to 0,$$

and thus $(\theta(g_n) - I)^* x_n^* \overset{w^*}{\to} 0$, and by property (G), $(\theta(g_n) - I)^* x_n^* \overset{w}{\to} 0$; so by Theorem 6.7, $\|(\theta(g_n) - I)^2\| \to 0$ and $\rho((\theta(g_n) - I)^2) \to 0$. Thus, for any strongly continuous representation $\theta : G \to \mathcal{L}(X)$, we have $\rho((\theta(g_n) - I)^2) \to 0$. By the spectral mapping theorem, $\rho(\theta(g_n) - I) \to 0$ and, by [8, Theorem 3.3], θ is norm continuous. ∎

Remarks 6.9. (1) The first assertion of the theorem remains true if G is not assumed to be locally compact but if θ is norm bounded (see Corollary 5.6(2)).

(2) If G is compact (so that θ is clearly norm bounded), then the second assertion of the theorem remains true if G is not assumed to be abelian (using Corollary 2.5 instead of [8, Theorem 3.3]).

(3) The second part in the particular case of strongly continuous one-parameter groups (and, actually, also one-parameter semigroups) is proved in [21] and also in the Ph.D thesis of T. Coulhon. It applies, in particular, to spaces such as $\mathbb{L}^\infty(\Omega)$.

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