STUDIA MATHEMATICA 208 (1) (2012)

Multidimensional decay in the van der Corput lemma

by

Michael Ruzhansky (London)

Abstract. We establish a multidimensional decay of oscillatory integrals with degenerate stationary points, gaining the decay with respect to all space variables. This bridges the gap between the one-dimensional decay for degenerate stationary points given by the classical van der Corput lemma and the multidimensional decay for non-degenerate stationary points given by the stationary phase method. Complex-valued phase functions as well as phases and amplitudes of limited regularity are considered. Conditions for estimates to be uniform in parameter are also given.

1. Introduction. This paper is devoted to estimates for oscillatory integrals of the type

$$I(\lambda) = \int_{\mathbb{R}^N} e^{i\lambda\Phi(x)} a(x)\, dx,$$

where the support of $a \in C_0^\infty(\mathbb{R}^N)$ is sufficiently small. An estimate for $I(\lambda)$ as $\lambda \to \infty$ is well-known in one dimension $N = 1$ as the van der Corput lemma. If Φ is real-valued and $|\Phi^{(k)}(x)| \geq 1$ on the support of a, then $|I(\lambda)| \leq c_k \lambda^{-1/k}$ for $k \geq 2$, or for $k = 1$ and $\Phi'(x)$ monotonic. In this case the bound c_k is also independent of Φ and λ (see e.g. Sogge [So] or Stein [St]), and the decay rate is sharp. This result plays a crucial role in various areas of analysis. For example, it is closely related to sublevel set estimates of the form

$$\operatorname{meas}\{s \in \operatorname{supp} a : |\Phi(s)| \leq t\} \leq c_k t^{1/k},$$

where Φ is a function as above, with numerous applications in partial differential equations, microlocal analysis, harmonic analysis, etc.

A multidimensional version of these results would be of great value, but presents many difficulties. It is known that for dimensions $N \geq 1$, if, for example, $|\partial^\alpha \Phi| \geq 1$ on $\operatorname{supp} a$, then $|I(\lambda)| \leq c_\alpha |\lambda|^{-1/|\alpha|}$. The decay rate here is sharp, but the constant c_α may depend on Φ and the estimate does not scale well. Again, such an estimate is closely related to the multilinear

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B37, 35L56.
Key words and phrases: van der Corput lemma, stationary phase, sublevel set estimates.

DOI: 10.4064/sm208-1-1

sublevel set problem (see e.g. Phong, Stein and Sturm [PSS]). Certain parameter dependent sublevel set estimates were recently established and used by Kamotski and Ruzhansky [KR] in the analysis of elliptic and hyperbolic systems with multiplicities, to yield Sobolev space estimates for relevant classes of oscillatory integrals and for the solutions.

There are different versions of the van der Corput lemma but still with one-dimensional rate of decay. For example, Christ, Carbery and Wright [CCW] and Carbery and Wright [CW] proposed versions of the van der Corput lemma for functions of several variables, in formulations where also the constant in the estimate is independent of the phase function. This aspect is of significant importance for applications, allowing one to investigate various perturbation and other properties of the integrals. However, there, the decay rate of the corresponding oscillatory integral is always one-dimensional and the non-degeneracy of only one (higher-order) derivative is assumed.

At the same time, the decay rate exhibited in many problems of interest is better than one-dimensional. If one compares this with the case of non-degenerate stationary points of Φ, the stationary phase method will readily yield the decay rate $|I(\lambda)| \leq C\lambda^{-N/2}$. However, if a stationary point degenerates, the situation becomes much more delicate (see e.g. Hörmander [H, Chapter 7]), and no good estimates are available in general.

The aim of the present paper is to bridge the gap between the van der Corput lemma and the estimates provided by the stationary phase method. On one hand, the standard van der Corput lemma works well for degeneracies of high orders but produces only one-dimensional decay rate. On the other hand, the stationary phase method produces a multidimensional decay rate, but does not work well for degenerate stationary points. The result of this paper (Theorem 2.1) was announced in [Ru] without proof in a slightly different setting.

Thus, the main features of the result we are after here are:

- to obtain a multidimensional decay rate, but
- allow degenerate stationary points;
- allow complex-valued phase functions;
- allow dependence on parameters, with estimates, uniform in the parameter;
- allow low regularity phase and amplitude, keeping track of the number of derivatives needed for the estimates.

The result of this paper yields a multidimensional decay rate for degenerate stationary points. We identify a class of functions for which this can be achieved. These functions have certain convexity type properties. It is clear that certain convexity conditions are necessary to ensure the multidimensional decay rate. In fact, conditions of the one-dimensional van der Corput lemma guarantee that the function (or some derivative of the function) is convex in one dimension. Thus, it is natural that an analogue of convexity also appears in several dimensions to ensure that we gain one-dimensional decays in all directions. It is then a question of putting all these rates together to yield a full multidimensional decay, which will turn out to be N times better than the standard van der Corput estimate.

In what follows we will also allow the phase function Φ to be complex-valued and to depend on an arbitrary set of parameters. These two situations often happen in applications to partial differential equations, in particular in the analysis of solutions represented as oscillatory integrals, leading to the dispersive and to the subsequent Strichartz estimates. Thus, the complex phase corresponds to the fact that characteristics of the analysed evolution equations may be complex (see e.g. Trèves [T]). At the same time, the dependence of the phase and of the amplitude on parameters is also essential, and is related to uniform sublevel set estimates. Also, in applications to the Strichartz estimates for hyperbolic equations of high orders considered by Ruzhansky and Smith [RS], a parameter is essential to encode the information on low order perturbations of the equation, in order to establish the dispersive estimates for solutions uniformly over such perturbations. At the same time, in hyperbolic equations with time dependent coefficients (e.g. considered by Matsuyama and Ruzhansky [MR]), the parameter encodes the information on the perturbations of the limiting behaviour of the coefficients, again allowing one to obtain dispersive estimates uniformly over such perturbations. Another such application to hyperbolic systems with oscillating coefficients appeared in Ruzhansky and Wirth [RW]. We will leave out these and other applications outside the scope of this short paper.

We will use the standard multi-index notation: $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$. We will also use the standard convention to denote all constants by C although they may have different values on different occasions.

2. Multidimentional decay of oscillatory integrals. The following theorem is the main result that establishes the multidimensional decay rate for a class of oscillatory integrals.

THEOREM 2.1. *Consider the oscillatory integral*

$$I(\lambda, \nu) = \int_{\mathbb{R}^N} e^{i\lambda\Phi(x,\nu)} a(x, \nu)\chi(x)\, dx,$$

where $N \geq 1$, and ν is a parameter. Let $\gamma \geq 2$ be an integer. Assume that

(A1) *there exists a sufficiently small $\delta > 0$ such that $\chi \in C_0^\infty(B_{\delta/2}(0))$, where $B_{\delta/2}(0)$ is the ball with radius $\delta/2$ around 0;*

(A2) *$\Phi(x,\nu)$ is a complex-valued function such that $\operatorname{Im}\Phi(x,\nu)\geq 0$ for all $x\in\operatorname{supp}\chi$ and all parameters ν;*

(A3) *for some fixed $z\in\operatorname{supp}\chi$, the function*

$$F(\rho,\omega,\nu):=\Phi(z+\rho\omega,\nu),\qquad |\omega|=1,$$

satisfies the following conditions: for each $\mu=(\omega,\nu)$, $F(\cdot,\mu)$ is of class $C^{\gamma+1}$ on $\operatorname{supp}\chi$, and if its γth order Taylor expansion in ρ at 0 is

$$F(\rho,\mu)=\sum_{j=0}^{\gamma}a_j(\mu)\rho^j+R_{\gamma+1}(\rho,\mu),$$

where $R_{\gamma+1}$ is the remainder term, then

(F1) *$a_0(\mu)=a_1(\mu)=0$ for all μ;*

(F2) *there exists a constant $C>0$ such that $\sum_{j=2}^{\gamma}|a_j(\mu)|\geq C$ for all μ;*

(F3) *for each μ, $|\partial_\rho F(\rho,\mu)|$ is increasing in ρ for $0<\rho<\delta$;*

(F4) *for each $k\leq\gamma+1$, $\partial_\rho^k F(\rho,\mu)$ is bounded uniformly in $0<\rho<\delta$ and μ;*

(A4) *for each multi-index α with $|\alpha|\leq[N/\gamma]+1$, there exists a constant $C_\alpha>0$ such that $|\partial_x^\alpha a(x,\nu)|\leq C_\alpha$ for all $x\in\operatorname{supp}\chi$ and all ν.*

Then there exists a constant $C=C_{N,\gamma}>0$ such that

(2.1) $$|I(\lambda,\nu)|\leq C(1+\lambda)^{-N/\gamma}\quad \text{for all } \lambda\in[0,\infty) \text{ and all } \nu.$$

Theorem 2.1 obviously includes the case where a and Φ depend on different sets of parameters. In this case we may let ν run over the whole space of parameters. Also, $[N/\gamma]$ stands for the integer part of N/γ.

In Theorem 2.1, if $\gamma=2$ we recover the decay given by the stationary phase method and if $N=1$ we recover the decay given by the van der Corput lemma.

Condition (F1) is not restrictive, since $a_0(\mu)$ can be taken out of the integral, and non-zero $a_1(\mu)$ would actually give a faster decay rate. Indeed, if $a_1(\mu)\neq 0$, we could integrate by parts under the integral with respect to ρ any number of times, giving a decay in λ of any power. Thus, avoiding this trivial situation, the situation with $a_1(\mu)=0$ is our main interest.

We note that assumption (F3) is of non-strict increase, i.e. we assume that

$$|\partial_\rho F(\rho_1,\mu)|\leq|\partial_\rho F(\rho_2,\mu)|\quad \text{for all } 0<\rho_1<\rho_2<\delta.$$

We also note that assumption (A3), or rather (F3), can be viewed as an analogue of a convexity assumption. Indeed, if F is real-valued, then (F3) implies that the second order derivative $\partial_\rho^2 F(\rho,\mu)$ does not change sign for $0<\rho<\delta$, because $\partial_\rho F(0,\mu)=0$ by (F1). In turn, this is ensured if the

Hesse matrix of second order derivatives $\nabla^2\Phi$ is sign-definite, i.e. $\nabla^2\Phi \geq 0$ or $\nabla^2\Phi \leq 0$ on $\operatorname{supp}\chi$. We note that compared to non-degenerate critical points in the stationary phase method, critical points may degenerate here.

REMARK 2.2. Thus, if Φ is real-valued, in place of (A3) we can make the following assumption for Theorem 2.1 to hold. Assume that for all ν we have $\Phi(\cdot,\nu) \in C^{\gamma+1}$ on $\operatorname{supp}\chi$ and that for some fixed $z \in \operatorname{supp}\chi$ and uniformly for all parameters ν we have

- (F1) $\Phi(z,\nu) = 0$, $\nabla_z\Phi(z,\nu) = 0$;
- (F2) there exists some $C > 0$ such that, for all ω with $|\omega| = 1$, the sum of multilinear forms satisfies

$$\sum_{j=2}^{\gamma} |\nabla_z^j \Phi(z,\nu)(\overbrace{\omega,\ldots,\omega}^{j})| \geq C > 0; \tag{2.2}$$

- (F3) the Hesse matrix $\nabla^2\Phi(\cdot,\nu)$ is non-negative or non-positive on $\operatorname{supp}\chi$ (with the same sign for all ν);
- (F4) $|\nabla_x^k\Phi(x,\nu)| \leq C_k < \infty$ for all $k \leq \gamma+1$ and $x \in \operatorname{supp}\chi$.

Moreover, we note that the statement of the theorem can be easily generalised to the following setting concerning assumption (F3):

REMARK 2.3. Suppose first that Φ is real-valued. If $D^2\Phi$ is not sign-definite on $\operatorname{supp}\chi$, we can restrict to a subspace where it is. Indeed, let V be a d-dimensional affine subspace of $\mathbb{R}^n$ (or a smooth surface if we change variables appropriately, e.g. by the Morse lemma) such that $z \in V$ and $D^2\Phi$ is sign-definite on $V \cap \operatorname{supp}\chi$. Then instead of (2.1) we have

$$|I(\lambda,\nu)| \leq C(1+\lambda)^{-d/\gamma} \quad \text{for all } \lambda \in [0,\infty) \text{ and all } \nu. \tag{2.3}$$

There are different trivial reformulations of this assumption, for example by looking at the number of non-negative and non-positive eigenvalues of $D^2\Phi$ over $\operatorname{supp}\chi$ (or its affine subspace through z).

The same conclusion (2.3) is true if Φ is complex-valued and if we replace (F3) by the assumption that $|\partial_\rho F(\rho,\mu)|$ is increasing in ρ for $0 < \rho < \delta$, for all parameters $\mu = (\omega,\nu)$ with $z + \rho\omega \in V \cap \operatorname{supp}\chi$.

Finally, let us make a remark on the sharpness of estimate (2.1). We can observe that if Φ is a sum of monomials x_j^γ, $j = 1,\ldots,N$, then estimate (2.1) reduces to the standard one-dimensional van der Corput lemma in each j, with each dimension giving a contribution of $(1+\lambda)^{-1/\gamma}$, which is sharp in general.

Proof of Theorem 2.1. It is clear that (2.1) holds for $0 \leq \lambda \leq 1$ since $|I(\lambda,\nu)|$ is bounded for such λ, in view of assumptions (A1), (A2) and (A4). So, we may consider the case where $\lambda \geq 1$. Let $z \in \mathbb{R}^N$ be as in (A3), and set $x = z + \rho\omega$, where $\omega \in \mathbb{S}^{N-1}$, $\rho > 0$. For $N = 1$, we use $\mathbb{S}^0 = \{-1, 1\}$. Then we can write

$$I(\lambda,\nu) = \int_{\mathbb{S}^{N-1}} \int_0^\infty e^{i\lambda\Phi(z+\rho\omega,\nu)} a(z+\rho\omega,\nu)\chi(z+\rho\omega)\rho^{N-1}\, d\rho\, d\omega.$$

It suffices to prove (2.1) for the inner integral.

Choose a function $\theta \in C_0^\infty([0,\infty))$, $0 \le \theta(s) \le 1$ for all s, such that $\theta(s)$ is identically 1 for $0 \le s \le 1/2$ and is identically zero for $s \ge 1$. Then with our notation $F(\rho,\omega,\nu) = \Phi(z+\rho\omega,\nu)$, we split the inner integral into the sum of the two integrals

$$I_1(\lambda,\nu,\omega,z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} a(z+\rho\omega,\nu)\chi(z+\rho\omega)\theta(\lambda^{1/\gamma}\rho)\rho^{N-1}\, d\rho,$$

$$I_2(\lambda,\nu,\omega,z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} a(z+\rho\omega,\nu)\chi(z+\rho\omega)(1-\theta)(\lambda^{1/\gamma}\rho)\rho^{N-1}\, d\rho.$$

Let us first estimate $I_1 = I_1(\lambda,\nu,\omega,z)$. Since $\theta(\lambda^{1/\gamma}\rho) = 0$ for $\lambda^{1/\gamma}\rho \ge 1$, changing variable $\tau = \lambda^{1/\gamma}\rho$ we have

$$|I_1| \le C\int_0^\infty \theta(\lambda^{1/\gamma}\rho)\rho^{N-1}\, d\rho = C\int_0^\infty \tau^{N-1}\lambda^{-(N-1)/\gamma}\theta(\tau)\lambda^{-1/\gamma}\, d\tau,$$

which yields

$$|I_1| \le C\lambda^{-N/\gamma}\int_0^1 \tau^{N-1}\, d\tau \le C\lambda^{-N/\gamma}. \tag{2.4}$$

In order to estimate $I_2 = I_2(\lambda,\nu,\omega,z)$, let us first establish a useful estimate for functions F satisfying condition (F3). We claim that under condition (A3), or rather under (F1)–(F4), there exist constants $C, C_m > 0$ such that

$$|\partial_\rho F(\rho,\mu)| \ge C\rho^{\gamma-1}, \tag{2.5}$$

$$|\partial_\rho^m F(\rho,\mu)| \le C_m\rho^{1-m}|\partial_\rho F(\rho,\mu)|, \tag{2.6}$$

for all $0 < \rho < \delta$, all parameters μ, and all $m \le \gamma + 1$. First, note that for $0 < \rho \le 1$ and $m = \gamma + 1$, estimate (2.6) follows from (2.5) and assumption (F4). So we may only consider $m \le \gamma$.

Now, assumption (F2) implies that

$$\pi(\rho,\mu) := \sum_{j=2}^{\gamma} j|a_j(\mu)|\rho^{j-1} \ge C\rho^{\gamma-1}. \tag{2.7}$$

Thus, in order to prove (2.5), it suffices to show that

$$|\partial_\rho F(\rho,\mu)| \ge C\pi(\rho,\mu) \quad \text{for all } 0 < \rho < \delta \text{ and all } \mu. \tag{2.8}$$

For $1 \le m \le \gamma$, we have, using (A3),

$$\partial_\rho^m F(\rho,\mu) = \sum_{k=0}^{\gamma-m} \frac{(k+m)!}{k!} a_{k+m}(\mu)\rho^k + R_{m,\gamma-m}(\rho,\mu), \tag{2.9}$$

where

$$R_{m,\gamma-m}(\rho,\mu) = \int_0^\rho \partial_s^{\gamma+1} F(s,\mu) \frac{(\rho-s)^{\gamma-m}}{(\gamma-m)!}\, ds$$

is the remainder term of the $(\gamma-m)$th Taylor expansion of $\partial_\rho^m F(\rho,\mu)$. By (F4) and (2.7), we get

$$|R_{m,\gamma-m}(\rho,\mu)| \le C_{\gamma,m}\rho^{\gamma-m+1} \le C_{\gamma,m}\pi(\rho,\mu)\rho^{2-m} \quad \text{for } 0<\rho<\delta. \tag{2.10}$$

Hence, for $0<\rho<\delta$, we have

$$\begin{aligned} |\partial_\rho F(\rho,\mu)| &= \Big|\sum_{k=0}^{\gamma-1}(k+1)a_{k+1}(\mu)\rho^k + R_{1,\gamma-1}(\rho,\mu)\Big| \\ &\ge \Big|\sum_{j=2}^{\gamma} j a_j(\mu)\rho^{j-1}\Big| - |R_{1,\gamma-1}(\rho,\mu)| \\ &\ge \Big|\sum_{j=2}^{\gamma} j a_j(\mu)\rho^{j-1}\Big| - C_\gamma \pi(\rho,\mu)\rho. \end{aligned}$$

It now follows from assumptions (F1) and (F3) that

$$\begin{aligned} |\partial_\rho F(\rho,\mu)| &= \max_{0\le\sigma\le\rho} |\partial_\rho F(\sigma,\mu)| \\ &\ge \max_{0\le\sigma\le\rho}\Big|\sum_{j=2}^{\gamma} j a_j(\mu)\sigma^{j-1}\Big| - \max_{0\le\sigma\le\rho} C_\gamma\pi(\sigma,\mu)\sigma \\ &= \max_{0\le\bar\sigma\le1}\Big|\sum_{j=2}^{\gamma} j a_j(\mu)\rho^{j-1}\bar\sigma^{j-1}\Big| - C_\gamma\pi(\rho,\mu)\rho, \end{aligned}$$

since $\pi(\sigma,\mu)\sigma = \sum_{j=2}^{\gamma} j|a_j(\mu)|\sigma^j$ achieves its maximum on $0\le\sigma\le\rho$ at $\sigma=\rho$. Noting that

$$\max_{0\le\bar\sigma\le1}\Big|\sum_{j=2}^{\gamma} z_j\bar\sigma^{j-1}\Big| \quad \text{and} \quad \sum_{j=2}^{\gamma}|z_j|$$

are both norms on $\mathbb{C}^{\gamma-1}$, and hence are equivalent, we immediately get

$$\begin{aligned} |\partial_\rho F(\rho,\mu)| &\ge C\sum_{j=2}^{\gamma} j|a_j(\mu)|\rho^{j-1} - C_\gamma\pi(\rho,\mu)\rho \\ &\ge (C - C_\gamma\delta)\pi(\rho,\mu) \ge C\pi(\rho,\mu) \end{aligned}$$

for some constant $C>0$ if δ is sufficiently small. This completes the proof of (2.8).

To prove (2.6), we will use the representation (2.9). Since $1\le m\le\gamma$, it follows from the definition of $\pi(\rho,\mu)$ that

$$\left|\sum_{k=0}^{\gamma-m}\frac{(k+m)!}{k!}a_{k+m}(\mu)\rho^k\right|\le C_m\pi(\rho,\mu)\rho^{1-m},$$

which, together with (2.10) and (2.8), yields

$$|\partial_\rho^m F(\rho,\mu)|\le C_{m,\delta}\rho^{1-m}|\partial_\rho F(\rho,\mu)|\quad \text{for } 0<\rho<\delta.$$

This completes the proof of the claimed estimates (2.5) and (2.6).

Let us now come back to the estimate for I_2. Define the operator

$$L:=(i\lambda\partial_\rho F(\rho,\omega,\nu))^{-1}\frac{\partial}{\partial\rho},$$

which clearly satisfies the useful identity $L(e^{i\lambda F(\rho,\omega,\nu)})=e^{i\lambda F(\rho,\omega,\nu)}$. Denoting the adjoint of L by L^*, we have, for each $l\in\mathbb{N}\cup\{0\}$,

$$I_2=\int_0^\infty e^{i\lambda F(\rho,\omega,\nu)}(L^*)^l[a(z+\rho\omega,\nu)\chi(z+\rho\omega)(1-\theta)(\lambda^{1/\gamma}\rho)\rho^{N-1}]\,d\rho.$$

Now,

$$(L^*)^l=\left(\frac{i}{\lambda}\right)^l\sum C_{s_1,\dots,s_p,p,r,l}\frac{\partial_\rho^{s_1}F\cdots\partial_\rho^{s_p}F}{(\partial_\rho F)^{l+p}}(\rho,\omega,\nu)\frac{\partial^r}{\partial\rho^r},$$

where the sum is over all integers $s_1,\dots,s_p,p,r\ge 0$ such that $s_1+\dots+s_p+r-p=l$. From (2.5) and (2.6) it follows that

$$\left|\frac{\partial_\rho^{s_1}F\dots\partial_\rho^{s_p}F}{(\partial_\rho F)^{l+p}}(\rho,\omega,\nu)\right|\le C\rho^{p-s_1-\dots-s_p-l\gamma+l}=C\rho^{r-l\gamma}.$$

Also, it is easy to see that for $r\le[N/\gamma]+1$, we have

$$\left|\frac{\partial^r}{\partial\rho^r}[a(z+\rho\omega,\nu)\chi(z+\rho\omega)(1-\theta)(\lambda^{1/\gamma}\rho)\rho^{N-1}]\right|\le C_N\rho^{N-1-r}\widetilde{\chi}(\lambda,\rho),\tag{2.11}$$

where $\widetilde{\chi}(\lambda,\rho)$ is a smooth function in ρ which is zero for $\lambda^{1/\gamma}\rho<1/2$. Let us now take $l=[N/\gamma]+1$, so that $N-l\gamma<0$. Then we can estimate

$$\begin{aligned}|I_2|&\le C_N\lambda^{-l}\int_0^\infty\sum C_{s_1,\dots,s_p,p,r,l}\,\rho^{r-l\gamma}\,\rho^{N-1-r}\,\widetilde{\chi}(\lambda,\rho)\,d\rho\\&\le C_N\lambda^{-l}\int_{\frac12\lambda^{-1/\gamma}}^\infty\rho^{N-1-l\gamma}\,d\rho=C_N\lambda^{-l}\left[\frac{\rho^{N-l\gamma}}{N-l\gamma}\right]_{\frac12\lambda^{-1/\gamma}}^\infty=C_{N,\gamma}\lambda^{-N/\gamma}.\end{aligned}$$

Combining this estimate with estimate (2.4) for I_1, we obtain the desired estimate (2.1). This completes the proof of the theorem. ■

We note that in the proof we showed that if F satisfies conditions (F1)–(F4), it also satisfies estimates (2.5) and (2.6). A version of this part of the argument was discussed by Sugimoto [Su] for real-valued analytic functions without dependence on μ, where the analysis was based on Cauchy's integral formula for analytic functions (see also Randol [Ra] and Beals [B]). The proof that we give for (2.5) and (2.6) works in the generality required for Theorem 2.1, tracing the parameter, but more importantly, eliminating the Cauchy integral argument, which allows us to relax the analyticity assumptions.

In fact, let us also briefly indicate a smooth version of these estimates. Suppose that a function $F(\cdot,\mu)$ is smooth in the first variable, and that it satisfies conditions (F1)–(F3), as well as condition (F4) for all $m\in\mathbb{N}$. Then we claim that for sufficiently small $\delta>0$, estimates (2.5) and (2.6) are satisfied also for all $m\in\mathbb{N}$.

Indeed, we already proved (2.5) and we also proved (2.6) for $m\le\gamma$. It remains to consider the case $m>\gamma$. Since $\gamma+1-m\le 0$, from (F4) it trivially follows that for $0<\rho<\delta$ we have a stronger estimate

$$|\partial_\rho^m F(\rho,\mu)|\le C_m\le C_{m,\delta}\rho^{\gamma+1-m}\le C_{m,\delta}\rho^{2-m}|\partial_\rho F(\rho,\mu)|,$$

where the last inequality is a consequence of (2.5).

Acknowledgements. The author thanks James Smith for discussions, especially on applications of this result in [RS].

The author was supported by the Leverhulme Research Fellowship and by EPSRC grant EP/E062873/01. Subsequent work was supported by the EPSRC Leadership Fellowship EP/G007233/1.

References

[B] R. M. Beals, *L^p boundedness of Fourier integral operators*, Mem. Amer. Math. Soc. 38 (1982), no. 264.

[CCW] A. Carbery, M. Christ and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. 12 (1999), 981–1015.

[CW] A. Carbery and J. Wright, *What is van der Corput's lemma in higher dimensions?*, in: Proc. 6th Int. Conf. on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. 2002, Vol. Extra, 13–26.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, Springer, Berlin, 1983.

[KR] I. Kamotski and M. Ruzhansky, *Regularity properties, representation of solutions and spectral asymptotics of systems with multiplicities*, Comm. Partial Differential Equations 32 (2007), 1–35.

[MR] T. Matsuyama and M. Ruzhansky, *Asymptotic integration and dispersion for hyperbolic equations*, Adv. Differential Equations 15 (2010), 721–756.

[PSS] D. H. Phong, E. M. Stein and J. Sturm, *Multilinear level set operators, oscillatory integral operators, and Newton polyhedra*, Math. Ann. 319 (2001), 573–596.

[Ra] B. Randol, *On the asymptotic behavior of the Fourier transform of the indicator function of a convex set*, Trans. Amer. Math. Soc. 139 (1969), 279–285.

[Ru] M. Ruzhansky, *Pointwise van der Corput lemma for functions of several variables*, Funct. Anal. Appl. 43 (2009), 75–77.

[RS] M. Ruzhansky and J. Smith, *Dispersive and Strichartz estimates for hyperbolic equations with constant coefficients*, MSJ Mem. 22, Math. Soc. Japan, Tokyo, 2010.

[RW] M. Ruzhansky and J. Wirth, *Dispersive estimates for hyperbolic systems with time-dependent coefficients*, J. Differential Equations 251 (2011), 941–969.

[So] C. D. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge Tracts in Math. 105, Cambridge Univ. Press, Cambridge, 1993.

[St] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, NJ, 1993.

[Su] M. Sugimoto, *A priori estimates for higher order hyperbolic equations*, Math. Z. 215 (1994), 519–531.

[T] F. Trèves, *Introduction to Pseudo-Differential and Fourier Integral Operators*, Vol. II, Plenum Press, 1982.

Michael Ruzhansky
Department of Mathematics
Imperial College London
180 Queen's Gate
London SW7 2AZ, United Kingdom
E-mail: m.ruzhansky@imperial.ac.uk

Received April 8, 2011
Revised version December 19, 2011 (7161)