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Optimal embeddings of critical Sobolev–Lorentz–Zygmund spaces

by

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Abstract. We establish the embedding of the critical Sobolev–Lorentz–Zygmund space $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ into the generalized Morrey space $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ with an optimal Young function Φ. As an application, we obtain the almost Lipschitz continuity for functions in $H^{n/p+1}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. O'Neil's inequality and its reverse play an essential role in the proofs of the main theorems.

1. Introduction and main theorems. In this paper, we consider optimal embeddings of the critical Sobolev–Lorentz–Zygmund space $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ into generalized Morrey spaces $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$, where $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ and $\lambda_1, \dots, \lambda_m$ are non-negative numbers with $m \in \mathbb{N}$, and Φ is a Young function. One of our main purposes is to investigate the optimal Young function Φ for which the embedding $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ holds.

The Sobolev–Lorentz–Zygmund space $H^{s}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined in terms of the Lorentz–Zygmund space $L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ as a Bessel potential space, $H^{s}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n) := (1-\Delta)^{-s/2} L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. The spaces $H^{s}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ extend Sobolev–Lorentz spaces and Sobolev spaces since $L_{p,q,0,\dots,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ and $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, where $L_p(\mathbb{R}^n)$ and $L_{p,q}(\mathbb{R}^n)$ denote the Lebesgue space and the Lorentz space, respectively. We give the definitions of those function spaces and related properties in Section 2.

We consider the optimal vanishing and growth orders of the local integrals $\int_E |u(x)|^r\,dx$ as $|E| \to 0$ or $|E| \to \infty$ for functions u in $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. In Suzuki–Wadade [SW], the authors gave the optimal growth order of the local integrals for functions in $H^{n/p}_{p,q}(\mathbb{R}^n)$:

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Theorem A ([SW]). *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \infty$. Then there exists a positive constant C such that the inequality*

$$\Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \le C|E|^{1/r-1/p}\|u\|_{H^{n/p}_{p,q}} \tag{1.1}$$

holds for all $u \in H^{n/p}_{p,q}(\mathbb{R}^n)$ and all measurable sets E if and only if $p > r$ or $p = r \ge q$.

In Theorem A, the necessity of the condition $p > r$ or $p = r \ge q$ comes from the part $|E| \to \infty$ in (1.1). In fact, the vanishing order $|E|^{1/r-1/p}$ as $|E| \to 0$ turns out not to be optimal, and in [SW], the authors also proved the following:

Theorem B ([SW]). *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \infty$. Then there exist positive constants δ and C such that for all $u \in H^{n/p}_{p,q}(\mathbb{R}^n)$ and all measurable sets E satisfying $|E| < \delta$,*

$$\Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \le C|E|^{1/r}\log(1/|E|)^{1/q'}\|u\|_{H^{n/p}_{p,q}}$$

where $q' := q/(q-1)$.

Theorem B was originally obtained by Brézis–Wainger [BW] when $p = q$, which corresponds to the critical Sobolev space $H^{n/p}_p(\mathbb{R}^n)$. Ozawa [Oz] gave an alternative proof of Theorem B when $p = q$. We also refer to Sawano–Wadade [SaWa], where the authors proved similar embeddings for the critical Sobolev–Morrey space.

Our first goal in this paper is to extend both Theorem A and Theorem B to functions in $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. Concerning an extension of Theorem A, one can show that the inequality (1.1) with $\|u\|_{H^{n/p}_{p,q}}$ replaced by $\|u\|_{H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}}$ holds if and only if $p > r$ or $p = r \ge q$ without any modification of the proof of Theorem A in [SW]. Therefore, we omit its proof in this paper. However, when we consider an extension of Theorem B to $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, the vanishing order as $|E| \to 0$ depends on the exponents $\lambda_1, \dots, \lambda_m$.

To state our main theorems, we define multiple logarithmic functions by

$$\ell_l(t) := \underbrace{\ell_1 \circ \cdots \circ \ell_1}_{l}(t) \quad \text{for } t \ge c_l \quad \text{with} \quad \ell_1(t) := \log t,$$

where the constants $c_l > 0$ are determined by $\ell_l(c_l) = 1$. Our first result is:

Theorem 1.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ and let $\lambda_1, \dots, \lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Assume one of the following conditions holds:*

(A) *there exists* $0 \le j \le m-1$ *such that*

$$\lambda_1 = \cdots = \lambda_j = \frac{1}{q'} \quad \textit{and} \quad \lambda_{j+1} > \frac{1}{q'},$$

(B) *there exists* $0 \le j \le m-1$ *such that*

$$\lambda_1 = \cdots = \lambda_j = \frac{1}{q'} \quad \textit{and} \quad \lambda_{j+1} < \frac{1}{q'},$$

(C) $\lambda_1 = \cdots = \lambda_m = 1/q'$,

where (A) *and* (B) *are understood as* $\lambda_1 > 1/q'$ *and* $\lambda_1 < 1/q'$ *respectively when* $j = 0$. *Then there exist positive constants* C *and* δ *such that for all* $u \in H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ *and all measurable sets* E *satisfying* $|E| < \delta$,

$$(1.2)\qquad \Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \le \begin{cases} C|E|^{1/r}\,\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \textit{if } (\mathrm{A}) \textit{ holds},\\ C|E|^{1/r}\,\ell_{j+1}(1/|E|)^{1/q'-\lambda_{j+1}} & \\ \quad\times \prod_{l=j+2}^m \ell_l(1/|E|)^{-\lambda_l}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \textit{if } (\mathrm{B}) \textit{ holds},\\ C|E|^{1/r}\ell_{m+1}(1/|E|)^{1/q'}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \textit{if } (\mathrm{C}) \textit{ holds}, \end{cases}$$

where in the middle inequality, the right-hand side is understood as $C|E|^{1/r}\ell_m(1/|E|)^{1/q'-\lambda_m}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}}$ *when* $j = m-1$.

As a special case $m = 1$ of Theorem 1.1, we obtain the following corollary:

Corollary 1.2. *Let* $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ *and* $\lambda \ge 0$. *Then there exist positive constants* C *and* δ *such that for all* $u \in H^{n/p}_{p,q,\lambda}(\mathbb{R}^n)$ *and all measurable sets* E *satisfying* $|E| < \delta$,

$$(1.3)\qquad \Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \le \begin{cases} C|E|^{1/r}\,\|u\|_{H^{n/p}_{p,q,\lambda}} & \textit{if } \lambda > \frac{1}{q'},\\ C|E|^{1/r}\log(1/|E|)^{1/q'-\lambda}\|u\|_{H^{n/p}_{p,q,\lambda}} & \textit{if } \lambda < \frac{1}{q'},\\ C|E|^{1/r}\log\big(\log(1/|E|)\big)^{1/q'}\|u\|_{H^{n/p}_{p,q,\lambda}} & \textit{if } \lambda = \frac{1}{q'}, \end{cases}$$

where the constants C *and* δ *are independent of* E.

Note that Theorem B corresponds to the middle inequality in (1.3) with $\lambda = 0$. Furthermore, Corollary 1.2 tells us that the exponent $\lambda = 1/q'$ is a threshold so that the logarithmic vanishing order as $|E| \to 0$ appears for the local integrals of functions in $H^{n/p}_{p,q,\lambda}(\mathbb{R}^n)$.

Theorem 1.1 can be regarded as giving the embedding of $H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ into a generalized Morrey space. Generalized Morrey spaces have been extensively studied: see for instance Kurata–Nishigaki–Sugano [KNS], Nakai

[N1, N2] and Sawano–Sugano–Tanaka [SST1, SST2]. Let Φ be a *Young function*, that is, $\Phi : [0,\infty) \to [0,\infty)$ is a continuous function satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Then for a locally integrable function u on $\mathbb{R}^n$, the norm of the generalized Morrey space $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ is given by

$$\|u\|_{\mathcal{M}_{\Phi,r}} := \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} \Phi(|Q|)\biggl(\frac{1}{|Q|}\int_Q |u(x)|^r\,dx\biggr)^{1/r},$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of dyadic cubes in $\mathbb{R}^n$. The spaces $\mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ extend Morrey spaces and in particular Lebesgue spaces. As an immediate consequence of Theorem 1.1 and Theorem A with $\|u\|_{H^{n/p}_{p,q}}$ replaced by $\|u\|_{H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}}$, we obtain the following embeddings:

COROLLARY 1.3. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, $1 \le r < \infty$ and let $\lambda_1,\dots,\lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Define Young functions Φ by*

$$(1.4)\quad \Phi(t) := \begin{cases} (1+t)^{1/p} & \text{if (A) holds},\\ (1+t)^{1/p}\,\ell_{j+1}(c_{j+1}+1/t)^{\lambda_{j+1}-1/q'}\prod_{l=j+2}^{m}\ell_l(c_l+1/t)^{\lambda_l} & \text{if (B) holds},\\ (1+t)^{1/p}\,\ell_{m+1}(c_{m+1}+1/t)^{-1/q'} & \text{if (C) holds}.\end{cases}$$

Then the continuous embedding $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ holds if and only if $p > r$ or $p = r \ge q$.

As another application of Theorem 1.1, we investigate the Lipschitz type continuity for functions in $H^{n/p+1}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. It is well-known that

$$H^{n/p+\alpha}_p(\mathbb{R}^n) \hookrightarrow C^\alpha(\mathbb{R}^n) \quad \text{for } 0<\alpha<1 \quad \text{but} \quad H^{n/p+1}_p(\mathbb{R}^n) \not\hookrightarrow \mathrm{Lip}(\mathbb{R}^n),$$

where $C^\alpha(\mathbb{R}^n)$ and $\mathrm{Lip}(\mathbb{R}^n)$ denote the Hölder space and the Lipschitz space, respectively. Instead, functions in $H^{n/p+1}_p(\mathbb{R}^n)$ exhibit almost Lipschitz continuity (see Brézis–Wainger [BW]). Based on this fact, we next clarify how the exponents $\lambda_1,\dots,\lambda_m$ influence the Lipschitz type continuity for functions in $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. Our second theorem reads as follows:

THEOREM 1.4. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 < q \le \infty$, and let $\lambda_1,\dots,\lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Assume one of the conditions* (A)–(C) *of Theorem* 1.1 *holds. Then there exist positive constants C and δ such that* *for all* $u \in H^{n/p+1}_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ *and all* x *and* y *satisfying* $|x-y|<\delta$,

$$|u(x)-u(y)| \le \begin{cases} C|x-y|\,\|u\|_{H^{n/p+1}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (A) holds,} \\ C|x-y|\ell_{j+1}\Big(\frac{1}{|x-y|}\Big)^{1/q'-\lambda_{j+1}} & \\ \quad\times \prod_{l=j+2}^{m} \ell_l\Big(\frac{1}{|x-y|}\Big)^{-\lambda_l} \|u\|_{H^{n/p+1}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (B) holds,} \\ C|x-y|\ell_{m+1}\Big(\frac{1}{|x-y|}\Big)^{1/q'} \|u\|_{H^{n/p+1}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (C) holds.} \end{cases}$$

The case $m=1$ in Theorem 1.4 yields the following corollary:

Corollary 1.5. *Let* $n\in\mathbb{N}$, $1<p<\infty$, $1<q\le\infty$ *and* $\lambda\ge 0$. *Then there exist positive constants* C *and* δ *such that for all* $u\in H^{n/p+1}_{p,q,\lambda}(\mathbb{R}^n)$ *and all* x *and* y *satisfying* $|x-y|<\delta$,

$$(1.5)\qquad |u(x)-u(y)| \le \begin{cases} C|x-y|\,\|u\|_{H^{n/p+1}_{p,q,\lambda}} & \text{if } \lambda>1/q', \\ C|x-y|\log\Big(\frac{1}{|x-y|}\Big)^{1/q'-\lambda} \|u\|_{H^{n/p+1}_{p,q,\lambda}} & \text{if } \lambda<1/q', \\ C|x-y|\log\Big(\log\Big(\frac{1}{|x-y|}\Big)\Big)^{1/q'} \|u\|_{H^{n/p+1}_{p,q,\lambda}} & \text{if } \lambda=1/q'. \end{cases}$$

In [BW], the middle inequality in (1.5) with $p=q$ and $\lambda=0$ was proved. Moreover, Corollary 1.5 tells us that the exponent $\lambda=1/q'$ is a threshold so that $H^{n/p}_{p,q,\lambda}(\mathbb{R}^n)$ embeds into $\mathrm{Lip}(\mathbb{R}^n)$.

Finally, we consider the optimality of the inequalities (1.2) in Theorem 1.1 with respect to the vanishing orders as $|E|\to 0$, which also implies the optimality of the Young functions (1.4) in Corollary 1.3. We will find that the vanishing orders as $|E|\to 0$ are optimal in terms of multiple logarithmic functions. Our final theorem is stated as follows:

Theorem 1.6. *Let* $n\in\mathbb{N}$, $1<p<\infty$, $1<q\le\infty$, $1<r<\infty$, *and let* $\lambda_1,\ldots,\lambda_m$ *be non-negative numbers with* $m\in\mathbb{N}$. *Take* $k\ge m$ *with* $k\in\mathbb{N}$ *and* $\varepsilon>0$. *Assume one of the conditions* (A)–(C) *of Theorem* 1.1 *holds.*

(i) *If* $q<\infty$, *then there exist* $u\in H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ *and positive constants* C *and* δ *such that for all measurable sets* E *satisfying* $|E|<\delta$,

$$
(1.6)\quad \Big(\int_E |u(x)|^r\,dx\Big)^{1/r}
\ge \begin{cases}
C|E|^{1/r} & \textit{if (A) holds,}\\
C|E|^{1/r}\ell_{j+1}(1/|E|)\displaystyle\prod_{l=j+1}^{m}\ell_l(1/|E|)^{-\lambda_l} & \\
\quad\times\displaystyle\prod_{l=j+1}^{k-1}\ell_l(1/|E|)^{-1/q}\ell_k(1/|E|)^{-1/q-\varepsilon} & \textit{if (B) holds,}\\
C|E|^{1/r}\ell_{m+1}(1/|E|)\displaystyle\prod_{l=m+1}^{k}\ell_l(1/|E|)^{-1/q}\ell_{k+1}(1/|E|)^{-1/q-\varepsilon} & \textit{if (C) holds.}
\end{cases}
$$

(ii) *If $q=\infty$, then there exist $u\in H^{n/p}_{p,\infty,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ and positive constants C and δ such that for all measurable sets E satisfying $|E|<\delta$,*

$$
\Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \ge \begin{cases}
C|E|^{1/r} & \textit{if (A) holds,}\\
C|E|^{1/r}\ell_{j+1}(1/|E|)\displaystyle\prod_{l=j+1}^{m}\ell_l(1/|E|)^{-\lambda_l} & \textit{if (B) holds,}\\
C|E|^{1/r}\ell_{m+1}(1/|E|) & \textit{if (C) holds.}
\end{cases}
$$

Theorem 1.6 implies that the vanishing orders as $|E|\to 0$ for the inequalities (1.2) in Theorem 1.1 are best possible when $q=\infty$ and they are also sharp even when $q<\infty$ in terms of multiple logarithmic functions. It is worth noting that the last two inequalities in (1.6) become sharper as $k\in\mathbb{N}$ gets larger.

Inequalities characterizing critical function spaces (in terms of Sobolev embedding) such as Sobolev–Lorentz spaces, Sobolev-Morrey spaces, Besov spaces, Triebel–Lizorkin spaces and functions of bounded mean oscillation have been extensively studied: see for instance Brézis–Wainger [BW], Chen–Zhu [ChZ], Edmunds–Triebel [ET], Machihara–Ozawa–Wadade [MOW], Nagayasu–Wadade [NW], Ogawa [Og], Ogawa–Ozawa [OgOz], Ozawa [Oz], Sawano–Wadade [SaWa] and Wadade [W1, W2, W3]. In those papers, the authors established critical embeddings by proving Trudinger–Moser type inequalities, Gagliardo–Nirenberg type inequalities, Brézis–Gallouët–Wainger type inequalities and logarithmic Hardy inequalities.

Our main subject is the optimal embedding of the critical Sobolev–Lorentz–Zygmund space into generalized Morrey spaces, which is regarded as one of the characterizations for the critical Sobolev–Lorentz–Zygmund space. As far as we know, this kind of embedding is poorly known compared to embeddings related to Trudinger–Moser type inequalities etc. We will discuss relations between those critical embeddings in a forthcoming paper.

This paper is organized as follows. In Section 2 we give the definition of Sobolev–Lorentz–Zygmund spaces and collect the elementary properties of rearrangements of functions. We prove the main theorems in Section 3.

2. Preliminaries. In this section, we first recall the definition of Lorentz–Zygmund spaces. To this end, we define the rearrangement of a measurable function. For a measurable function f on $\mathbb{R}^n$, let $f_* : [0,\infty) \to [0,\infty]$ be the distribution function of f defined by

$$f_*(\lambda) := |\{x \in \mathbb{R}^n;\ |f(x)| > \lambda\}| \quad \text{for } \lambda \geq 0,$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$; then the *rearrangement* $f^* : [0,\infty) \to [0,\infty]$ of f is defined by

$$f^*(t) := \inf\{\lambda > 0;\ f_*(\lambda) \leq t\} \quad \text{for } t \geq 0.$$

Moreover, $f^{**} : (0,\infty) \to [0,\infty]$ denotes the average function of f^* defined by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(\tau)\,d\tau \quad \text{for } t > 0.$$

In what follows, we assume $f^*(t) < \infty$ for all $t > 0$. Then f^* is right-continuous and non-increasing on $(0,\infty)$, and hence f^{**} is continuous and non-increasing on $(0,\infty)$ with $f^*(t) \leq f^{**}(t)$ for all $t > 0$.

We now introduce Lorentz–Zygmund spaces by using rearrangements. Let $1 \leq p,\ q \leq \infty$, and let $\lambda_1, \ldots, \lambda_m$ be non-negative numbers with $m \in \mathbb{N}$. Then the *Lorentz–Zygmund space* $L_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ is a function space equipped with the norm

$$\|f\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}} := \left(\int_0^\infty \Big(t^{1/p}\prod_{l=1}^m \ell_l(c_l + 1/t)^{\lambda_l} f^*(t)\Big)^q \frac{dt}{t}\right)^{1/q},$$

where $\ell_l(t) := \ell_1 \circ \cdots \circ \ell_1(t)$ (l-fold composition) for $t \geq c_l$ with $\ell_1(t) := \log t$, and the constants $c_l > 0$ are determined by $\ell_l(c_l) = 1$. When $q = \infty$, the norm $\|f\|_{L_{p,\infty,\lambda_1,\ldots,\lambda_m}}$ can be defined by the usual modification. Note that Lorentz–Zygmund spaces generalize Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ since $\|f\|_{L_{p,q,0,\ldots,0}} = \|f\|_{L_{p,q}}$.

Replacing f^* by f^{**} in $\|f\|_{L_{p,q}}$ we obtain another equivalent norm on $L_{p,q}(\mathbb{R}^n)$ if $p \neq 1$. Indeed, the following Hardy inequality guarantees the equivalence:

$$\left(\int_0^\infty \left(\frac{t^{1/p}}{t}\int_0^t f(s)\,ds\right)^q \frac{dt}{t}\right)^{1/q} \leq p'\left(\int_0^\infty (t^{1/p}f(t))^q\,\frac{dt}{t}\right)^{1/q} \tag{2.1}$$

for non-negative measurable functions f, where $p' := p/(p-1)$. For the proof of (2.1), see O'Neil [O, Lemma 2.3] and references therein. Furthermore, since f^* and f^{**} are non-increasing functions in $(0,\infty)$, we get the following decay estimates: for any $t>0$,

$$f^*(t) \le \left(\frac{q}{p}\right)^{1/q} t^{-1/p}\,\|f\|_{L_{p,q}}, \tag{2.2}$$

and hence if $p>1$, together with (2.1), we obtain, for any $t>0$,

$$f^{**}(t) \le p'\left(\frac{q}{p}\right)^{1/q} t^{-1/p}\,\|f\|_{L_{p,q}}.$$

Next, we recall the pointwise rearrangement inequality for the convolution of functions proved by O'Neil [O, Theorem 1.7]: for measurable functions f and g on $\mathbb{R}^n$,

$$(f*g)^{**}(t) \le t\,f^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,ds \quad \text{for } t>0. \tag{2.3}$$

Moreover, we will make use of the reverse O'Neil inequality established in Kozono–Sato–Wadade [KSW, Lemma 2.2]: there exists a positive constant C such that

$$(f*g)^{**}(t) \ge C\Big(t\,f^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,ds\Big) \tag{2.4}$$

for all $t>0$ and all measurable functions f and g on $\mathbb{R}^n$ which are both non-negative, radially symmetric and non-increasing in the radial direction.

In this paper, we frequently use the *Bessel potential* $G_s*f := (1-\Delta)^{-s/2}f$ and the *Riesz potential* $I_s*f := (-\Delta)^{-s/2}f$ for $0<s<n$. More precisely, the kernel functions I_s and G_s are defined respectively by

$$I_s(x) := \frac{\Gamma((n-s)/2)}{2^s\pi^{n/2}\Gamma(s/2)}|x|^{-(n-s)},$$

$$G_s(x) := \frac{1}{(4\pi)^{s/2}\Gamma(s/2)}\int_0^\infty e^{-\pi|x|^2/t-t/(4\pi)}\,t^{-(n-s)/2}\,\frac{dt}{t},$$

for $x\in\mathbb{R}^n\setminus\{0\}$, where Γ denotes the Gamma function. Based on the Lorentz–Zygmund space, we define the *Sobolev–Lorentz–Zygmund space* $H^s_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ by

$$H^s_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n) := (I-\Delta)^{-s/2}L_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n) = G_s * L_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n),$$

equipped with the norm $\|u\|_{H^s_{p,q,\lambda_1,\ldots,\lambda_m}} := \|(I-\Delta)^{s/2}u\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}}$. The spaces $H^s_{p,q,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$ extend Sobolev–Lorentz spaces $H^s_{p,q}(\mathbb{R}^n)$ and in particular Sobolev spaces $H^s_p(\mathbb{R}^n)$ since $L_{p,q,0,\ldots,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ and $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

We now collect the elementary properties of I_s and G_s in the following lemma.

LEMMA 2.1. *Let $n \in \mathbb{N}$ and $0 < s < n$.*

(i) *I_s and G_s are non-negative, radially symmetric and non-increasing in the radial direction, so that $I_s^*(t) = I_s(x)$ and $G_s^*(t) = G_s(x)$ if $|x| = (t/\omega_n)^{1/n} > 0$, where $\omega_n := \frac{2\pi^{\pi/2}}{n\Gamma(n/2)}$ denotes the volume of the unit ball in $\mathbb{R}^n$.*

(ii) *$G_s(x) \le I_s(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, which implies $G_s^*(t) \le I_s^*(t)$, $G_s^{**}(t) \le I_s^{**}(t)$ for all $t > 0$, and*

$$\lim_{|x|\downarrow 0} \frac{G_s(x)}{I_s(x)} = \lim_{t\downarrow 0} \frac{G_s^*(t)}{I_s^*(t)} = 1.$$

(iii) *$\|G_s\|_{L_1(\mathbb{R}^n)} = 1$ and there exists a positive constant C such that*

$$G_s(x) \le \begin{cases} C|x|^{-(n-s)} & \text{for } x \in \mathbb{R}^n \setminus \{0\}, \\ Ce^{-|x|} & \text{for } x \in \mathbb{R}^n \text{ with } |x| \ge 1. \end{cases}$$

Since the facts in Lemma 2.1 are well-known, we omit the detailed proof (see Stein [St] for instance). Furthermore, we refer to Almgren–Lieb [AL], Bennett–Sharpley [BS] and Kokilashvili–Krbec [KK] for further information about rearrangements.

3. Proof of main theorems

Proof of Theorem 1.1. First, letting $(1-\Delta)^{n/(2p)}u = f$, we have $u = G_{n/p} * f$, where $G_{n/p}$ denotes the Bessel kernel. Thus the inequality (1.2) can be written equivalently as

$$\Big(\int_E |G_{n/p} * f(x)|^r\,dx\Big)^{1/r} \le \begin{cases} C|E|^{1/r}\|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} & \text{if (A) holds,} \\ C|E|^{1/r}\ell_{j+1}(1/|E|)^{1/q'-\lambda_{j+1}} \prod_{l=j+2}^{m} \ell_l(1/|E|)^{-\lambda_l}\|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} & \text{if (B) holds,} \\ C|E|^{1/r}\ell_{m+1}(1/|E|)^{1/q'}\|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} & \text{if (C) holds,} \end{cases}$$

for all $f \in L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$ and all measurable sets E of small measure.

By O'Neil's inequality (2.3), we obtain

$$\Big(\int_E |G_{n/p} * f(x)|^r\,dx\Big)^{1/r} = \Big(\int_0^{|E|} (G_{n/p} * f)^*(t)^r\,dt\Big)^{1/r}$$
$$\le \Big(\int_0^{|E|} \big(t\,G_{n/p}^{**}(t) f^{**}(t)\big)^r\,dt\Big)^{1/r} + \Big(\int_0^{|E|} \Big(\int_t^\infty G_{n/p}^*(s) f^*(s)\,ds\Big)^r\,dt\Big)^{1/r}$$

$$\leq \Big(\int_0^{|E|} \big(t\, G^{**}_{n/p}(t) f^{**}(t)\big)^r \, dt \Big)^{1/r} + \Big(\int_0^{|E|} \Big(\int_t^{|E|} G^{*}_{n/p}(s) f^{*}(s)\, ds \Big)^r \, dt \Big)^{1/r}$$
$$+ \Big(\int_0^{|E|} \Big(\int_{|E|}^{\infty} G^{*}_{n/p}(s) f^{*}(s)\, ds \Big)^r \, dt \Big)^{1/r}$$
$$=: I_1 + I_2 + I_3.$$

We first estimate I_1. For small $t > 0$, by the decay estimate (2.2) and Lemma 2.1,

$$\begin{aligned} tG^{**}_{n/p}(t) f^{**}(t) &= \frac{1}{t} \int_0^t G^{*}_{n/p}(s)\, ds \int_0^t f^{*}(s)\, ds \\ &\leq \frac{C}{t} \int_0^t s^{-1/p'}\, ds \int_0^t s^{-1/p}\, ds\, \|f\|_{L_{p,q}} \\ &= C\|f\|_{L_{p,q}} \leq C\|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}}, \end{aligned}$$

and so $I_1 \leq C|E|^{1/r} \|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}}$.

For I_2, by using (2.2) and Lemma 2.1, we have

$$\begin{aligned} I_2 &\leq C\Big(\int_0^{|E|} \Big(\int_t^{|E|} s^{-1/p'-1/p}\, ds \Big)^r \, dt \Big)^{1/r} \|f\|_{L_{p,q}} \\ &\leq C\bigg(\int_0^{|E|} \Big(\log \frac{|E|}{t} \Big)^r \, dt \bigg)^{1/r} \|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} \\ &= C\bigg(\int_0^{1} \Big(\log \frac{1}{s} \Big)^r \, ds \bigg)^{1/r} |E|^{1/r} \|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} = C|E|^{1/r} \|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}}. \end{aligned}$$

Finally, we estimate I_3. For small $\delta > 0$, we have

$$\begin{aligned} I_3 &= |E|^{1/r} \int_{|E|}^{\infty} G^{*}_{n/p}(s) f^{*}(s)\, ds \\ &= |E|^{1/r} \int_{|E|}^{\delta} G^{*}_{n/p}(s) f^{*}(s)\, ds + |E|^{1/r} \int_{\delta}^{\infty} G^{*}_{n/p}(s) f^{*}(s)\, ds =: I_{31} + I_{32}. \end{aligned}$$

By using (2.2) and Lemma 2.1 again, we see that for any $\alpha > 1/p'$,

$$I_{32} \leq C|E|^{1/r} \int_{\delta}^{\infty} s^{-\alpha-1/p}\, ds\, \|f\|_{L_{p,q}} \leq C|E|^{1/r} \|f\|_{L_{p,q,\lambda_1,\dots,\lambda_m}}.$$

Furthermore, by Lemma 2.1 and Hölder's inequality,

$$I_{31} \le C|E|^{1/r} \int_{|E|}^{\delta} s^{-1/p'-1/p} \prod_{l=1}^{m} \ell_l(1/s)^{-\lambda_l} s^{1/p} \prod_{l=1}^{m} \ell_l(1/s)^{\lambda_l} f^*(s)\,ds \tag{3.1}$$
$$\le C|E|^{1/r} \underbrace{\bigg(\int_{|E|}^{\delta} \prod_{l=1}^{m} \ell_l(1/s)^{-\lambda_l q'} \frac{ds}{s}\bigg)^{1/q'}}_{J} \|f\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}}.$$

By applying L'Hôpital's rule, we can investigate the growth orders as $|E| \to 0$ of J under conditions (A)–(C). We obtain

$$J \le \begin{cases} C & \text{if (A) holds,} \\ C\ell_{j+1}(1/|E|)^{1/q'-\lambda_{j+1}} \prod_{l=j+2}^{m} \ell_l(1/|E|)^{-\lambda_l} & \text{if (B) holds,} \\ C\ell_{m+1}(1/|E|)^{1/q'} & \text{if (C) holds,} \end{cases}$$

and hence

$$I_{31} \le \begin{cases} C|E|^{1/r}\|f\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (A) holds,} \\ C|E|^{1/r}\ell_{j+1}(1/|E|)^{1/q'-\lambda_{j+1}} \prod_{l=j+2}^{m} \ell_l(1/|E|)^{-\lambda_l}\|f\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (B) holds,} \\ C|E|^{1/r}\ell_{m+1}(1/|E|)^{1/q'}\|f\|_{L_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (C) holds.} \end{cases}$$

Summing up all the estimates above, we obtain the desired conclusions. ∎

Corollary 1.3 is an immediate consequence of Theorem 1.1 and Theorem A with $\|u\|_{H^{n/p}_{p,q}}$ replaced by $\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}}$:

Proof of Corollary 1.3. First, assume $p > r$ or $p = r \ge q$. Then by applying Theorem 1.1 and Theorem A with $\|u\|_{H^{n/p}_{p,q}}$ replaced by $\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}}$, we see that for any measurable set E,

$$\Big(\int_E |u(x)|^r\,dx\Big)^{1/r} \tag{3.2}$$
$$\le \begin{cases} C|E|^{1/r}(1+|E|)^{-1/p}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (A) holds,} \\ C|E|^{1/r}(1+|E|)^{-1/p}\ell_{j+1}(c_{j+1}+1/|E|)^{1/q'-\lambda_{j+1}} & \\ \quad \times \prod_{l=j+2}^{m} \ell_l(c_l+1/|E|)^{-\lambda_l}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (B) holds,} \\ C|E|^{1/r}(1+|E|)^{-1/p}\ell_{m+1}(c_{m+1}+1/|E|)^{1/q'}\|u\|_{H^{n/p}_{p,q,\lambda_1,\ldots,\lambda_m}} & \text{if (C) holds,} \end{cases}$$

which implies continuous embeddings $H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\Phi,r}(\mathbb{R}^n)$ with Young functions (1.4). Conversely, since the conditions $p > r$ or $p = r \geq q$ are necessary for Theorem A with $\|u\|_{H^{n/p}_{p,q}}$ replaced by $\|u\|_{H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}}$, they are also necessary for the inequalities (3.2) to hold. This finishes the proof of Corollary 1.3. ∎

Theorem 1.4 will be proved by utilizing Theorem 1.1:

Proof of Theorem 1.4. We only consider the case of condition (C) since the other cases can be treated in quite the same way. Let x and y be distinct points in $\mathbb{R}^n$, and let Q be a closed cube in $\mathbb{R}^n$ with vertices x and y with edge length $\rho = |x-y|$. For any $z \in Q$, we have

$$u(z) - u(x) = \int_0^1 \nabla u\,(tz + (1-t)x) \cdot (z-x)\,dt,$$

and so

$$|u(z) - u(x)| \leq \sqrt{n}\,\rho \int_0^1 |\nabla u(tz + (1-t)x)|\,dt. \tag{3.3}$$

Defining $u_Q := |Q|^{-1}\int_Q u(z)\,dz$ and integrating (3.3) with respect to z over Q, we obtain

$$\begin{aligned} |u_Q - u(x)| &\leq \frac{1}{|Q|}\int_Q |u(z) - u(x)|\,dz \\ &\leq \sqrt{n}\,\rho^{1-n}\int_0^1\int_Q |\nabla u\,(tz + (1-t)x)|\,dz\,dt \\ &= \sqrt{n}\,\rho^{1-n}\int_0^1 t^{-n} \int_{tQ+(1-t)x} |\nabla u(\zeta)|\,d\zeta\,dt. \end{aligned} \tag{3.4}$$

Here, applying Theorem 1.1 with $r = 1$ we obtain, for any small $|Q|$,

$$\begin{aligned} \int_{tQ+(1-t)x} |\nabla u(\zeta)|\,d\zeta &\leq C|tQ|\ell_{m+1}(1/|tQ|)^{1/q'}\|\nabla u\|_{H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}} \\ &\leq Ct^n\rho^n\ell_{m+1}\left(\frac{1}{t^n\rho^n}\right)^{1/q'}\|u\|_{H^{n/p+1}_{p,q,\lambda_1,\dots,\lambda_m}}. \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5) yields, for any small $|Q|$,

$$\begin{aligned} |u_Q - u(x)| &\leq C\rho\int_0^1 \ell_{m+1}\left(\frac{1}{t^n\rho^n}\right)^{1/q'} dt\,\|u\|_{H^{n/p+1}_{p,q,\lambda_1,\dots,\lambda_m}} \\ &\leq C\rho\ell_{m+1}(1/\rho)^{1/q'}\|u\|_{H^{n/p+1}_{p,q,\lambda_1,\dots,\lambda_m}}. \end{aligned} \tag{3.6}$$

Interchanging the roles of x and y, we obtain (3.6) with x replaced by y, which gives the desired conclusion. ∎

Finally, we prove Theorem 1.6. The reverse O'Neil inequality (2.4) is an essential tool to estimate local integrals from below.

Proof of Theorem 1.6. First, we consider the case $q < \infty$. Assume condition (A) holds, and define $f_0(x) := |x|^{\alpha n}\chi_{\{x\in\mathbb{R}^n;\,|x|<\delta\}}(x)$, where α is any number satisfying $-1/p < \alpha < 0$, and $\delta > 0$ will be chosen small enough such that

$$f_0^*(t) = \tilde{f}_0((t/\omega_n)^{1/n}) \simeq g_0(t) := t^\alpha \chi_{(0,\delta)}(t)$$

for all $t > 0$, where $\tilde{f}_0(|x|) = f_0(x)$, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. That is, there exist positive constants C and $\tilde{C}$ such that

$$\tilde{C} g_0(t) \le f_0^*(t) \le C g_0(t) \tag{3.7}$$

for all $t > 0$. By the definition of the Lorentz–Zygmund norm and the right estimate in (3.7), since $1/p + \alpha > 0$ we obtain

$$\begin{aligned}\|f_0\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} &\le C\Big(\int_0^\infty \Big(t^{1/p}\prod_{l=1}^m \ell_l(c_l+1/t)^{\lambda_l} g_0(t)\Big)^q \frac{dt}{t}\Big)^{1/q}\\ &\le C\Big(\int_0^\delta \Big(t^{1/p+\alpha}\prod_{l=1}^m \ell_l(1/t)^{\lambda_l}\Big)^q \frac{dt}{t}\Big)^{1/q} \le C\Big(\int_0^\delta t^{\frac{q}{2}(1/p+\alpha)-1}\,dt\Big)^{1/q} < \infty,\end{aligned}$$

which implies $f_0 \in L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, or equivalently $u_0 := G_{n/p} * f_0 \in H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. On the other hand, for any measurable set E satisfying $|E| < \delta/2$, by the left estimate in (3.7), the Hardy inequality (2.1), the reverse O'Neil inequality (2.4) and Lemma 2.1, we see that

$$\begin{aligned}&\int_E |G_{n/p} * f_0(x)|^r\,dx \\ &\quad= \int_0^{|E|} (G_{n/p} * f_0)^*(t)^r\,dt \ge C\int_0^{|E|} (G_{n/p} * f_0)^{**}(t)^r\,dt\\ &\quad\ge C\int_0^{|E|}\Big(t\,G_{n/p}^{**}(t)f_0^{**}(t) + \int_t^\infty G_{n/p}^*(s)f_0^*(s)\,ds\Big)^r dt\\ &\quad\ge C\int_0^{|E|}\Big(\int_t^\delta G_{n/p}^*(s)f_0^*(s)\,ds\Big)^r dt \ge C\int_0^{|E|}\Big(\int_t^\delta G_{n/p}^*(s)g_0(s)\,ds\Big)^r dt\\ &\quad\ge C\int_0^{|E|}\Big(\int_t^\delta s^{-1/p'}g_0(s)\,ds\Big)^r dt \ge C\int_0^{|E|}\Big(\int_{\delta/2}^\delta s^{\alpha-1/p'}\,ds\Big)^r dt = C|E|,\end{aligned} \tag{3.8}$$

as desired.

Next, assume (B) holds, and define

$$f_{\varepsilon,k}(x) := \prod_{l=1}^{j} \ell_l(1/|x|)^{-1} \prod_{l=j+1}^{m} \ell_l(1/|x|)^{-\lambda_l} \times \prod_{l=j+1}^{k-1} \ell_l(1/|x|)^{-1/q} \ell_k(1/|x|)^{-1/q-\varepsilon} |x|^{-n/p} \chi_{\{x\in\mathbb{R}^n;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough. It is easy to see that $f_{\varepsilon,k}$ are non-negative, radially symmetric and non-increasing in the radial direction. Thus there exists $\delta > 0$ such that

$$\begin{aligned} f^*_{\varepsilon,k}(t) &= \tilde{f}_{\varepsilon,k}((t/\omega_n)^{1/n}) \\ &\simeq g_{\varepsilon,k}(t) := \prod_{l=1}^{j} \ell_l(1/t)^{-1} \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l} \\ &\qquad \times \prod_{l=j+1}^{k-1} \ell_l(1/t)^{-1/q}\, \ell_k(1/t)^{-1/q-\varepsilon} t^{-1/p} \chi_{(0,\delta)}(t) \end{aligned} \tag{3.9}$$

for all $t > 0$. Then by (3.9), we have

$$\begin{aligned} \|f_{\varepsilon,k}\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} &\le C\Big(\int_0^\infty \Big(t^{1/p} \prod_{l=1}^{m} \ell_l(c_l + 1/t)^{\lambda_l} g_{\varepsilon,k}(t)\Big)^q \frac{dt}{t}\Big)^{1/q} \\ &\le C\Big(\int_0^\delta \prod_{l=1}^{k-1} \ell_l(1/t)^{-1} \ell_k(1/t)^{-1-q\varepsilon} \frac{dt}{t}\Big)^{1/q} < \infty, \end{aligned}$$

which implies $f_{\varepsilon,k} \in L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, or equivalently $u_{\varepsilon,k} := G_{n/p} * f_{\varepsilon,k} \in H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. By using L'Hôpital's rule, we see that there exists a small positive constant $\tilde{\delta} < \delta$ such that

$$\begin{aligned} &\int_t^\delta s^{-1/p'} g_{\varepsilon,k}(s)\, ds \\ &\quad\simeq \ell_{j+1}(1/t) \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(1/t)^{-1/q}\, \ell_k(1/t)^{-1/q-\varepsilon} \end{aligned} \tag{3.10}$$

for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates as in (3.8) and using (3.9) and (3.10), for any measurable set E with $|E| < \tilde{\delta}$ we have

$$\int_E |G_{n/p} * f_{\varepsilon,k}(x)|^r\,dx \ge C \int_0^{|E|} \Big(\int_t^\delta s^{-1/p'} g_{\varepsilon,k}(s)\,ds\Big)^r\,dt$$

$$\ge C \int_0^{|E|} \Big(\ell_{j+1}(1/t) \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(1/t)^{-1/q}\,\ell_k(1/t)^{-1/q-\varepsilon}\Big)^r\,dt$$

$$\ge C|E|\Big(\ell_{j+1}(1/|E|) \prod_{l=j+1}^{m} \ell_l(1/|E|)^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(1/|E|)^{-1/q}\,\ell_k(1/|E|)^{-1/q-\varepsilon}\Big)^r,$$

where the last inequality can be derived by noticing that the function

$$\ell_{j+1}(1/t) \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l} \prod_{l=j+1}^{k-1} \ell_l(1/t)^{-1/q}\,\ell_k(1/t)^{-1/q-\varepsilon}$$

is decreasing for small $t > 0$.

Next, assume (C) holds, and define

$$f_{\varepsilon,k}(x) := \prod_{l=1}^{m} \ell_l(1/|x|)^{-1} \times \prod_{l=m+1}^{k} \ell_l(1/|x|)^{-1/q}\,\ell_{k+1}(1/|x|)^{-1/q-\varepsilon}|x|^{-n/p}\chi_{\{x\in\mathbb{R}^n;\,|x|<\delta\}}(x).$$

We have, for some small $\delta > 0$,

$$f^*_{\varepsilon,k}(t) = \tilde f_{\varepsilon,k}((t/\omega_n)^{1/n}) \simeq g_{\varepsilon,k}(t) := \prod_{l=1}^{m} \ell_l(1/t)^{-1} \times \prod_{l=m+1}^{k} \ell_l(1/t)^{-1/q}\,\ell_{k+1}(1/t)^{-1/q-\varepsilon} t^{-1/p}\chi_{(0,\delta)}(t) \tag{3.11}$$

for all $t > 0$. Then by (3.11), we obtain

$$\|f_{\varepsilon,k}\|_{L_{p,q,\lambda_1,\dots,\lambda_m}} \le C\Big(\int_0^\infty \Big(t^{1/p} \prod_{l=1}^{m} \ell_l(c_l + 1/t)^{\lambda_l} g_{\varepsilon,k}(t)\Big)^q \frac{dt}{t}\Big)^{1/q}$$

$$\le C\Big(\int_0^\delta \prod_{l=1}^{k} \ell_l(1/t)^{-1}\ell_{k+1}(1/t)^{-1-q\varepsilon}\,\frac{dt}{t}\Big)^{1/q} < \infty,$$

which implies $f_{\varepsilon,k} \in L_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, or equivalently $u_{\varepsilon,k} := G_{n/p} * f_{\varepsilon,k} \in H^{n/p}_{p,q,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. Moreover there exists a small positive constant $\tilde\delta < \delta$ such that

$$\int_t^\delta s^{-1/p'} g_{\varepsilon,k}(s)\,ds \simeq \ell_{m+1}(1/t) \prod_{l=m+1}^{k} \ell_l(1/t)^{-1/q}\,\ell_{k+1}(1/t)^{-1/q-\varepsilon} \tag{3.12}$$

for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates as in (3.8) and using (3.11) and (3.12), for any measurable set E with $|E| < \tilde{\delta}$ we have

$$\begin{aligned}
&\int_E |G_{n/p} * f_{\varepsilon,k}(x)|^r\,dx \\
&\qquad \geq C \int_0^{|E|} \Big(\int_t^\delta s^{-1/p'} g_{\varepsilon,k}(s)\,ds\Big)^r\,dt \\
&\qquad \geq C \int_0^{|E|} \Big(\ell_{m+1}(1/t) \prod_{l=m+1}^{k} \ell_l(1/t)^{-1/q}\,\ell_{k+1}(1/t)^{-1/q-\varepsilon}\Big)^r\,dt \\
&\qquad \geq C|E|\Big(\ell_{m+1}(1/|E|) \prod_{l=m+1}^{k} \ell_l(1/|E|)^{-1/q}\,\ell_{k+1}(1/|E|)^{-1/q-\varepsilon}\Big)^r,
\end{aligned}$$

where the last inequality can be derived by noticing that the function

$$\ell_{m+1}(1/t) \prod_{l=m+1}^{k} \ell_l(1/t)^{-1/q}\,\ell_{k+1}(1/t)^{-1/q-\varepsilon}$$

is decreasing for small $t > 0$.

We proceed to the case $q = \infty$. If (A) holds, we can argue in the same way as for $q < \infty$, so we omit this subcase.

Next, assume (B) holds, and define

$$f_0(x) := \prod_{l=1}^{j} \ell_l(1/|x|)^{-1} \prod_{l=j+1}^{m} \ell_l(1/|x|)^{-\lambda_l} |x|^{-n/p} \chi_{\{x\in\mathbb{R}^n;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough such that

$$\begin{aligned}
f_0^*(t) &= \tilde{f}_0((t/\omega_n)^{1/n}) \\
&\simeq g_0(t) := \prod_{l=1}^{j} \ell_l(1/t)^{-1} \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l}\, t^{-1/p} \chi_{(0,\delta)}(t)
\end{aligned} \tag{3.13}$$

for all $t > 0$. By (3.13), we obtain $f_0 \in L_{p,\infty,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$, or equivalently $u_0 := G_{n/p} * f_0 \in H^{n/p}_{p,\infty,\lambda_1,\dots,\lambda_m}(\mathbb{R}^n)$. Moreover, there exists a small positive constant $\tilde{\delta} < \delta$ such that

$$\int_t^\delta s^{-1/p'} g_0(s)\,ds \simeq \ell_{j+1}(1/t) \prod_{l=j+1}^{m} \ell_l(1/t)^{-\lambda_l}$$

for all $0 < t < \tilde{\delta}$. Thus by carrying out the same estimates as in (3.8), for any measurable set E with $|E| < \tilde{\delta}$ we have

$$\begin{aligned}\int_E |G_{n/p} * f_0(x)|^r\,dx &\geq C\int_0^{|E|}\Big(\int_t^{\delta} s^{-1/p'} g_0(s)\,ds\Big)^r dt\\ &\geq C\int_0^{|E|}\Big(\ell_{j+1}(1/t)\prod_{l=j+1}^{m}\ell_l(1/t)^{-\lambda_l}\Big)^r dt\\ &\geq C|E|\Big(\ell_{j+1}(1/|E|)\prod_{l=j+1}^{m}\ell_l(1/|E|)^{-\lambda_l}\Big)^r.\end{aligned}$$

Finally, assume (C) holds and define

$$f_0(x) := \prod_{l=1}^{m}\ell_l(1/|x|)^{-1}|x|^{-n/p}\chi_{\{x\in\mathbb{R}^n;\,|x|<\delta\}}(x),$$

where $\delta > 0$ will be taken small enough such that

$$f_0^*(t) = \tilde{f}_0((t/\omega_n)^{1/n}) \simeq g_0(t) := \prod_{l=1}^{m}\ell_l(1/t)^{-1}t^{-1/p}\chi_{(0,\delta)}(t) \tag{3.14}$$

for all $t > 0$. By (3.14), we obtain $f_0 \in L_{p,\infty,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$, or equivalently $u_0 := G_{n/p} * f_0 \in H^{n/p}_{p,\infty,\lambda_1,\ldots,\lambda_m}(\mathbb{R}^n)$. Moreover, there exists a small positive constant $\tilde{\delta} < \delta$ such that $\int_t^{\delta} s^{-1/p'} g_0(s)\,ds \simeq \ell_{m+1}(1/t)$ for all $0 < t < \tilde{\delta}$. Therefore, by carrying out the same estimates as in (3.8), for any measurable set E with $|E| < \tilde{\delta}$ we have

$$\begin{aligned}\int_E |G_{n/p} * f_0(x)|^r\,dx &\geq C\int_0^{|E|}\Big(\int_t^{\delta} s^{-1/p'} g_0(s)\,ds\Big)^r dt\\ &\geq C\int_0^{|E|}\ell_{m+1}(1/t)^r\,dt \geq C|E|\,\ell_{m+1}(1/|E|)^r.\end{aligned}$$

This completes the proof of Theorem 1.6. ∎

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