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Constructing non-compact operators into c_0

by

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Abstract. We prove that for each dense non-compact linear operator $S : X \to Y$ between Banach spaces there is a linear operator $T : Y \to c_0$ such that the operator $TS : X \to c_0$ is not compact. This generalizes the Josefson–Nissenzweig Theorem.

By the Josefson–Nissenzweig Theorem [8], [9] (see also [6], [3], [4, XII], [2], and [7, 3.27] for alternative proofs and generalizations), for each infinite-dimensional Banach space Y the weak* sequential convergence and norm convergence in the dual Banach space Y^* are distinct. This allows us to find a sequence $(y_n^*)_{n\in\omega}$ of norm-one functionals in Y^* that converges to zero in the weak* topology. Such functionals determine a non-compact operator $T : Y \to c_0$ that assigns to each $y \in Y$ the vanishing sequence $(y_n^*(y))_{n\in\omega} \in c_0$. Thus each infinite-dimensional Banach space Y admits a non-compact operator $T : Y \to c_0$ into the Banach space c_0.

The following theorem (which is a crucial ingredient in the topological classification [1] of closed convex sets in Fréchet spaces) says a bit more:

Theorem 1. *For any dense non-compact operator $S : X \to Y$ between Banach spaces there is an operator $T : Y \to c_0$ such that the composition $TS : X \to c_0$ is non-compact.*

By an *operator* we understand a continuous linear operator. An operator $T : X \to Y$ is *dense* if $T(X)$ is dense in Y.

The proof of Theorem 1 uses the famous Rosenthal ℓ_1 Theorem [10] (see also [4, XI] and [2]) saying that any bounded sequence in a Banach space X contains a subsequence which is either weakly Cauchy or ℓ_1-basic.

A sequence $(x_n)_{n\in\omega}$ in a Banach space $(X, \|\cdot\|)$ is called ℓ_1-*basic* if there are constants $0 < c \le C < \infty$ such that for each real sequence $(\alpha_n)_{n\in\omega} \in \ell_1$,

$$c \sum_{n\in\omega} |\alpha_n| \le \Big\| \sum_{n\in\omega} \alpha_n x_n \Big\| \le C \sum_{n\in\omega} |\alpha_n|.$$

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Proof of Theorem 1. Assume that $S : X \to Y$ is a dense non-compact operator. Let $(e_n)_{n\in\omega}$ be the standard Schauder basis of the Banach space c_0 and $(e_n^*)_{n\in\omega}$ be the dual basis in the dual space $c_0^* = \ell_1$. To construct the operator $T : Y \to c_0$ with TS non-compact, we shall consider three cases.

1. First we assume that the following condition holds:

(i) there is an ℓ_1-basic sequence $(y_n^*)_{n\in\omega}$ in Y^* such that the sequence $(S^*y_n^*)_{n\in\omega}$ is ℓ_1-basic and weak* null in X^*.

In this case we define the operator $T : Y \to c_0$ by $T : y \mapsto (y_n^*(y))_{n\in\omega}$. Observe that the dual operator $T^* : c_0^* \to Y^*$ maps the nth coordinate functional $e_n^* \in c_0^*$ to y_n^*. Consequently, the sequence

$$(S^*y_n^*)_{n\in\omega} = ((TS)^*e_n^*)_{n\in\omega},$$

being ℓ_1-basic, is not totally bounded in Y^*, which implies that the dual operator $(TS)^* : c_0^* \to X^*$ is not compact. By the Schauder Theorem [5, 7.7], the operator $TS : X \to c_0$ is not compact either.

2. Assume that the condition (i) does not hold but

(ii) there is an ℓ_1-basic sequence $(y_n^*)_{n\in\omega}$ in Y^* whose image $(S^*y_n^*)_{n\in\omega}$ is ℓ_1-basic in X^*.

In this case, by [4, XII Exercise 3(i)], the condition (ii) combined with the negation of (i) implies the existence of an ℓ_1-basic sequence $(x_n)_{n\in\omega}$ in X whose image $(Sx_n)_{n\in\omega}$ is an ℓ_1-basic sequence in Y. Arguing as in the proof of the Josefson–Nissenzweig Theorem [4, p. 223], we can construct a bounded linear operator $T : Y \to c_0$ such that $T(Sx_n) = e_n \in c_0$ for all $n \in \omega$. Since the operator TS is not compact, we are done.

3. Assume that (ii) does not hold. Since the operator S is not compact, its dual $S^* : Y^* \to X^*$ is not compact either (see [5, 7.7]). This means that the image $S^*(B^*)$ of the closed unit ball $B^* \subset Y^*$ is not totally bounded in X^*. Consequently, B^* contains a sequence $(y_n^*)_{n\in\omega}$ whose image $(S^*y_n^*)_{n\in\omega}$ is ε-separated for some $\varepsilon > 0$. The latter means that $\|S^*(y_n^* - y_m^*)\| \geq \varepsilon$ for all $n \neq m$.

By the Rosenthal ℓ_1 Theorem, $(S^*y_n^*)_{n\in\omega}$ contains a subsequence which is either weak Cauchy or ℓ_1-basic. We lose no generality assuming that the entire sequence $(S^*y_n^*)_{n\in\omega}$ is either weak Cauchy or ℓ_1-basic.

3a. First we assume that the sequence $(S^*y_n^*)_{n\in\omega}$ is weak Cauchy. Then it is weak* Cauchy and being a subset of the weakly* compact set $S^*(B^*)$ it weakly* converges to some point $x_\infty^* \in S^*(B^*)$. Fix any point $y_\infty^* \in B^*$ with $S^*(y_\infty^*) = x_\infty^*$. The density of the operator $S : X \to Y$ implies the injectivity of the dual operator $S^* : Y^* \to X^*$. The weak* compactness of the closed unit ball $B^* \subset Y^*$ guarantees that $S^*|B^* : B^* \to X^*$ is a homeomorphic embedding for the weak* topologies on B^* and X^*. Now we see that the weak* convergence of the sequence $(S^*y_n^*)_{n\in\omega}$ to $S^*y_\infty^*$ implies the weak* convergence of the sequence $(y_n^* - y_\infty^*)_{n\in\omega}$ to zero.

Then the bounded operator $T : Y \to c_0$, $T : y \mapsto \big((y_n^* - y_\infty^*)(y)\big)_{n\in\omega}$, is well-defined. Since the set $\{(TS)^*(e_n^*)\}_{n\in\omega} = \{S^*(y_n^* - y_\infty^*)\}_{n\in\omega}$ is ε-separated, the operator $(TS)^* : c_0^* \to X^*$ is not compact and hence $TS : X \to c_0$ is not compact either.

3b. Finally, assume that $(S^*y_n^*)_{n\in\omega}$ is an ℓ_1-basic sequence in X^*. By [5, Proposition 5.10] (the lifting property of ℓ_1), the sequence $(y_n^*)_{n\in\omega}$ is ℓ_1-basic in Y^*, which contradicts our assumption that the condition (ii) fails. ∎

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