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Asymptotically cyclic quasianalytic contractions

by

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Abstract. The study of quasianalytic contractions, motivated by the hyperinvariant subspace problem, is continued. Special emphasis is put on the case when the contraction is asymptotically cyclic. New properties of the functional commutant are explored. Analytic contractions and bilateral weighted shifts are discussed as illuminating examples.

1. Introduction. In this paper we continue the study of quasianalytic contractions initiated and carried out in [Kér01], [Kér11] and [Kér13]. These investigations are motivated by the Invariant Subspace Problem (ISP) and the Hyperinvariant Subspace Problem (HSP), since in the setting of asymptotically non-vanishing contractions these problems can be reduced to special classes of quasianalytic contractions.

We recall that (ISP) asks about the existence of a non-trivial invariant subspace $\mathcal{M}$ of an arbitrary (bounded linear) operator T acting on a (complex) Hilbert space $\mathcal{H}$, while (HSP) asks whether there exists a non-trivial hyperinvariant subspace $\mathcal{N}$ of a non-scalar T. The subspace (closed linear manifold) $\mathcal{N}$ is *hyperinvariant* for T if it is invariant for every operator C commuting with T: $C\mathcal{N} \subset \mathcal{N}$ whenever $CT = TC$, and it is *non-trivial* if $\mathcal{N} \neq \{0\}$ and $\mathcal{N} \neq \mathcal{H}$. The invariant subspace lattice of T is denoted by $\operatorname{Lat} T$, and the hyperinvariant subspace lattice of T is denoted by $\operatorname{Hlat} T$.

(ISP) and (HSP) are arguably the most challenging open questions in operator theory. Studying these problems we may assume that $\dim \mathcal{H} = \aleph_0$ and T is an absolutely continuous (a.c.) contraction, that is, $T = T_u \oplus T_c$ where T_u is an a.c. unitary operator and T_c is a completely non-unitary (c.n.u.) contraction. The latter means that $\|T_c\| \leq 1$ and there is no non-zero invariant subspace $\mathcal{M}$ such that the restriction $T_c|\mathcal{M}$ is unitary.

Let $\mathcal{L}(\mathcal{H})$ stand for the C^*-algebra of all operators acting on $\mathcal{H}$. Assuming that $T \in \mathcal{L}(\mathcal{H})$ is an a.c. contraction, the Sz.-Nagy–Foiaş functional calculus

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Φ_T is a contractive, unital algebra-homomorphism from the Hardy space H^∞ of bounded analytic functions on the open unit disc $\mathbb{D}$ into $\mathcal{L}(\mathcal{H})$, which is continuous in the weak-$*$ topologies and satisfies the condition $T = \Phi_T(\chi) = \chi(T)$, where $\chi(z) = z$ is the identical function. It is also worth mentioning that Φ_T is monotone in the sense that $\|f(T)x\| \leq \|g(T)x\|$ for every $x \in \mathcal{H}$ (in notation: $f(T) \overset{a}{\prec} g(T)$) whenever $|f(z)| \leq |g(z)|$ for every z in the unit disc $\mathbb{D}$ (in notation: $f \overset{a}{\prec} g$).

Another tool in the study of T is its unitary asymptote. We recall that the pair (X, V) is a *unitary asymptote* of T if V is a unitary operator acting on a Hilbert space $\mathcal{K}$ and $X\colon \mathcal{H} \to \mathcal{K}$ is a linear transformation satisfying the conditions

$$\bigvee_{n=1}^{\infty} V^{-n} X\mathcal{H} = \mathcal{K}, \qquad \|Xh\| = \lim_{n\to\infty} \|T^n h\| \quad \text{for every } h \in \mathcal{H}, \qquad XT = VX.$$

The *nullspace* of X is the hyperinvariant subspace of stable vectors:

$$\mathcal{H}_0(T) = \Big\{h \in \mathcal{H} : \lim_{n\to\infty} \|T^n h\| = 0\Big\}.$$

We say that T is *asymptotically non-vanishing* if $\mathcal{H}_0(T) \neq \mathcal{H}$. Moreover, $T \in C_{1\cdot}$ if $\mathcal{H}_0(T) = \{0\}$; $T \in C_{0\cdot}$ if $\mathcal{H}_0(T) = \mathcal{H}$; $T \in C_{\cdot 1}$ if $T^* \in C_{1\cdot}$; $T \in C_{\cdot 0}$ if $T^* \in C_{0\cdot}$; and $C_{ij} = C_{i\cdot} \cap C_{\cdot j}$ $(i, j = 0, 1)$. For further properties of unitary asymptotes we refer to [Kér13] and [NFBK, Chapter IX]. Our basic reference for the theory of contractions is [NFBK].

Our paper is organized in the following way. In Section 2 we introduce a local version of the quasianalytic spectral set and exhibit its connection with the residual set. In Section 3 the fundamental properties of quasianalytic contractions are summarized including their asymptotic behaviour. Asymptotically cyclic quasianalytic contractions are studied in Section 4, where conditions equivalent to the existence of a non-trivial hyperinvariant subspace are given. For such a contraction T the commutant $\{T\}'$ can be identified with a function algebra $\mathcal{F}(T)$, the so-called functional commutant. Answering a question posed in [Kér11] we show in Section 5 that $\mathcal{F}(T)$ can be a pre-Douglas algebra only in the case when $\mathcal{F}(T) = H^\infty$. We also prove similarity invariance of $\mathcal{F}(T)$ and find its representation in the functional model. The last two sections are devoted to special classes of operators, where quasianalytic contractions naturally arise. Namely, we study analytic contractions in Section 6 and bilateral weighted shifts in Section 7.

2. Local quasianalytic spectral set. Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction and let (X, V) be a unitary asymptote of T. It is known that $V \in \mathcal{L}(\mathcal{K})$ is an a.c. unitary operator, that is, the spectral measure E of V is a.c. with respect to the normalized Lebesgue measure m on the unit circle $\mathbb{T}$. The *residual set* $\omega(T)$ of T is the measurable support of E. For any $x, y \in \mathcal{H}$, $w_{x,y} \in L^1(\mathbb{T})$ is the *asymptotic density function* of T at x and y: $E_{Xx,Xy} = w_{x,y}\, dm$. The measurable set

$$\omega(T,x) = \{\zeta \in \mathbb{T} : w_{x,x}(\zeta) > 0\}$$

is the *local residual set* of T at x. (It is easy to check that $w_{x,y}$ and $\omega(T,x)$ are independent of the choice of (X,V).) It is worth mentioning that $\mathcal{H}_\omega(T) = \{x \in \mathcal{H} : \omega(T,x) = \omega(T)\}$ is a dense G_δ-set in $\mathcal{H}$ (see [NFBK, Lemma IX.2.15]).

Given a decreasing sequence $F = \{f_n\}_{n=1}^\infty$ in H^∞ ($f_{n+1} \overset{a}{\prec} f_n$ for every n), consider the limit function φ_F on $\mathbb{T}$, defined by $\varphi_F(\zeta) = \lim_{n\to\infty} |f_n(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$, and the measurable set $N_F = \{\zeta \in \mathbb{T} : \varphi_F(\zeta) > 0\}$. Then the sequence $F(T) = \{f_n(T)\}_{n=1}^\infty$ of operators is also decreasing ($f_{n+1}(T) \overset{a}{\prec} f_n(T)$ for every n) and the set

$$\mathcal{H}_0(T,F) = \Big\{x \in \mathcal{H} : \lim_{n\to\infty} \|f_n(T)x\| = 0\Big\}$$

of stable vectors for $F(T)$ is a hyperinvariant subspace of T.

For measurable subsets α and β of $\mathbb{T}$, we write $\alpha = \beta$, $\alpha \neq \beta$ and $\alpha \subset \beta$ if $m(\alpha \triangle \beta) = 0$, $m(\alpha \triangle \beta) > 0$ and $m(\alpha \setminus \beta) = 0$ respectively, that is, $\chi_\alpha = \chi_\beta$, $\chi_\alpha \neq \chi_\beta$ and $\chi_\alpha \le \chi_\beta$ respectively for the corresponding characteristic functions as elements of the Banach space $L^1(\mathbb{T})$.

We say that T is *quasianalytic on a measurable subset* α *of* $\mathbb{T}$ *at a vector* $x \in \mathcal{H}$ if $x \notin \mathcal{H}_0(T,F)$ whenever F is non-vanishing on α, that is, $N_F \cap \alpha \neq \emptyset$. Let $\mathcal{A}(T,x)$ be the system of sets α with this property and set $a(T,x) = \sup\{m(\alpha) : \alpha \in \mathcal{A}(T,x)\}$. Taking a sequence $\{\alpha_n\}_{n=1}^\infty$ in $\mathcal{A}(T,x)$ so that $\lim_{n\to\infty} m(\alpha_n) = a(T,x)$, it is easy to see that $\pi(T,x) = \bigcup_{n=1}^\infty \alpha_n$ will be the largest element of $\mathcal{A}(T,x)$. The set $\pi(T,x)$ is called the *local quasianalytic spectral set of* T *at* x. (Note that $\pi(T,x)$ is uniquely determined up to sets of measure 0.) We recall from [Kér11] that T is quasianalytic on α if $\mathcal{H}_0(T,F) = \{0\}$ whenever $N_F \cap \alpha \neq \emptyset$; the (global) *quasianalytic spectral set* $\pi(T)$ is the largest element of $\mathcal{A}(T)$, the system of sets on which T is quasianalytic. The following statement follows immediately from the definitions.

Proposition 1. *The set* $\pi(T)$ *is the largest measurable set such that* $\pi(T) \subset \pi(T,x)$ *for every non-zero* $x \in \mathcal{H}$.

The next lemma states that local stability is determined by the asymptotic density function.

Lemma 2. *Let* $F = \{f_n\}_{n=1}^\infty$ *be a decreasing sequence in* H^∞ *and* $x \in \mathcal{H}$.

(a) *If* $\lim_{n\to\infty} \|f_n(T)x\| = 0$ *then* $\varphi_F w_{x,x} = 0$.

(b) *If* $\varphi_F w_{x,x} = 0$ *then there exists an increasing mapping* $\tau\colon \mathbb{N} \to \mathbb{N}$ *such that* $\lim_{n\to\infty} \|T^{\tau(n)} f_n(T)x\| = 0$.

Proof. Part (a) readily follows from the equalities

$$\lim_{n\to\infty} \|Xf_n(T)x\|^2 = \lim_{n\to\infty} \|f_n(V)Xx\|^2 = \lim_{n\to\infty} \int_{\mathbb{T}} |f_n|^2 w_{x,x}\, dm = \int_{\mathbb{T}} \varphi_F^2 w_{x,x}\, dm.$$

Since there exists an increasing $\tau\colon \mathbb{N} \to \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \|Xf_n(T)x\|^2 = \lim_{n\to\infty} \|T^{\tau(n)} f_n(T)x\|,$$

the same equalities yield (b) too.

Note that $G = \{\chi^{\tau(n)} f_n\}_{n=1}^{\infty}$ is also a decreasing sequence with $\varphi_G = \varphi_F$. ∎

The following theorem establishes a connection between the local and global spectral invariants introduced before.

Theorem 3. *For every non-zero* $x \in \mathcal{H}$ *we have*

$$\pi(T) \subset \pi(T,x) = \omega(T,x) \subset \omega(T).$$

Proof. Let $F = \{f_n\}_{n=1}^{\infty}$ be a decreasing sequence with $N_F \cap \omega(T,x) \neq \emptyset$. Then $\varphi_F w_{x,x} \neq 0$ implies $\lim_{n\to\infty} \|f_n(T)x\| > 0$ by Lemma 2. Thus T is quasianalytic on $\omega(T,x)$ at x, and so $\omega(T,x) \subset \pi(T,x)$.

Setting $\alpha = \mathbb{T} \setminus \omega(T,x)$, let $\vartheta \in H^\infty$ be such that $|\vartheta| = \chi_\alpha + \frac{1}{2}\chi_{\mathbb{T}\setminus\alpha}$, and form the decreasing sequence $F = \{\vartheta^n\}_{n=1}^{\infty}$ with $\varphi_F = \chi_\alpha$. By Lemma 2, $\varphi_F w_{x,x} = 0$ yields the existence of an increasing $\tau\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \|T^{\tau(n)} f_n(T)x\| = 0$. Then $G = \{\chi^{\tau(n)}\vartheta^n\}_{n=1}^{\infty}$ is a decreasing sequence with $N_G = \alpha$ and $x \in \mathcal{H}_0(T,G)$. Therefore $\pi(T,x) \subset \omega(T,x)$. ∎

As a consequence we obtain conditions for the existence of a non-trivial hyperinvariant subspace. (Statement (b) below already appears in [Kér01].)

Corollary 4.

(a) *If* $\omega(T,x) \neq \omega(T)$ *for some non-zero* $x \in \mathcal{H}$ *and* $F = \{f_n\}_{n=1}^{\infty}$ *is a decreasing sequence with* $N_F = \omega(T) \setminus \omega(T,x)$, *then* $x \in \mathcal{H}_0(T,F)$ *and* $\mathcal{H}_0(T,F) \cap \mathcal{H}_\omega(T) = \emptyset$, *therefore* $\mathcal{H}_0(T,F)$ *is a non-trivial hyperinvariant subspace of* T.

(b) *If* $\pi(T) \neq \omega(T)$ *then* $\operatorname{Hlat} T$ *is non-trivial.*

Remark 5. We know that $\omega(T,x) = \omega(T)$ for every $x \in \mathcal{H}_\omega(T)$, which is a dense G_δ-set in $\mathcal{H}$. On the other hand, it may happen that $\pi(T,x) \neq \pi(T)$ for every non-zero $x \in \mathcal{H}$. Indeed, let α_1 and α_2 be sets of positive measure on $\mathbb{T}$ such that $\alpha_1 \neq \alpha \neq \alpha_2$ for $\alpha = \alpha_1 \cap \alpha_2$. For $j = 1,2$, let $T_j \in \mathcal{L}(\mathcal{H}_j)$ be an a.c. contraction satisfying $\pi(T_j) = \omega(T_j) = \alpha_j$. (Its existence follows from the results of Section 3.) Form the orthogonal sum $T = T_1 \oplus T_2 \in \mathcal{L}(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2)$. For a non-zero $x = x_1 \oplus x_2 \in \mathcal{H}$ the local residual set $\omega(T, x)$ is α_1 if $x_2 = 0$, α_2 if $x_1 = 0$, and $\alpha_1 \cup \alpha_2$ if $x_1 \neq 0 \neq x_2$. On the other hand, $\pi(T) = \alpha$.

It is known that the local asymptotic density function, and so the local quasianalytic spectral set as well, can be expressed in terms of the resolvent as a non-tangential limit (see [ARS07, Lemma 2.2].)

PROPOSITION 6. *Given any $x \in \mathcal{H}$ we have*

$$\operatorname{nt-}\lim_{z\to\zeta}(1 - |z|^2)\|(I - \overline{z}T)^{-1}x\|^2 = w_{x,x}(\zeta) \quad \textit{for a.e. } \zeta \in \mathbb{T}.$$

Proof. For the sake of completeness we sketch the proof, which is based on the representation of the unitary asymptote in the dilation space.

Let $U_{T^*,+} \in \mathcal{L}(\mathcal{K}_{*,+})$ be the minimal isometric dilation of T^*. Then $U_* = (U_{T^*,+})^*$ is the minimal coisometric extension of T. Taking the Wold decomposition $U_{T^*,+} = S_n \oplus R_*^* \in \mathcal{L}(\mathcal{K}_{*,+} = \mathcal{S}_* \oplus \mathcal{R}_*)$, where S_n is a unilateral shift of some multiplicity n and R_* is unitary, we obtain the decomposition $U_* = S_n^* \oplus R_*$. The pair (X_*, R_*) is a unitary asymptote of T, where $X_* = P_{\mathcal{R}_*}|\mathcal{H}$.

Given $x \in \mathcal{H}$ we have

$$\begin{aligned}&\operatorname{nt-}\lim_{z\to\zeta}(1 - |z|^2)\|(I - \overline{z}R_*)^{-1}X_*x\|^2 \\ &= \operatorname{nt-}\lim_{z\to\zeta}\int_{\mathbb{T}}\frac{1 - |z|^2}{|1 - \overline{z}s|^2}w_{x,x}(s)\,dm(s) = w_{x,x}(\zeta) \quad \text{for a.e. } \zeta \in \mathbb{T}.\end{aligned}$$

Notice that the Poisson kernel appears in the integral. Using tools from harmonic analysis it can be shown that, for every $y \in \mathcal{S}_*$,

$$\operatorname{nt-}\lim_{z\to\zeta}(1 - |z|^2)\|(I - \overline{z}S_n^*)^{-1}y\|^2 = 0 \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

Now the statement follows from the decomposition

$$(I - \overline{z}T)^{-1}x = (I - \overline{z}U_*)^{-1}x = (I - \overline{z}S_n^*)^{-1}P_{\mathcal{S}_*}x \oplus (I - \overline{z}R_*)^{-1}X_*x. \ \blacksquare$$

3. Quasianalytic contractions. An a.c. contraction $T \in \mathcal{L}(\mathcal{H})$ is *quasianalytic* if $\pi(T) = \omega(T) \neq \emptyset$. In view of Corollary 4, in the setting of asymptotically non-vanishing contractions, (HSP) can be reduced to the case when T is quasianalytic.

For the sake of convenience and easy reference, in the following theorem we collect some fundamental statements on quasianalytic contractions. For their proofs we refer to [Kér01] and [Kér11].

First we recall some definitions. The *simple unilateral shift* $S \in \mathcal{L}(H^2)$ is defined by $Sf = \chi f$. An operator $A \in \mathcal{L}(\mathcal{H})$ is a *quasiaffine transform* of $B \in \mathcal{L}(\mathcal{K})$, in notation: $A \prec B$, if there exists a *quasiaffinity* (i.e. an injective transformation with dense range) $Q \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $QA = BQ$. The operators A and B are *quasisimilar*, in notation: $A \sim B$, if $A \prec B$ and $B \prec A$. A function $f \in H^\infty$ is *partially inner* if $|f(0)| < 1 = \|f\|_\infty$ and the set $\Omega(f) = \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$ is of positive measure. A partially inner function f is *regular* if $\alpha \subset \Omega(f)$ and $m(\alpha) = 0$ imply $m(f(\alpha)) = 0$; or equivalently, if $f(\alpha)$ is measurable whenever $\alpha \subset \Omega(f)$ is measurable.

THEOREM 7. *The operators T, T_1 and T_2 below are all a.c. contractions.*

(a) *The unilateral shift $S \in \mathcal{L}(H^2)$ is quasianalytic with $\pi(S) = \mathbb{T}$.*
(b) *If $T \prec S$, then T is quasianalytic with $\pi(T) = \mathbb{T}$.*
(c) *If T is quasianalytic, then its inflation*

$$T^{(n)} = \underbrace{T \oplus \cdots \oplus T}_{n \text{ terms}}$$

is quasianalytic with $\pi(T^{(n)}) = \pi(T)$ $(n \in \mathbb{N})$.
(d) *If T is quasianalytic and $\mathcal{M}$ is a non-zero invariant subspace of T, then $T|\mathcal{M}$ is quasianalytic with $\pi(T|\mathcal{M}) = \pi(T)$.*
(e) *If T is quasianalytic and f is a regular partially inner function satisfying $\Omega(f) \cap \pi(T) \neq \emptyset$, then $f(T)$ is quasianalytic with $\pi(f(T)) = f(\Omega(f) \cap \pi(T))$.*
(f) *If $T_1 \sim T_2$ and T_1 is quasianalytic, then so is T_2 and $\pi(T_2) = \pi(T_1)$.*

On the basis of these statements a lot of examples of quasianalytic contractions can be constructed.

We show that quasianalycity determines the asymptotic behaviour of a contraction.

THEOREM 8. *If T is a quasianalytic contraction, then $T \in C_{10}$.*

Proof. Since $\pi(T) \neq \emptyset$ and $F = \{\chi^n\}_{n=1}^\infty$ is a decreasing sequence with $N_F = \mathbb{T}$, we infer that $T \in C_{1\cdot}$.

Suppose that $T \notin C_{\cdot 0}$. Then $\mathcal{H}_1 = \mathcal{H}_0(T^*)^\perp$ is a non-zero invariant subspace of T and $T_1 = T|\mathcal{H}_1 \in C_{11}$. Hence T_1 is quasisimilar to an a.c. unitary operator V_1. By Theorem 7(d)&(f) it follows that V_1 is quasianalytic, which is impossible since $\pi(V_1) = \emptyset \neq \omega(V_1)$. Therefore $T \in C_{\cdot 0}$. ∎

4. Asymptotically cyclic contractions. Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction, and let (X, V) be a unitary asymptote of T. We say that T is *asymptotically cyclic* if the a.c. unitary operator $V \in \mathcal{L}(\mathcal{K})$ is *cyclic*, that is, $\bigvee_{n=0}^\infty V^n y = \mathcal{K}$ for some $y \in \mathcal{K}$. It is known that V is cyclic exactly when its commutant $\{V\}' = \{D \in \mathcal{L}(\mathcal{K}) : DV = VD\}$ is abelian. The universal property of the unitary asymptote implies that for every $C \in \{T\}'$ there is a unique $D \in \{V\}'$ such that $XC = DX$, and the mapping $\gamma \colon \{T\}' \to \{V\}'$, $C \mapsto D$, is a contractive, unital algebra-homomorphism. Hence $\{T\}'$ is abelian if so is $\{V\}'$ and γ is injective, which is evidently true if $T \in C_{1\cdot}$. (Injectivity of γ was studied in [GK11].)

We give a sufficient condition for the contraction T to be asymptotically cyclic. First we fix some notation. Given $A \in \mathcal{L}(\mathcal{E})$ and $B \in \mathcal{L}(\mathcal{F})$,

$$\mathcal{I}(A, B) = \{Q \in \mathcal{L}(\mathcal{E}, \mathcal{F}) : QA = BQ\}$$

is the set of transformations intertwining A with B. The operators A and B are *unitarily equivalent*, in notation: $A \cong B$, if $\mathcal{I}(A, B)$ contains a unitary transformation. Moreover, A and B are *similar*, in notation: $A \approx B$, if $\mathcal{I}(A, B)$ contains an *affinity* (invertible transformation). Finally, A *can be injected into* B, in notation: $A \overset{i}{\prec} B$, if $\mathcal{I}(A, B)$ contains an injection. The minimal unitary extension of the *simple unilateral shift* $S \in \mathcal{L}(H^2)$ is the *simple bilateral shift* $\widetilde{S} \in \mathcal{L}(L^2(\mathbb{T}))$, defined by $\widetilde{S}f = \chi f$.

Proposition 9. *If $T \in \mathcal{L}(\mathcal{H})$ is a contraction and $T \prec S$, then T is asymptotically cyclic and $V|(X\mathcal{H})^- \cong S$.*

Proof. The relation $T \prec S$ immediately implies that $T \in C_{10}$; in particular, T is a.c. and the unitary asymptote V acts on a non-zero space $\mathcal{K}$. Suppose that V is not cyclic. Then $S_2 = S \oplus S \overset{i}{\prec} T$ by [Kér07, Theorem 1] (see also [NFBK, Theorem IX.3.2]). Thus $S_2 \overset{i}{\prec} S$, which is impossible by [NF74, Theorem 5]. Therefore V is cyclic, that is, T is asymptotically cyclic.

Let $Q \in \mathcal{I}(T, S)$ be a quasiaffinity, and let $\widetilde{Q} \in \mathcal{I}(T, \widetilde{S})$ be defined by $\widetilde{Q}h = Qh$ $(h \in \mathcal{H})$. There exists a unique $Y \in \mathcal{I}(V, \widetilde{S})$ such that $\widetilde{Q} = YX$. It is known that $\ker Y$ is reducing for V, $(Y\mathcal{K})^-$ is reducing for $\widetilde{S}$, and $V|(\ker Y)^\perp \cong \widetilde{S}|(Y\mathcal{K})^-$. Since $(Y\mathcal{K})^- \supset (Q\mathcal{H})^- = H^2$, it follows that $(Y\mathcal{K})^- = L^2(\mathbb{T})$, and so $V|(\ker Y)^\perp \cong \widetilde{S}$. Taking into account that V is cyclic, we infer that $\ker Y = \{0\}$, thus Y is a quasiaffinity. The relations $Y(X\mathcal{H})^- \subset (YX\mathcal{H})^- = (Q\mathcal{H})^- = H^2$ imply that $(X\mathcal{H})^-$ is a non-trivial invariant subspace of V. Since $\bigvee_{n=0}^\infty V^{-n}(X\mathcal{H})^- = \mathcal{K}$, we conclude that $V|(X\mathcal{H})^- \cong S$. ■

We note that $S_2 \prec \widetilde{S}$, and so $T \prec \widetilde{S}$ does not imply that T is asymptotically cyclic. Indeed, $Q \in \mathcal{I}(S_2, \widetilde{S})$ defined by $Q(f \oplus g) = \vartheta f + g$ is a quasiaffinity provided $\vartheta \in L^\infty(\mathbb{T})$ is a.e. non-zero and $\int_{\mathbb{T}} \log|\vartheta|\, dm = -\infty$.

The set of asymptotically cyclic, quasianalytic contractions acting on the Hilbert space $\mathcal{H}$ is denoted by $\mathcal{L}_0(\mathcal{H})$. If T is cyclic then so is V (but not conversely), hence (ISP) in the setting of quasianalytic contractions can be reduced to the class $\mathcal{L}_0(\mathcal{H})$.

Proposition 10. *If $T \in \mathcal{L}_0(\mathcal{H})$ then*

(i) *$\{T\}'$ is abelian, and*
(ii) *every non-zero $C \in \{T\}'$ is injective.*

Proof. If $T \in \mathcal{L}_0(\mathcal{H})$ then $T \in C_{10}$ by Theorem 8, and so γ is injective. Since $\{V\}'$ is abelian it follows that $\{T\}'$ is abelian too. For the proof of (ii) see [Kér01, Proposition 23]. ∎

PROPOSITION 11. *If $T_1, T_2 \in \mathcal{L}_0(\mathcal{H})$ and $T_1T_2 = T_2T_1$, then $\{T_1\}' = \{T_2\}'$.*

Proof. Fix any $C \in \{T_1\}'$. Since $T_2 \in \{T_1\}'$, the commutativity of $\{T_1\}'$ yields $CT_2 = T_2C$, that is, $C \in \{T_2\}'$. ∎

We have a lot of information on the structure of a contraction if its residual set covers the unit circle. Hence it is worth considering the special class

$$\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}.$$

In the next theorem we summarize important properties of operators in $\mathcal{L}_1(\mathcal{H})$; for the proof we refer to [NFBK, Section IX.3]. We recall that $\mathrm{Lat_s}\, T$ stands for the set of those invariant subspaces $\mathcal{M}$ for which the restriction $T|\mathcal{M}$ is similar to S. The range of the functional calculus Φ_T is denoted by $H^\infty(T)$, and the algebra $\mathcal{W}(T)$ is the closure of $H^\infty(T)$ in the weak operator topology. Finally, T is called *reflexive* if $C \in \mathcal{W}(T)$ whenever $\mathrm{Lat}\, C \supset \mathrm{Lat}\, T$.

THEOREM 12. *If $T \in \mathcal{L}_1(\mathcal{H})$ then*

(i) $\bigvee \mathrm{Lat_s}\, T = \mathcal{H}$,
(ii) *Φ_T is an isometry,*
(iii) *$H^\infty(T) = \mathcal{W}(T)$, and*
(iv) *T is reflexive.*

Examples of operators in $\mathcal{L}_1(\mathcal{H})$ are provided by the following propositions.

PROPOSITION 13. *If $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $T \prec S$, then $T \in \mathcal{L}_1(\mathcal{H})$ and $H^\infty(T) = \{T\}'$.*

Proof. By Theorem 7(b) and Proposition 9 it follows that $T \in \mathcal{L}_1(\mathcal{H})$. For the proof of $H^\infty(T) = \{T\}'$ see [Kér11, Proposition 5.3]. ∎

PROPOSITION 14. *If $T \in \mathcal{L}_1(\mathcal{H})$, then $T|\mathcal{M} \in \mathcal{L}_1(\mathcal{M})$ for every non-zero invariant subspace $\mathcal{M}$ of T.*

Proof. The restriction $T|\mathcal{M}$ is quasianalytic with $\pi(T|\mathcal{M}) = \pi(T) = \mathbb{T}$ by Theorem 7(d). Since T is asymptotically cyclic, so is $T|\mathcal{M}$, since its unitary asymptote is a direct summand of V. ∎

There is a strong connection between the classes $\mathcal{L}_0(\mathcal{H})$ and $\mathcal{L}_1(\mathcal{H})$. The following statement is the content of [KT12, Theorem 1].

THEOREM 15. *For every $T_0 \in \mathcal{L}_0(\mathcal{H})$ we can find $T_1 \in \mathcal{L}_1(\mathcal{H})$ such that $T_0T_1 = T_1T_0$; hence $\{T_0\}' = \{T_1\}'$ and so* $\mathrm{Hlat}\, T_0 = \mathrm{Hlat}\, T_1$.

Therefore, (HSP) in $\mathcal{L}_0(\mathcal{H})$ can be reduced to $\mathcal{L}_1(\mathcal{H})$, where Theorem 12 provides a lot of information on the operator. If $\{T\}' = H^\infty(T)$, then

$\operatorname{Hlat} T = \operatorname{Lat} T$ is non-trivial. However, if $\{T\}' \neq H^\infty(T)$ then shift-type invariant subspaces are not hyperinvariant.

PROPOSITION 16. *Let $T \in \mathcal{L}_1(\mathcal{H})$ be such that $\{T\}' \neq H^\infty(T)$. Then*

$$\operatorname{Lat} C \cap \operatorname{Lat_s} T = \emptyset \quad \text{for every } C \in \{T\}' \setminus H^\infty(T).$$

Proof. Let $C \in \{T\}'$ be such that $C\mathcal{M} \subset \mathcal{M}$ for some $\mathcal{M} \in \operatorname{Lat_s} T$. Since $\{T|\mathcal{M}\}' = H^\infty(T|\mathcal{M})$ by Proposition 13 and $C|\mathcal{M} \in \{T|\mathcal{M}\}'$, there exists $f \in H^\infty$ such that $C|\mathcal{M} = f(T|\mathcal{M}) = f(T)|\mathcal{M}$. In view of Proposition 10(ii), the relations $C - f(T) \in \{T\}'$ and $(C - f(T))|\mathcal{M} = 0$ yield $C = f(T)$. ∎

COROLLARY 17. *For any $T \in \mathcal{L}_1(\mathcal{H})$,*

$$\{T\}' = H^\infty(T) \quad \text{if and only if} \quad \operatorname{Hlat} T = \operatorname{Lat} T.$$

The following theorem states that if non-trivial hyperinvariant subspaces exist, then they can be derived from shift-invariant subspaces.

THEOREM 18. *Let $T \in \mathcal{L}_1(\mathcal{H})$ with $\{T\}' \neq H^\infty(T)$. Then the following statements are equivalent:*

(i) $\operatorname{Hlat} T$ *is non-trivial;*
(ii) *there exists $\mathcal{M} \in \operatorname{Lat_s} T$ such that $\bigvee \{C\mathcal{M} : C \in \{T\}'\} \neq \mathcal{H}$;*
(iii) *there exists $\mathcal{S} \subset \operatorname{Lat_s} T$ such that $\mathcal{H} \neq \bigvee \mathcal{S} \in \operatorname{Hlat} T$.*

Proof. Assume that $\mathcal{N}$ is a non-trivial hyperinvariant subspace of T. Since $T|\mathcal{N} \in \mathcal{L}_1(\mathcal{N})$, there exists a subspace $\mathcal{M} \in \operatorname{Lat_s}(T|\mathcal{N}) \subset \operatorname{Lat_s} T$ included in $\mathcal{N}$ (see Theorem 12 and Proposition 14). It is clear that $\mathcal{N}_0 = \bigvee \{C\mathcal{M} : C \in \{T\}'\}$ is a hyperinvariant subspace satisfying the conditions $\mathcal{M} \subset \mathcal{N}_0 \subset \mathcal{N}$, in particular $\mathcal{N}_0$ is non-trivial.

For any $C \in \{T\}'$ and $\lambda \in \mathbb{C}$, we have $(C - \lambda I)\mathcal{M} \vee \mathcal{M} = C\mathcal{M} \vee \mathcal{M}$. Hence $\mathcal{N}_0 = \bigvee \{C\mathcal{M} : C \in \{T\}' \text{ invertible}\}$. However, if $C \in \{T\}'$ is invertible, then $T|C\mathcal{M} \approx T|\mathcal{M} \approx S$ and so $C\mathcal{M} \in \operatorname{Lat_s}(T)$. ∎

It is known that the unilateral shift S is *cellular-indecomposable*, that is, the intersection of any two non-zero invariant subspaces of S is non-zero. A contraction $T \in \mathcal{L}_1(\mathcal{H})$ is called *quasiunitary* if X has dense range and so it is a quasiaffinity, where (X, V) is a unitary asymptote of T (see [Kér01, Section 5].)

PROPOSITION 19. *If $T \in \mathcal{L}_1(\mathcal{H})$, then the following conditions are equivalent:*

(i) T *is not quasiunitary;*
(ii) $T \prec S$;
(iii) T *is cellular-indecomposable.*

Proof. Let (X, V) be a unitary asymptote of T; we know that $\mathcal{L}(\mathcal{K}) \ni V \cong \widetilde{S}$. The subspace $(X\mathcal{H})^-$ is invariant for V and $\bigvee_{n=1}^\infty V^{-n}(X\mathcal{H})^- = \mathcal{K}$.

Hence, if $(X\mathcal{H})^- \neq \mathcal{K}$ then $V|(X\mathcal{H})^- \cong S$, and so (i) implies (ii). The converse follows from Proposition 9.

If T is not quasiunitary, then $S \prec \widetilde{S}$ implies $T \prec \widetilde{S}$. Hence $T \prec \widetilde{S}$ always holds. Therefore, (ii) and (iii) are equivalent by the result of [Tak90]. ∎

If T is not quasiunitary, then $\operatorname{Hlat} T = \operatorname{Lat} T$ is a rich lattice containing $\operatorname{Lat_s} T$ and $(\ker(T^* - \overline{\lambda} I))^\perp$ $(\lambda \in \mathbb{D})$ because $S^* \prec T^*$ (see Propositions 13, 19 and Theorem 12). Hence (HSP) in $\mathcal{L}_1(\mathcal{H})$ can be reduced to the quasiunitary case. Propositions 14, 19 and Theorem 12 yield:

Proposition 20. *If $T \in \mathcal{L}_1(\mathcal{H})$ is quasiunitary, then there exist $\mathcal{M}_1, \mathcal{M}_2 \in \operatorname{Lat_s} T$ such that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$.*

5. Functional commutant. Let $T \in \mathcal{L}(\mathcal{H})$ be an asymptotically cyclic a.c. contraction, and assume that $T \in C_{1\cdot}$ and $\omega(T) = \mathbb{T}$. Let (X, V) be a unitary asymptote of T, and consider the contractive algebra-homomorphism $\gamma\colon \{T\}' \to \{V\}'$, $C \mapsto D$, where $XC = DX$, which is injective because $T \in C_{1\cdot}$. The functional calculus $\Phi\colon L^\infty(\mathbb{T}) \to \{V\}'$, $f \mapsto f(V)$ is an isomorphism between the corresponding Banach algebras. The composition

$$\widehat{\gamma}_T = \Phi^{-1} \circ \gamma\colon \{T\}' \to L^\infty(\mathbb{T})$$

is also an injective, contractive, unital algebra-homomorphism. It can be easily checked that $\widehat{\gamma}_T$ is independent of the special choice of (X, V). Indeed, for $j = 1, 2$ let (X_j, V_j) be a unitary asymptote of T, and let γ_j, Φ_j be defined as before. There exist unitary transformations $Y_1 \in \mathcal{I}(V_1, V_2)$ and $Y_2 \in \mathcal{I}(V_2, V_1)$ such that $X_2 = Y_1 X_1$, $X_1 = Y_2 X_2$ and $Y_2 = {Y_1}^{-1}$. Given any $C \in \{T\}'$ we have $X_j C = D_j X_j = f_j(V_j) X_j$. Hence

$$f_2(V_1)X_1 = Y_2 f_2(V_2) Y_1 X_1 = Y_2 f_2(V_2) X_2 = Y_2 X_2 C = X_1 C = f_1(V_1) X_1,$$

and so $f_2(V_1) = f_1(V_1)$, whence $f_2 = f_1$.

The uniquely determined $\widehat{\gamma}_T$ is called the *functional mapping* of T, and its range $\mathcal{F}(T)$ is called the *functional commutant* of T. Since $\widehat{\gamma}_T(f(T)) = f$ for every $f \in H^\infty$, we see that $\mathcal{F}(T)$ is a subalgebra of $L^\infty(\mathbb{T})$ containing H^∞. It is natural to ask the following questions. Which function algebras $H^\infty \subset \mathcal{A} \subset L^\infty(\mathbb{T})$ are attainable as a functional commutant: $\mathcal{A} = \mathcal{F}(T)$, and what kind of information on the behaviour of T can be derived from the properties of $\widehat{\gamma}_T$ and $\mathcal{F}(T)$? We recall that the function algebra $\mathcal{A}$ is called *quasianalytic* if $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$ whenever f is a non-zero element of $\mathcal{A}$. The following statement was proved in [Kér11, Proposition 4.2].

Proposition 21. *If $T \in \mathcal{L}_1(\mathcal{H})$, then $\mathcal{F}(T)$ is quasianalytic.*

It is clear that $\mathcal{F}(T) = H^\infty$ exactly when $\{T\}' = H^\infty(T)$, and this happens in particular if $T \prec S$. (For a more complete characterization of this case see [Kér11, Theorem 5.2].)

If $T \in \mathcal{L}_1(\mathcal{H})$ and $\mathcal{F}(T) \neq H^\infty$, then the closure $\mathcal{F}(T)^-$ contains $H^\infty + C(\mathbb{T})$ (see [Gar07, Theorems IX.1.4 and IX.2.2]); thus $\mathcal{F}(T)^-$ is not quasianalytic, and so $\mathcal{F}(T)$ is not closed, or equivalently, $\widehat{\gamma}_T$ is not bounded from below.

We recall that $\eta \in H^\infty$ is an *inner function* if $|\eta(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$. Let H^∞_{i} stand for the multiplicative semigroup of all inner functions. Given a subsemigroup $\mathcal{B}$ of H^∞_{i}, the algebra $\overline{\mathcal{B}} \cdot H^\infty$ generated by $\overline{\mathcal{B}}$ (the set of conjugates of functions in $\mathcal{B}$) and H^∞ is clearly quasianalytic. The closure $(\overline{\mathcal{B}} \cdot H^\infty)^-$ is called the *Douglas algebra* induced by $\mathcal{B}$. By the celebrated Chang–Marshall theorem every closed subalgebra $\mathcal{A}$ of $L^\infty(\mathbb{T})$ containing H^∞ is a Douglas algebra (see [Gar07, Theorem IX.3.1]). Therefore, $\mathcal{F}(T)^- = (\overline{\mathcal{B}} \cdot H^\infty)^-$ with $\mathcal{B} = \{\eta \in \mathcal{F}(T)^- \cap H^\infty_{\mathrm{i}} : \overline{\eta} \in \mathcal{F}(T)^-\}$. We note that $\mathcal{B}$ can be replaced by a semigroup generated by interpolating Blaschke products (see [Gar07, Theorems IX.3.2 and IX.3.4]). The question which pre-Douglas algebras $\overline{\mathcal{B}} \cdot H^\infty$ arise as functional commutants was posed in [Kér11]. The next theorem settles this problem.

THEOREM 22. *The only attainable pre-Douglas algebra is H^∞.*

Proof. Fix $T \in \mathcal{L}_1(\mathcal{H})$, and assume that $\mathcal{F}(T) \neq H^\infty$. If the spectrum $\sigma(T)$ covers the closed unit disc $\mathbb{D}^-$, then $\overline{\eta} \notin \mathcal{F}(T)$ for every non-constant $\eta \in H^\infty_{\mathrm{i}}$ (see [Kér11, Proposition 4.4]), hence $\mathcal{F}(T)$ cannot be a pre-Douglas algebra.

Assume now that $\sigma(T) \neq \mathbb{D}^-$. Select a point $a \in \mathbb{D} \setminus \sigma(T)$, and consider the operator $A = (T - aI)^{-1} \in \{T\}'$ and the function $g = \widehat{\gamma}_T(A) = (\chi - a)^{-1} \in \mathcal{F}(T)$. Since $\widetilde{A} := \exp A = \sum_{n=0}^\infty (n!)^{-1} A^n \in \{T\}'$, it follows that

$$\widetilde{g} := \widehat{\gamma}_T(\widetilde{A}) = \sum_{n=0}^\infty (n!)^{-1} g^n = \exp g$$

belongs to $\mathcal{F}(T)$. The function $\widetilde{g}$, defined on $\mathbb{T}$, has an analytic extension $G(z) = \exp(1/(z-a))$ defined for $z \in \mathbb{C} \setminus \{a\}$. It is clear that a is an essential isolated singularity of G. Assume that we can find functions $h \in H^\infty$ and $k \in H^\infty_{\mathrm{i}}$ so that $\widetilde{g} = h\overline{k} = h/k$. Let H and K be analytic extensions of h and k, respectively, onto $\mathbb{D}$. Then H/K is a meromorphic function on $\mathbb{D}$, and

$$\operatorname*{nt-lim}_{z \to \zeta} \frac{H(z)}{K(z)} = \frac{h(\zeta)}{k(\zeta)} = \widetilde{g}(\zeta) = G(\zeta)$$

for every $\zeta \in \omega_0$, where $m(\mathbb{T} \setminus \omega_0) = 0$. Let Ω be a domain in $\mathbb{D}$ bounded by an open arc α_1 on $\mathbb{T}$ and a closed segment α_2, such that $a \notin \Omega^-$. Let ψ be a conformal mapping of $\mathbb{D}$ onto Ω, and consider the bounded analytic function

$$F = (H \circ \psi) - (G \circ \psi)(K \circ \psi)$$

on $\mathbb{D}$. By Carathéodory's theorem, ψ can be extended to a homeomorphism of $\mathbb{D}^-$ onto Ω^-. Since the Jordan curve $\partial\Omega$ is rectifiable, the set $\omega_1 = \psi^{-1}(\omega_0 \cap \alpha_1) \subset \mathbb{T}$ is of positive measure. For every $\zeta \in \omega_1$, there is an $r_\zeta \in (0,1)$ such that $I_\zeta = \{r\psi(\zeta) : r_\zeta \le r < 1\} \subset \Omega$. Then the arc $C_\zeta = \psi^{-1}(I_\zeta) \subset \mathbb{D}$ terminates at ζ, and $F(z)$ converges to 0 when z tends to ζ along C_ζ. We conclude that $\text{nt-lim}_{z\to\zeta} F(z) = 0$ by Lindelöf's theorem (see [CL66, Theorem 2.3]). Hence the theorem of F. and M. Riesz implies that F is identically zero (see [CL66, Theorem 2.5]). Therefore, $G = H/K$, which is impossible since H/K is meromorphic on $\mathbb{D}$ and $a \in \mathbb{D}$ is an essential singularity. ∎

A special case of the following property of the functional commutant has been exploited in the previous proof.

PROPOSITION 23. *If $f \in \mathcal{F}(T)$, $r > \|\widehat{\gamma}_T^{-1}(f)\|$ and φ is analytic on $r\mathbb{D}$, then $\varphi \circ f \in \mathcal{F}(T)$.*

Proof. Consider the Taylor expansion $\varphi(z) = \sum_{n=0}^\infty c_n z^n$ $(z \in r\mathbb{D})$, where $\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|} \le r$. Setting $C = \widehat{\gamma}_T^{-1}(f) \in \{T\}'$, we know that $\widetilde{C} = \varphi(C) = \sum_{n=0}^\infty c_n C^n \in \{T\}'$ (convergence in norm), and so

$$\varphi(f) = \sum_{n=0}^\infty c_n f^n = \widehat{\gamma}_T(\widetilde{C}) \in \mathcal{F}(T). \ ∎$$

We recall that $H^\infty \subset \mathcal{A} \subset L^\infty$ is a *generalized Douglas algebra* if for every $f \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, $|\lambda| > \|f\|_\infty$ implies $(f-\lambda)^{-1} \in \mathcal{A}$. These algebras were introduced and studied in [Tol92], where Gelfand's theory of maximal ideals and the theory of Douglas algebras were carried over to such algebras. We know that $\mathcal{F}(T)$ is a generalized Douglas algebra if and only if $\widehat{\gamma}_T$ preserves the spectral radius, and in that case $\sigma(T) = \mathbb{T}$ (see [Kér11, Theorems 5.5 and 5.6]). It remains open which (if any) generalized Douglas algebras other than H^∞ arise as a functional commutant $\mathcal{F}(T)$ of a contraction $T \in \mathcal{L}_1(\mathcal{H})$. Are there any quasianalytic generalized Douglas algebras other than H^∞ at all?

We provide an example of an operator $T \in \mathcal{L}_1(\mathcal{H})$ such that $\mathcal{F}(T) \ne H^\infty$ and $\mathcal{F}(T) \cap \overline{H_{\mathrm{i}}^\infty} = \mathbb{C}\mathbb{1}$. (Here $\mathbb{1}$ denotes the constant 1 function.)

EXAMPLE 24. We consider an extended version of [Kér11, Example 5.8]. Given $0 \le \delta < 1$, set $G_\delta = \{re^{it} : \sqrt{\delta} < r < 1,\ 0 < t < \pi\}$. Let η_δ denote the conformal mapping of $\mathbb{D}$ onto G_δ satisfying $\eta_\delta(\zeta) = \zeta$ for $\zeta = 1, i, -1$. Forming the regular partially inner function $\vartheta_\delta = \eta_\delta^2$, consider the analytic Toeplitz operator $T_\delta \in \mathcal{L}(H^2)$ defined by $T_\delta f = \vartheta_\delta f$. It can be easily verified (see [Kér11]) that $T_\delta \in \mathcal{L}_1(H^2)$, $\sigma(T_\delta) = \{z \in \mathbb{C} : \delta \le |z| \le 1\}$ and

$$\mathcal{F}(T_\delta) = \{g \in L^\infty(\mathbb{T}) : g \circ \vartheta_{\delta,+} = h|\mathbb{T}_+ \text{ for some } h \in H^\infty\},$$

where $\mathbb{T}_+ = \{z \in \mathbb{C} : |z| = 1, \operatorname{Im} z > 0\}$ and $\vartheta_{\delta,+} = \vartheta_\delta|\mathbb{T}_+$.

Let $g_\delta \in L^\infty(\mathbb{T})$ be the inverse of $\vartheta_{\delta,+}$, that is, $g_\delta(\vartheta_{\delta,+}(\zeta)) = \zeta$ for every $\zeta \in \mathbb{T}_+$. Since $g_\delta(\mathbb{T} \setminus \{1\}) = \mathbb{T}_+$, it follows that $g_\delta \notin H^\infty$. (Indeed, assuming $g_\delta \in H^\infty$ choose a fractional linear function ψ transforming $\mathbb{T}_+$ onto $(0,1)$. Then $\mathrm{Im}(\psi \circ g_\delta) = 0$ a.e. on $\mathbb{T}$; taking the Poisson transform we infer that $\mathrm{Im}(\psi \circ g_\delta)$ is zero on $\mathbb{D}$. The Cauchy–Riemann equations show that $\psi \circ g_\delta$ is constant. Thus g_δ is constant, which is impossible because $g_\delta(\mathbb{T}\setminus\{1\}) = \mathbb{T}_+$.) On the other hand, the equality $g_\delta \circ \vartheta_{\delta,+} = \chi|\mathbb{T}_+$ implies that $g_\delta \in \mathcal{F}(T_\delta)$. Therefore $\mathcal{F}(T_\delta) \neq H^\infty$ for every $\delta \in [0,1)$. In particular, if $\delta = 0$ then $\sigma(T_0) = \mathbb{D}^-$, and so $\mathcal{F}(T_0) \cap \overline{H_{\mathrm{i}}^\infty} = \mathbb{C}\mathbb{1}$.

Though $\mathcal{F}(T_\delta) \neq H^\infty$, there is a connection between these algebras. Namely, for any $\delta \in [0,1)$, the commutation relation $T_\delta S = S T_\delta$ yields $\{T_\delta\}' = \{S\}'$ (see Proposition 11). It is known that $\{S\}' = H^\infty(S)$ and $\widehat{\gamma}_S$ is an isometry. Thus $\widehat{\gamma}_T \circ \widehat{\gamma}_S^{-1}$ is a contractive algebra-isomorphism from H^∞ onto $\mathcal{F}(T)$.

We show that the functional commutant is a similarity invariant. Actually, the following theorem contains a more general statement.

Theorem 25. *For $j = 1,2$, let $T_j \in \mathcal{L}_1(\mathcal{H}_j)$ have unitary asymptote (X_j, V_j). Assume that there exist $Y \in \mathcal{I}(T_1,T_2)$ and $Z \in \mathcal{I}(T_2,T_1)$ such that $ZY \neq 0$. Then:*

- (a) *Y and Z are injective,*
- (b) *$0 \neq \widehat{\gamma}_{T_1}(ZY) = \widehat{\gamma}_{T_2}(YZ) =: g$ belongs to $\mathcal{F}(T_1) \cap \mathcal{F}(T_2)$ and $g\mathcal{F}(T_1) \subset \mathcal{F}(T_2)$, $g\mathcal{F}(T_2) \subset \mathcal{F}(T_1)$,*
- (c) *in particular, if $ZY = I$, that is, $T_1 \approx T_2$, then $g = \mathbb{1}$ and $\mathcal{F}(T_1) = \mathcal{F}(T_2)$.*

Proof. By the universality property of unitary asymptotes there exist $A \in \mathcal{I}(V_1,V_2)$ and $B \in \mathcal{I}(V_2,V_1)$ such that $AX_1 = X_2Y$ and $BX_2 = X_1Z$. Since $X_1(ZY) = BX_2Y = (BA)X_1$, $ZY \in \{T_1\}'$ and $BA \in \{V_1\}'$, we infer that $\widehat{\gamma}_{T_1}(ZY) = g$ where $g(V_1) = BA$. Similarly, $AB = h(V_2)$ with $h = \widehat{\gamma}_{T_2}(YZ)$. The assumption $ZY \neq 0$ yields $0 \neq g \in \mathcal{F}(T_1)$. Since the function algebra $\mathcal{F}(T_1)$ is quasianalytic, we see that $g(V_1)$ is a quasiaffinity, and so B has dense range. Now the equalities $g(V_1)B = BAB = Bh(V_2) = h(V_1)B$ imply that $g(V_1) = h(V_1)$, whence $g = h$. Thus $YZ \in \{T_2\}'$ is also non-zero, and we conclude by Proposition 10(ii) that ZY and YZ are injective, hence so are Y and Z.

Given $f_1 \in \mathcal{F}(T_1)$, consider $C_1 = (\widehat{\gamma}_{T_1})^{-1}(f_1) \in \{T_1\}'$, $C_2 = YC_1Z \in \{T_2\}'$ and $\widehat{\gamma}_{T_2}(C_2) = f_2$. Then the equalities

$$\begin{aligned} f_2(V_2)X_2 &= X_2C_2 = X_2YC_1Z = AX_1C_1Z = Af_1(V_1)X_1Z \\ &= Af_1(V_1)BX_2 = f_1(V_2)ABX_2 = f_1(V_2)g(V_2)X_2 \end{aligned}$$

yield $f_2 = f_1 g$. Therefore $g\mathcal{F}(T_1) \subset \mathcal{F}(T_2)$, and in a similar way we find that $g\mathcal{F}(T_2) \subset \mathcal{F}(T_1)$.

Since Z is injective, the equality $ZY = I$ is equivalent to the invertibility of Z with $Y = Z^{-1}$. In that case $g = \widehat{\gamma}_{T_1}(ZY) = \mathbb{1}$, and so $\mathcal{F}(T_1) = \mathcal{F}(T_2)$. ∎

We conclude this section by providing a representation of the functional mapping in the functional model.

Let $\mathcal{E}, \mathcal{E}_*$ be Hilbert spaces, and let $\Theta\colon \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ be a purely contractive, analytic, inner and $*$-outer function. Then $\mathcal{H}(\Theta) = H^2(\mathcal{E}_*) \ominus \Theta H^2(\mathcal{E})$ is the corresponding model space, and the model operator $S(\Theta) \in \mathcal{L}(\mathcal{H}(\Theta))$ is defined by $S(\Theta)u = P_{\mathcal{H}(\Theta)}(\chi u)$, where $P_{\mathcal{H}(\Theta)} \in \mathcal{L}(H^2(\mathcal{E}_*))$ denotes the orthogonal projection onto $\mathcal{H}(\Theta)$. We know that $S(\Theta) \in C_{10}$, and every contraction of class C_{10} is unitarily equivalent to a model operator of this kind. Let us consider the measurable projection-valued function $\Delta_*(\zeta) = I - \Theta(\zeta)\Theta(\zeta)^*$ defined for a.e. $\zeta \in \mathbb{T}$, the subspace $\mathcal{R}_* = \Delta_* L^2(\mathcal{E}_*)$ in $L^2(\mathcal{E}_*)$, and the a.c. unitary operator $R_* \in \mathcal{L}(\mathcal{R}_*)$ defined by $R_* v = \chi v$. The pair (X_*, R_*) is a unitary asymptote of $S(\Theta)$, where $X_* \in \mathcal{L}(\mathcal{H}(\Theta), \mathcal{R}_*)$ is defined by $X_* u = \Delta_* u$. (For the characteristic properties of Δ_* see [Kér13].)

The spectral-multiplicity function of R_* is $\delta_*(\zeta) = \operatorname{rank} \Delta_*(\zeta)$. Hence $S(\Theta)$ is asymptotically cyclic exactly when $\delta_* \le 1$. The asymptotic density function of $S(\Theta)$ at $u, v \in \mathcal{H}(\Theta)$ is $w_{u,v}(\zeta) = \langle \Delta_*(\zeta)u(\zeta), v(\zeta)\rangle$. Thus the local residual set of $S(\Theta)$ at u is $\omega(S(\Theta), u) = \{\zeta \in \mathbb{T} : \Delta_*(\zeta)u(\zeta) \ne 0\}$, while the global residual set is $\omega(S(\Theta)) = \{\zeta \in \mathbb{T} : \Delta_*(\zeta) \ne 0\}$. In view of Proposition 1 and Theorem 3 we obtain the following characterization.

Proposition 26. *We have $S(\Theta) \in \mathcal{L}_1(\mathcal{H}(\Theta))$ if and only if*

(i) $\delta_*(\zeta) = 1$ *for a.e.* $\zeta \in \mathbb{T}$, *and*
(ii) $\Delta_*(\zeta)u(\zeta) \ne 0$ *for a.e.* $\zeta \in \mathbb{T}$ *whenever* $0 \ne u \in \mathcal{H}(\Theta)$.

By the Lifting Theorem, $C \in \{S(\Theta)\}'$ if and only if there exists a bounded analytic function $\Psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E}_*)$ such that $\Psi\Theta H^2(\mathcal{E}) \subset \Theta H^2(\mathcal{E})$ and $Cu = P_{\mathcal{H}(\Theta)}\Psi u$ for every $u \in \mathcal{H}(\Theta)$. We note that Ψ can be chosen so that $\|\Psi\|_\infty = \|C\|$. Furthermore, the condition $\Psi\Theta H^2(\mathcal{E}) \subset \Theta H^2(\mathcal{E})$ is equivalent to the existence of a bounded analytic function $\Psi_0\colon \mathbb{D} \to \mathcal{L}(\mathcal{E})$ such that $\Psi\Theta = \Theta\Psi_0$. Let $H^\infty(\Theta)$ stand for the set of all bounded analytic functions $\Psi\colon \mathbb{D} \to \mathcal{L}(\mathcal{E}_*)$ satisfying $\Psi\Theta H^2(\mathcal{E}) \subset \Theta H^2(\mathcal{E})$, and for any $\Psi \in H^\infty(\Theta)$ let C_Ψ be the operator in $\{S(\Theta)\}'$ defined by $C_\Psi u = P_{\mathcal{H}(\Theta)}\Psi u$ ($u \in \mathcal{H}(\Theta)$). Moreover, let $\Gamma_\Theta(\Psi)$ denote the function $\psi \in L^\infty(\mathbb{T})$ defined by $\Delta_*\Psi\Delta_* = \psi\Delta_*$. (Notice that $\dim \Delta_*(\zeta)\mathcal{E}_* = 1$ for a.e. $\zeta \in \mathbb{T}$.)

Theorem 27. *If $S(\Theta) \in \mathcal{L}_1(\mathcal{H}(\Theta))$, then for every $\Psi \in H^\infty(\Theta)$ we have:*

(i) $\Delta_*\Psi(I - \Delta_*) = 0$, *and*
(ii) $\widehat{\gamma}_{S(\Theta)}(C_\Psi) = \Gamma_\Theta(\Psi)$.

Proof. Fix $\Psi \in H^\infty(\Theta)$, and let $\psi = \Gamma_\Theta(\Psi)$. For any $u \in H^2(\mathcal{E})$ and $n \in \mathbb{N}$, we have

$$\Delta_* \Psi \Theta(\chi^{-n} u) = \chi^{-n} \Delta_* \Psi \Theta u = 0,$$

since $\Psi\Theta u \in \Theta H^2(\mathcal{E})$ and $\Delta_*\Theta = 0$. Thus $\Delta_*\Psi\Theta v = 0$ for every $v \in L^2(\mathcal{E})$, and so $\Delta_*\Psi\Theta = 0$. Since $\Theta(\zeta)$ is an isometry from $\mathcal{E}$ onto $\mathcal{E}_* \ominus \Delta_*(\zeta)\mathcal{E}_*$ for a.e. $\zeta \in \mathbb{T}$, it follows that $\Delta_*\Psi(I - \Delta_*) = 0$.

For every $u \in \mathcal{H}(\Theta)$, we have

$$\begin{aligned} X_* C_\Psi u &= \Delta_* P_{\mathcal{H}(\Theta)} \Psi u = \Delta_* \Psi u = \Delta_* \Psi(\Delta_* u + (I - \Delta_*)u) \\ &= \Delta_* \Psi \Delta_* u = \psi \Delta_* u = \psi(R_*) X_* u. \end{aligned}$$

Therefore, $\widehat{\gamma}_{S(\Theta)}(C_\Psi) = \psi$. ∎

6. Analytic contractions. In [ARS07] the multiplication operator on a general Hilbert space of analytic functions has been studied. Namely, let $\mathcal{H}_a$ be a Hilbert space of analytic functions defined on $\mathbb{D}$, with the usual vector space operations, satisfying the following conditions:

(i) for every $f \in \mathcal{H}_a$, we have $\chi f \in \mathcal{H}_a$ and $\|\chi f\| \le \|f\|$ $(\chi(z) = z)$;
(ii) for every $\lambda \in \mathbb{D}$, the evaluation $K_\lambda \colon \mathcal{H}_a \to \mathbb{C}$, $f \mapsto f(\lambda)$, is a bounded linear functional, and so there is a unique reproducing kernel $k_\lambda \in \mathcal{H}_a$ with the property $f(\lambda) = \langle f, k_\lambda \rangle$ $(f \in \mathcal{H}_a)$;
(iii) $\mathbb{1} \in \mathcal{H}_a$.

The operator $M_a \in \mathcal{L}(\mathcal{H}_a)$, $M_a f = \chi f$, is called an *analytic multiplication operator*. Since $M_a^* k_\lambda = \overline{\lambda} k_\lambda$ $(\lambda \in \mathbb{D})$ and $\bigvee\{k_\lambda : \lambda \in \mathbb{D}\} = \mathcal{H}_a$, it follows that M_a is a $C_{\cdot 0}$-contraction. Condition (iii), which yields $H^\infty \subset \mathcal{H}_a$, is not always assumed in [ARS07]; we suppose it here for simplicity. The boundary behaviour of functions in $\mathcal{H}_a$ is governed by the set

$$\Delta(\mathcal{H}_a) = \Big\{\zeta \in \mathbb{T} : \text{nt-}\varliminf_{\lambda \to \zeta} (1 - |\lambda|^2)^{-1} \|k_\lambda\|^{-2} > 0\Big\}.$$

Namely, it has been shown in [ARS07] that

(a) for every $f \in \mathcal{H}_a$, $\text{nt-}\lim_{z\to\zeta} f(z)$ exists for a.e. $\zeta \in \Delta(\mathcal{H}_a)$;
(b) there exists $f \in \mathcal{H}_a$ such that $\text{nt-}\lim_{z\to\zeta} f(z)$ does not exist for a.e. $\zeta \in \mathbb{T} \setminus \Delta(\mathcal{H}_a)$.

The measurable set $\Delta(\mathcal{H}_a)$ can be related to the quasianalytic spectral set of M_a.

Proposition 28. *We have $\Delta(\mathcal{H}_a) \subset \pi(M_a)$. Therefore M_a is quasianalytic whenever $\Delta(\mathcal{H}_a) = \omega(M_a) \neq \emptyset$.*

Proof. Pick a non-zero $f \in \mathcal{H}_a$. The inequality

$$\frac{|f(\lambda)|^2}{(1 - |\lambda|^2)\|k_\lambda\|^2} \le (1 - |\lambda|^2)\|(I - \overline{\lambda} M_a)^{-1} f\| \quad (\lambda \in \mathbb{D})$$

implies by Proposition 6 that

$$\text{nt-}\overline{\lim_{\lambda\to\zeta}} \frac{|f(\lambda)|^2}{(1-|\lambda|^2)\|k_\lambda\|^2} \leq w_{f,f}(\zeta) \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

By Proposition 3.3 of [ARS07] we know that

$$\text{nt-}\underline{\lim_{\lambda\to\zeta}} \frac{|f(\lambda)|^2}{(1-|\lambda|^2)\|k_\lambda\|^2} > 0 \quad \text{for a.e. } \zeta \in \Delta(\mathcal{H}_a).$$

Thus $\omega(M_a, f) \supset \Delta(\mathcal{H}_a)$, and so $\Delta(\mathcal{H}_a) \subset \pi(M_a)$ (see Proposition 1 and Theorem 3). ∎

Conditions ensuring $\Delta(\mathcal{H}_a) = \omega(M_a)$ are given in [ARS07].

It is easy to verify that the mapping $\lambda \mapsto k_\lambda$ is coanalytic, which means that the function $\varphi(\lambda) = \langle f, k_\lambda\rangle$ $(\lambda \in \mathbb{D})$ is analytic for every $f \in \mathcal{H}_a$. Hence M_a is an analytic operator in the sense of [CEP89]. We say that $T \in \mathcal{L}(\mathcal{H})$ is an *analytic contraction* if $\|T\| \leq 1$ and there exists a coanalytic function $\eta\colon \mathbb{D} \to \mathcal{H}$ satisfying:

(i) $T^*\eta(\lambda) = \overline{\lambda}\eta(\lambda)$ for every $\lambda \in \mathbb{D}$,
(ii) $\bigvee\{\eta(\lambda) : \lambda \in \mathbb{D}\} = \mathcal{H}$.

(We note that such contractions are called fully analytic in [CEP89].) The function η has an expansion $\eta(\lambda) = \sum_{n=0}^{\infty} \overline{\lambda}^n y_n$ $(\lambda \in \mathbb{D})$, where

$$\overline{\lim_{n\to\infty}} \|y_n\|^{1/n} \leq 1,\ \bigvee\{y_n\}_{n=0}^{\infty} = \mathcal{H},\ T^*y_n = y_{n-1} \text{ for } n \in \mathbb{N},\ T^*y_0 = 0.$$

We say that T is a *purely analytic contraction* if η can be chosen so that $y_0 \notin \bigvee\{y_n\}_{n=1}^{\infty}$. It can be easily verified that these are exactly those contractions which are unitarily equivalent to an analytic multiplication operator. We also note that T^* belongs to the Cowen–Douglas class $B_1(\mathbb{D})$ introduced in [CD78] if and only if T is an analytic contraction with approximate point spectrum $\sigma_{\mathrm{ap}}(T) = \mathbb{T}$ and with Fredholm index $\operatorname{ind} T = -1$. Surprisingly, rather general spectral conditions ensure the existence of *purely analytic invariant subspaces*, restriction to which is a purely analytic contraction. Namely, let $T \in \mathcal{L}(\mathcal{H})$ be an a.c. contraction with an isometric functional calculus Φ_T, and assume that the *extended right spectrum*

$$\widetilde{\sigma}_r(T) = \mathbb{D} \setminus \{\lambda \in \mathbb{D} : (T-\lambda I)\mathcal{H} = \mathcal{H} \text{ and } 0 < \dim\ker(T-\lambda I) < \infty\}$$

is *dominating* in $\mathbb{D}$, that is, a.e. $\zeta \in \mathbb{T}$ is a non-tangential cluster point of $\widetilde{\sigma}_r(T)$. Then there is a dense set $\mathcal{H}_0$ in $\mathcal{H}$ such that $\bigvee\{T^n x\}_{n=0}^{\infty}$ is a purely analytic invariant subspace of T for every $x \in \mathcal{H}_0$ (see [CEP89]).

It is not obvious how to identify the unitary asymptote of a general analytic multiplication operator M_a. This identification can be carried out in the special case when $\mathcal{H}_a$ is induced by a measure satisfying particular conditions considered in [ARS09]. Let μ be a finite positive Borel measure supported on $\mathbb{D}^-$, with $\mu(\mathbb{T}) > 0$. Let $\mathcal{P}$ stand for the algebra of complex polynomials, and $\mathcal{P}^2(\mu)$ for the closure of $\mathcal{P}$ in $L^2(\mu)$. We consider the cyclic subnormal operator $S_\mu \in \mathcal{L}(\mathcal{P}^2(\mu))$ defined by $S_\mu f = \chi f$. The following assumptions are made:

(i) $\mathcal{P}^2(\mu)$ is irreducible, i.e. contains no non-trivial characteristic function;

(ii) for every $\lambda \in \mathbb{D}$, the evaluation $K_\lambda \colon \mathcal{P} \to \mathbb{C}$, $p \mapsto p(\lambda)$, is a bounded linear functional; its continuous extension to $\mathcal{P}^2(\mu)$ is represented by $k_\lambda \in \mathcal{P}^2(\mu)$, i.e. $p(\lambda) = \langle p, k_\lambda \rangle$ $(p \in \mathcal{P})$.

By the results of [ARS09] (see also [Con91, Chapter VIII]), we know that for every $f \in \mathcal{P}^2(\mu)$, the function $\widetilde{f}(\lambda) = \langle f, k_\lambda \rangle$ is analytic on $\mathbb{D}$, $\widetilde{f}(\lambda) = f(\lambda)$ for μ-a.e. $\lambda \in \mathbb{D}$, and $\text{nt-lim}_{\lambda\to\zeta} \widetilde{f}(\lambda) = f(\zeta)$ for μ_0-a.e. $\zeta \in \mathbb{T}$. Here μ_0 denotes the restriction of μ to the Borel subsets of $\mathbb{T}$, which is a.c. with respect to m. Therefore, S_μ can be considered as an analytic multiplication operator. Furthermore, for $h = d\mu_0/dm$ we have

$$h(\zeta) = \text{nt-}\overline{\lim_{\lambda\to\zeta}}(1 - |\lambda|^2)^{-1}\|k_\lambda\|^{-2} \quad \text{for a.e. } \zeta \in \mathbb{T},$$

and $\varDelta(\mathcal{P}^2(\mu)) = \{\zeta \in \mathbb{T} : h(\zeta) > 0\}$. It is clear that (X, V) is a unitary asymptote of S_μ, where $V \in \mathcal{L}(L^2(\mu_0))$, $Vf = \chi f$ and $X \in \mathcal{L}(\mathcal{P}^2(\mu), L^2(\mu_0))$ is defined by $Xf = f|\mathbb{T}$. Thus S_μ is asymptotically cyclic. Since $\omega(S_\mu) = \varDelta(\mathcal{P}^2(\mu))$, it follows by Proposition 28 that S_μ is quasianalytic.

PROPOSITION 29. *If* $h(\zeta) > 0$ *for a.e.* $\zeta \in \mathbb{T}$, *then* $S_\mu \in \mathcal{L}_1(\mathcal{P}^2(\mu))$ *and* $\mathcal{F}(S_\mu) = X(\mathcal{P}^2(\mu) \cap L^\infty(\mu))$.

Proof. The last equality follows from Yoshino's theorem (see [Con91, Theorem II.5.4]). ∎

7. Bilateral weighted shifts. Weighted shifts always serve as a source of examples. Here we consider those bilateral weighted shifts which are C_{10}-contractions. As earlier, $\widetilde{S} \in \mathcal{L}(L^2(\mathbb{T}))$, $\widetilde{S}f = \chi f$, is the simple bilateral shift. The Fourier transformation $F \colon L^2(\mathbb{T}) \to l^2(\mathbb{Z})$, $f \mapsto \widehat{f}$, where $\widehat{f}(n) = \langle f, \chi^n \rangle$ $(n \in \mathbb{Z})$, is a Hilbert space isomorphism. Assume that $\beta \colon \mathbb{Z} \to (0, \infty)$ satisfies:

(i) $\beta(n) \geq \beta(n+1)$ for every $n \in \mathbb{Z}$,

(ii) $\lim_{n\to\infty} \beta(-n) = \infty$,

(iii)$_0$ $\lim_{n\to\infty} \beta(n) > 0$.

It is clear that

$$l^2(\beta) = \Big\{\xi \colon \mathbb{Z} \to \mathbb{C} : \|\xi\|_\beta^2 := \sum_{n=-\infty}^{\infty} |\xi(n)|^2 \beta(n)^2 < \infty\Big\}$$

is a dense linear manifold in $l^2(\mathbb{Z})$, which is a Hilbert space with the norm $\|\xi\|_\beta$. Hence

$$L^2(\beta) = \{f \in L^2(\mathbb{T}) : \widehat{f} \in l^2(\beta)\}$$

is a dense linear manifold in $L^2(\mathbb{T})$, which forms a Hilbert space with the norm $\|f\|_\beta := \|\widehat{f}\|_\beta$. It can be easily verified that $T_\beta \in \mathcal{L}(L^2(\beta))$ defined by $T_\beta f = \chi f$ is a C_{10}-contraction (see [NFBK, Section IX.2]). Furthermore, in this way we obtain all bilateral weighted shifts which are C_{10}-contractions. Since T_β is unitarily equivalent to $T_{c\beta}$ $(c > 0)$, we may assume that $\lim_{n\to\infty} \beta(n) = 1$. Moreover, in that case T_β is similar to $T_{\widetilde{\beta}}$, where $\widetilde{\beta}(-n) = \beta(-n)$ for $n > 0$ and $\widetilde{\beta}(n) = 1$ for $n \geq 0$. Therefore, without restricting generality, condition (iii)$_0$ can be replaced by

(iii) $\beta(n) = 1$ for every $n \in \mathbb{Z}_+$.

Obviously, $(X_\beta, \widetilde{S})$ is a unitary asymptote of T_β, where $X_\beta \colon L^2(\beta) \to L^2(\mathbb{T})$, $f \mapsto f$, is a quasiaffinity. Thus T_β is asymptotically cyclic and quasiunitary, with $\omega(T_\beta) = \mathbb{T}$.

The special form of X_β implies that, for any $\phi \in \mathcal{F}(T_\beta)$, the operator $M_{\phi,\beta} := (\widehat{\gamma}_{T_\beta})^{-1}(\phi) \in \{T_\beta\}'$ acts as multiplication: $M_{\phi,\beta} f = \phi f$ $(f \in L^2(\beta))$. Clearly, $\mathcal{F}(T_\beta) \subset L^\infty(\mathbb{T}) \cap L^2(\beta)$. The following characterization follows from the Closed Graph Theorem.

PROPOSITION 30. *The functional commutant $\mathcal{F}(T_\beta)$ consists of all measurable functions $\phi\colon \mathbb{T} \to \mathbb{C}$ satisfying $\phi L^2(\beta) \subset L^2(\beta)$.*

The previous discussion shows that T_β belongs to the class $\mathcal{L}_1(L^2(\beta))$ exactly when the function space $L^2(\beta)$ is quasianalytic, that is, when $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$ whenever f is a non-zero element of $L^2(\beta)$. This happens if $\beta(-n)$ increases sufficiently fast as $n \to \infty$.

PROPOSITION 31. *The function space $L^2(\beta)$ is quasianalytic if and only if*

$$\sum_{n=1}^{\infty} n^{-2} \log \beta(-n) = \infty.$$

Proof. Suppose that $\sum_{n=1}^{\infty} n^{-2} \log \beta(-n) = \infty$, and consider a non-zero function $f \in L^2(\beta)$. For any $n \in \mathbb{N}$, we have

$$F_n := \Big[\sum_{k=n}^{\infty} |\widehat{f}(-k)|^2\Big]^{1/2} \leq \frac{1}{\beta(-n)} \Big[\sum_{k=n}^{\infty} |\widehat{f}(-k)|^2 \beta(-k)^2\Big]^{1/2} \leq \frac{\|f\|_\beta}{\beta(-n)},$$

whence

$$\sum_{n=1}^{\infty} \frac{\log F_n}{n^2} \leq (\log \|f\|_\beta) \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{\log \beta(-n)}{n^2} = -\infty.$$

By [Beu77, Corollary III.4.2] we infer that $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$.

Assume now that $\sum_{n=1}^{\infty} n^{-2}\log\beta(-n) < \infty$, and set $W(n) = \beta(-|n|)^2$ $(n \in \mathbb{Z})$. Since $\sum_{n=-\infty}^{\infty}(\log W(n))/(n^2+1) < \infty$, $W(n) \geq 1$ for every $n \in \mathbb{Z}$, and $\lim_{|n|\to\infty} W(n) = \infty$, by [Koo98, Corollary] there exists a non-identically zero sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that $\sum_{n=-\infty}^{\infty}|a_n|W(n) < \infty$ and the continuous function $f = \sum_{n=-\infty}^{\infty} a_n\chi^n$ satisfies $f(e^{it}) = 0$ whenever $h \leq |t| \leq \pi$, where $h \in (0,\pi)$ is an arbitrarily prescribed number. Notice that, by uniform convergence, $\widehat{f}(n) = a_n$ for every $n \in \mathbb{Z}$, and so f is non-zero. On the other hand, the relations

$$\sum_{n=-\infty}^{\infty}|\widehat{f}(n)|\beta(n)^2 \leq \sum_{n=-\infty}^{\infty}|a_n|W(n) < \infty \quad \text{and} \quad \lim_{|n|\to\infty}\widehat{f}(n) = 0$$

imply that $\sum_{n=-\infty}^{\infty}|\widehat{f}(n)|^2\beta(n)^2 < \infty$, and so $f \in L^2(\beta)$. ∎

The invertibility of T_β is controlled by the number

$$\delta_\beta := \inf\{\beta(n+1)/\beta(n) : n \in \mathbb{Z}\} \in \mathbb{R}_+.$$

Namely, T_β is invertible if and only if $\delta_\beta > 0$. The non-invertible case is well understood.

Proposition 32. *If $\delta_\beta = 0$, then $\mathcal{F}(T_\beta) = H^\infty$ and so $\operatorname{Hlat} T_\beta = \operatorname{Lat} T_\beta$ is non-trivial.*

Proof. For the reader's convenience we sketch the short proof. Given $\phi \in \mathcal{F}(T_\beta)$, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we have

$$\langle M_{\phi,\beta}\chi^k, \chi^{k-n}\rangle_\beta = \widehat{\phi}(-n)\beta(k-n)^2,$$

whence

$$|\widehat{\phi}(-n)| \leq \|M_{\phi,\beta}\|\beta(k)/\beta(k-n).$$

Since $\delta_\beta = 0$, we infer that $\widehat{\phi}(-n) = 0$. ∎

Remarks 33. (a) If $\beta(-n) = \exp(n^2)$ $(n \in \mathbb{N})$, then $T_\beta \in \mathcal{L}_1(L^2(\beta))$, $\{T_\beta\}' = H^\infty(T_\beta)$, but T_β is not a quasiaffine transform of S, since T_β^* is injective. This example can be contrasted with Proposition 13.

(b) The sequence β can be chosen so that

$$\delta_\beta = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\log\beta(-n)}{n^2} < \infty.$$

In that case the functional commutant $\mathcal{F}(T_\beta)$ is a quasianalytic algebra, while the C_{10}-contraction T_β is not quasianalytic.

Now let us turn to the case when $\delta_\beta > 0$, and so $\delta_\beta = \|T_\beta^{-1}\|^{-1}$. Let $r_\beta := r(T_\beta^{-1})^{-1}$ be the inner spectral radius of T_β. It is easy to verify that

$$0 < \delta_\beta \leq r_\beta \leq \Big(\varlimsup_{n\to\infty}\beta(-n)^{1/n}\Big)^{-1} \leq \Big(\varliminf_{n\to\infty}\beta(-n)^{1/n}\Big)^{-1} =: R_\beta \leq 1.$$

It is known that the operators of the form $\sum_{n=-N}^{N} c_n T_\beta^n$ are dense in $\{T_\beta\}'$ in the strong operator topology (see [Shi74, Section 8, Corollary (b)]). Thus $\operatorname{Hlat} T_\beta = \operatorname{Lat} T_\beta \cap \operatorname{Lat} T_\beta^{-1}$, and the hyperinvariant subspaces of T_β may be called biinvariant.

The case $r_\beta = 1$ has been settled by Esterle, by providing a subspace $\mathcal{M}$ satisfying condition (ii) of Theorem 18. Namely, Theorem 5.7 of [Est97] can be stated in the following way.

Theorem 34 (Esterle). *If $r_\beta = 1$, then there exists $\mathcal{M} \in \operatorname{Lat_s} T_\beta$ such that $\widetilde{\mathcal{M}} = \bigvee\{C\mathcal{M} : C \in \{T_\beta\}'\} \neq L^2(\beta)$, and so $\widetilde{\mathcal{M}}$ is a non-trivial hyperinvariant subspace of T_β.*

For any $R \in (0,1)$, let $A(R) := \{z \in \mathbb{C} : R < |z| < 1\}$. It is known that the point spectrum of the adjoint satisfies the condition $A(R_\beta) \subset \sigma_p(T_\beta^*) \subset A(R_\beta)^-$ (see [Shi74, Section 5, Theorem 9]). Thus the (HSP) for bilateral weighted shifts, which are C_{10}-contractions, is open (up to our knowledge) in the case when

$$0 < \delta_\beta \le r_\beta < R_\beta = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\log \beta(-n)}{n^2} = \infty.$$

Under these conditions the functional commutant can be related to bounded analytic functions defined on an annulus. For $R \in (0,1)$, let $H^\infty(A(R))$ stand for the Banach algebra of bounded analytic functions on $A(R)$. We note that $\mathcal{F}(T_\beta)$ is an abelian Banach algebra with the norm $\|\phi\|_{\beta,\infty} := \|M_{\phi,\beta}\|$. In the next statement we consider this norm on $\mathcal{F}(T_\beta)$.

Proposition 35. *If $0 < \delta_\beta \le r_\beta < 1$, then the mapping*

$$\Lambda_\beta \colon \mathcal{F}(T_\beta) \to H^\infty(A(r_\beta)),\ \phi \mapsto \Phi, \quad \text{where} \quad \Phi(z) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) z^n,$$

is an injective and contractive algebra-homomorphism, while the mapping

$$\widetilde{\Lambda}_\beta \colon H^\infty(A(\delta_\beta)) \to \mathcal{F}(T_\beta),\ \Phi \to \phi,\ \text{where } \phi(\zeta) = \operatorname{nt-lim}_{z\to\zeta} \Phi(z) \text{ for a.e. } \zeta \in \mathbb{T},$$

is a bounded algebra-homomorphism; moreover

$$\Lambda_\beta \widetilde{\Lambda}_\beta \colon H^\infty(A(\delta_\beta)) \to H^\infty(A(r_\beta)), \qquad \Phi \mapsto \Phi|A(r_\beta).$$

In particular, if $0 < \delta_\beta = r_\beta < 1$ then Λ_β is an algebra-isomorphism.

Proof. For the sake of completeness we sketch the proof, which is an adaption of the proof of Theorem 10′ in Section 6 of [Shi74] to our situation, avoiding formal series.

Fix $\phi \in \mathcal{F}(T_\beta)$. The inequality in the proof of Proposition 32 shows that

$$|\widehat{\phi}(-n)| \le \|M_{\phi,\beta}\| \inf\{\beta(k)/\beta(k-n) : k \in \mathbb{Z}\} = \|M_{\phi,\beta}\| \cdot \|T_\beta^{-n}\|^{-1} \quad (n \in \mathbb{N}),$$

so $\overline{\lim}_{n\to\infty} |\widehat{\phi}(-n)|^{1/n} \le r_\beta$. Therefore, the Laurent series $\sum_{n=-\infty}^{\infty} \widehat{\phi}(n)z^n$ converges to an analytic function Φ on $A(r_\beta)$. Fix $z \in A(r_\beta)$. Since the linear functional $E_z\colon \mathcal{F}(T_\beta) \to \mathbb{C}$, $\phi \mapsto \Phi(z)$, is multiplicative and $\mathcal{F}(T_\beta)$ is an abelian Banach algebra, we infer that $\|E_z\| \le 1$, and so $|\Phi(z)| \le \|\phi\|_{\beta,\infty}$. Thus Λ_β is a contractive algebra-isomorphism.

Given any $\Phi \in H^\infty(A(\delta_\beta))$, consider the Laurent expansion $\Phi(z) = \sum_{n=-\infty}^{\infty} c_n z^n$. The function $\Phi_+(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic on $\mathbb{D}$, while $\Phi_-(z) = \sum_{n=1}^{\infty} c_{-n} z^{-n}$ is analytic on $\overline{\mathbb{C}} \setminus (\delta_\beta \mathbb{D})^-$. For any $r \in (\delta_\beta, 1)$, Φ_- is bounded on $A(r)$, hence $\Phi_+ = \Phi - \Phi_-$ is bounded on $A(r)$, and so Φ_+ is bounded on $\mathbb{D}$. By Fatou's theorem,

$$\phi(\zeta) = \text{nt-}\lim_{z\to\zeta} \Phi(z) = \text{nt-}\lim_{z\to\zeta} \Phi_+(z) + \Phi_-(\zeta)$$

exists for a.e. $\zeta \in \mathbb{T}$. Set $\phi_r(\zeta) = \Phi(r\zeta)$ $(r \in (\delta_\beta, 1),\ \zeta \in \mathbb{T})$. Lebesgue's dominating convergence theorem yields $\widehat{\phi}(n) = \lim_{r\to 1} \widehat{\phi}_r(n) = c_n r^n = c_n$ for every $n \in \mathbb{Z}$. (This argument shows that ϕ can be recovered from $\Lambda_\beta\phi$, and so Λ_β is injective.)

For any $N \in \mathbb{N}$, set

$$\sigma_N = \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) c_n \chi^n.$$

By a von Neumann-type inequality for an annulus we know that

$$\|\sigma_N(T_\beta)\| \le C_\beta \sup\{|\sigma_N(z)| : z \in A(\delta_\beta)\},$$

where C_β depends only on δ_β see [Shi74, Section 6, Proposition 23]. Since $|\sigma_N(r\zeta)| \le \|\phi_r\|_\infty \le \|\Phi\|_\infty$ for $r \in (\delta_\beta, 1)$ and $\zeta \in \mathbb{T}$, it follows that $\{M_{\sigma_N,\beta} = \sigma_N(T_\beta)\}_{N=1}^{\infty}$ is a bounded sequence of operators. Taking into account that

$$\langle \sigma_N \chi^k, \chi^l \rangle_\beta = \widehat{\sigma}_N(l-k)\beta(l)^2 \to c_{l-k}\beta(l)^2 \quad \text{as } N \to \infty\ (k, l \in \mathbb{Z}),$$

we conclude that $M_{\sigma_N,\beta} \in \{T_\beta\}'$ converges in the weak operator topology to an operator $M_{\psi,\beta}$ with $\psi \in \mathcal{F}(T)$, as $N \to \infty$. Since

$$\widehat{\psi}(l-k)\beta(l)^2 = \lim_{N\to\infty} \langle \sigma_N \chi^k, \chi^l \rangle_\beta = c_{l-k}\beta(l)^2 \quad (k, l \in \mathbb{Z}),$$

we find that $\widehat{\psi}(n) = c_n = \widehat{\phi}(n)$ $(n \in \mathbb{Z})$, and so $\phi = \psi \in \mathcal{F}(T_\beta)$. Clearly, $\|\phi\|_{\beta,\infty} \le C_\beta \|\Phi\|_\infty$, which means that $\widetilde{\Lambda}_\beta$ is a bounded algebra-homomorphism. The relation $(\Lambda_\beta \circ \widetilde{\Lambda}_\beta)\Phi = \Phi|A(r_\beta)$ readily follows from the previous discussions. ∎

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