

A Paley–Wiener type theorem for generalized non-quasianalytic classes

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Abstract. Let P be a hypoelliptic polynomial. We consider classes of ultradifferentiable functions with respect to the iterates of the partial differential operator $P(D)$ and prove that such classes satisfy a Paley–Wiener type theorem. These classes and the corresponding test spaces are nuclear.

1. Introduction. A *Paley–Wiener type theorem* is any theorem that deals with decay properties of the Fourier transform of a function or distribution.

Paley–Wiener type theorems and their applications to several areas of mathematical analysis have been studied in several settings: we refer for example to [BH], [BMT] and [DGM].

In this paper we continue the research begun in [FGJ, JH] on generalized non-quasianalytic classes, that is, classes of ultradifferentiable functions with respect to the iterates of a partial differential operator. Our aim is to establish a Paley–Wiener type theorem for such classes.

Classes of \mathcal{C}^∞ -functions defined in terms of the successive iterates of a partial differential operator appeared in 1960, when Komatsu [K], using tools introduced by Hörmander [H1], characterized when a smooth function $f \in \mathcal{C}^\infty(\Omega)$ in an open subset $\Omega \subset \mathbb{R}^N$ is real analytic in terms of the successive iterates of an elliptic partial differential operator $P(D)$. Kotake and Narasimhan extended this result to elliptic operators with analytic coefficients. See [KN, Theorem 1].

In 1973, Newberger and Zielezny [NZ] treated this problem in the setting of Gevrey classes. Research in this direction has been continued intensively by several authors, including Bolley, Camus, Rodino [BC, BCR], Langenbruch [L1–L4] and Zanghirati [Z1–Z4]. We also mention the recent contribu-

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tions by Bouzar and Chaili [BC1], Calvo and Hakobyan [CH] and Calvo and Rodino [CR]. Microlocal versions of this topic have been considered by Bolley, Camus and Mattera [BCM], Zanghirati [Z4], Bouzar and Chaili [BC3] and others.

All this research is related to the problem of iterates which consists in characterizing the functions in a given class of functions in terms of the behavior of the iterates of a fixed differential operator. See [BC1], [M], [K], [NZ] and also [BC2], [CR], [KN] and [Z3]. The author [JH] has studied the problem of iterates in the more general setting of classes of ultradifferentiable functions in the sense of Braun, Meise and Taylor.

The precise definition of the spaces of ultradifferentiable functions with respect to the iterates of P will be given in Section 2. In Section 3, we prove a Paley–Wiener type theorem for such classes. In Section 4 we prove that the spaces $\mathcal{E}_{P,(\omega)}(\Omega)$ and $\mathcal{E}_{P,\{\omega\}}(\Omega)$ and the corresponding test spaces $\mathcal{D}_{P,(\omega)}(\Omega)$ and $\mathcal{D}_{P,\{\omega\}}(\Omega)$ are nuclear whenever the polynomial P is hypoelliptic. The results in Sections 3 and 4 are new, even for the Gevrey case.

2. Generalized non-quasianalytic classes: $\mathcal{E}_{P,\omega}(\Omega)$. We follow the point of view of Braun–Meise–Taylor (see [BMT]). A *non-quasianalytic weight function* is an increasing continuous function $\omega : [0, \infty[\rightarrow [0, \infty[$ with the following properties:

- (α) There exists $L \geq 0$ with $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$.
- (β) $\int_1^\infty (\omega(t)/t^2) dt < \infty$.
- (γ) $\ln t = o(\omega(t))$ as t tends to ∞ , that is, $\lim_{t \rightarrow \infty} (\ln t)/\omega(t) = 0$.
- (δ) $\varphi : t \mapsto \omega(e^t)$ is convex.

We may assume that $\omega|_{[0,1]} \equiv 0$. The *Young conjugate* of φ is given by $\varphi^*(s) := \sup\{st - \varphi(t) : t \geq 0\}$. We refer the reader to [BMT] and [JH] for examples of weight functions and the definition of the spaces of ultradifferentiable functions and ultradistributions.

In what follows, Ω denotes an arbitrary open subset of \mathbb{R}^N , and $K \subset\subset \Omega$ means that K is a compact subset in Ω .

Following [JH], we consider smooth functions on an open set Ω such that there exists $C > 0$ such that for each $j \in \mathbb{N}_0$,

$$\|P^j(D)f\|_{2,K} \leq C \exp(\lambda\varphi^*(j/\lambda)),$$

where K is a compact subset in Ω , $\|\cdot\|_{2,K}$ denotes the \mathcal{L}^2 -norm on K and $P^j(D)$ is the j th iterate of the partial differential operator $P(D)$, i.e.,

$$P^j(D) = P(D) \underbrace{\circ \cdots \circ}_j P(D).$$

If $j = 0$, then

$$P^0(D)f = f.$$

Given a polynomial $P \in \mathbb{C}[z_1, \dots, z_N]$ of degree m ,

$$P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha,$$

the partial differential operator $P(D)$ is

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad \text{where } D^\alpha = \frac{1}{i} \partial^\alpha.$$

The spaces of ultradifferentiable functions with respect to the successive iterates of P are defined as follows.

Let ω be a weight function. Given a polynomial P , an open set Ω in \mathbb{R}^N , a compact subset $K \subset\subset \Omega$ and $\lambda > 0$, we define the seminorm

$$\|f\|_{K,\lambda} := \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp(-\lambda \varphi^*(j/\lambda))$$

and set

$$\mathcal{E}_{P,\omega}^\lambda(K) = \{f \in C^\infty(K) : \|f\|_{K,\lambda} < \infty\}.$$

$\mathcal{E}_{P,\omega}^\lambda(K)$ is a Banach space when endowed with the $\|\cdot\|_{K,\lambda}$ -norm.

The space of *ultradifferentiable functions of Beurling type with respect to the iterates of P* is

$$\mathcal{E}_{P,(\omega)}(\Omega) = \{f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < \infty \text{ for each } K \subset\subset \Omega \text{ and } \lambda > 0\}.$$

It is endowed with the topology given by

$$\mathcal{E}_{P,(\omega)}(\Omega) := \text{proj}_{K \subset\subset \Omega} \text{proj}_{\lambda > 0} \mathcal{E}_{P,\omega}^\lambda(K).$$

If $\{K_n\}_{n \in \mathbb{N}}$ is a compact exhaustion of Ω , this metrizable locally convex topology is defined by the fundamental system $\{\|\cdot\|_{K_n,n}\}_{n \in \mathbb{N}}$ of seminorms.

The space of *ultradifferentiable functions of Roumieu type with respect to the iterates of P* is defined by

$$\mathcal{E}_{P,\{\omega\}}(\Omega) = \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega \exists \lambda > 0 \text{ such that } \|f\|_{K,\lambda} < \infty\}.$$

Its topology is defined by

$$\mathcal{E}_{P,\{\omega\}}(\Omega) := \text{proj}_{K \subset\subset \Omega} \text{ind}_{\lambda > 0} \mathcal{E}_{P,\omega}^\lambda(K).$$

We write $\mathcal{E}_{P,\omega}(\Omega)$ when a statement holds in the Beurling and the Roumieu case. As in the Gevrey case, we call these classes *generalized non-quasianalytic classes*. The author has proved that the space $\mathcal{E}_{P,\omega}(\Omega)$ is complete if and only if P is hypoelliptic (see [JH, Theorem 3.3]). See [FGJ] and [JH] for more information and details.

3. A Paley–Wiener type theorem for $\mathcal{E}_{P,\omega}(\Omega)$. Let ω be a weight function and $m \geq 1$. It is easy to prove that $\sigma(t) := \omega(t^{1/m})$ is also a weight function. Moreover, $\varphi_\sigma^*(x) = \varphi_\omega^*(mx)$.

Suppose P is a hypoelliptic polynomial of order m . Our aim is to establish a Paley–Wiener type theorem for the generalized non-quasianalytic class $\mathcal{E}_{P,\omega(t^{1/m})}(\Omega)$. In order to guarantee the existence of compactly supported functions in this space we recall that the class of ultradifferentiable functions $\mathcal{E}_\omega(\Omega)$ is always contained in $\mathcal{E}_{P,\omega(t^{1/m})}(\Omega)$ where m is the degree of P (see [JH, Theorem 4.1]). As a consequence, $\mathcal{D}_\omega(\Omega)$ is a subset of $\mathcal{E}_{P,\omega(t^{1/m})}(\Omega)$.

Let K be a convex compact subset of \mathbb{R}^N . The supporting function of K is the function $H_K : \mathbb{R}^N \rightarrow \mathbb{R}$ given by $H_K(x) := \sup_{y \in K} x \cdot y$.

LEMMA 3.1. *Let ω a weight function, P a polynomial and $f \in \mathcal{C}^\infty(\mathbb{R}^N)$. The following statements hold:*

(1) *If there is $\lambda > 0$ satisfying*

$$C := \left(\int_{\mathbb{R}^N} |\widehat{f}(\xi)|^2 \exp(\lambda\omega(|P(\xi)|)) d\xi \right)^{1/2} < \infty,$$

then

$$(*) \quad \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{2,\mathbb{R}^N} \exp\left(-\frac{\lambda}{2}\varphi^*\left(\frac{2j}{\lambda}\right)\right) \leq \frac{C}{(2\pi)^{N/2}}.$$

(2) *Let K be a compact convex subset of \mathbb{R}^N and denote by $m(K)$ its Lebesgue measure. There is a constant $D > 0$ (depending on λ and ω) such that if $(*)$ holds and f has compact support contained in K , then for all $z \in \mathbb{C}^N$,*

$$|\widehat{f}(z)| \leq m(K)^{1/2} \frac{CD}{(2\pi)^{N/2}} \exp\left(H_K(\operatorname{Im} z) - \frac{\lambda}{4}\omega(|P(z)|)\right).$$

Proof. (1) By Plancherel's Theorem,

$$\begin{aligned} \|P^j(D)f\|_{2,\mathbb{R}^N} &= \frac{1}{(2\pi)^{N/2}} \|P(\xi)^j \widehat{f}(\xi)\|_{2,\mathbb{R}^N} \\ &\leq \frac{C}{(2\pi)^{N/2}} \sup_{|P(\xi)| \neq 0} \exp\left(j \ln |P(\xi)| - \frac{\lambda}{2}\omega(|P(\xi)|)\right) \leq \frac{C}{(2\pi)^{N/2}} \exp\left(\frac{\lambda}{2}\varphi^*\left(\frac{j}{\lambda/2}\right)\right). \end{aligned}$$

(2) By the Hölder inequality,

$$\begin{aligned} (3.1) \quad |\widehat{P^j(D)f}(z)| &\leq \int_K |P^j(D)f(t) \exp(-itz)| dt \\ &\leq m(K)^{1/2} \exp(H_K(\operatorname{Im} z)) \frac{C}{(2\pi)^{N/2}} \exp\left(\frac{\lambda}{2}\varphi^*\left(\frac{j}{\lambda/2}\right)\right). \end{aligned}$$

Suppose $|P(z)| > 1$; then

$$|\widehat{f}(z)| \leq m(K)^{1/2} \frac{C}{(2\pi)^{N/2}} \exp(H_K(\operatorname{Im} z)) \exp\left(\frac{\lambda}{2}\varphi^*\left(\frac{j}{\lambda/2}\right)\right) \frac{1}{|P(z)|^j}.$$

Now, we use $\varphi^{**} = \varphi$ and condition (γ) of the definition of weight functions to find t_0 such that

$$\ln t \leq \frac{\lambda}{4}\omega(t) + \ln t_0 \quad \forall t > 0.$$

Then

$$\begin{aligned} & \sup_{j \in \mathbb{N}_0} \left(j \ln |P(z)| - \frac{\lambda}{2} \varphi^* \left(\frac{j}{\lambda/2} \right) \right) \\ &= \frac{\lambda}{2} \sup_{j \in \mathbb{N}_0} \left(\frac{j+1}{\lambda/2} \ln |P(z)| - \varphi^* \left(\frac{j}{\lambda/2} \right) \right) - \ln |P(z)| \\ &\geq \frac{\lambda}{2} \sup_{x \geq 0} (x \ln |P(z)| - \varphi^*(x)) - \ln |P(z)| = \frac{\lambda}{2} (\varphi^{**}(\ln |P(z)|) - \ln |P(z)|) \\ &= \frac{\lambda}{2} \omega(|P(z)|) - \ln |P(z)| \geq \frac{\lambda}{4} \omega(|P(z)|) - \ln t_0. \end{aligned}$$

Taking the infimum we obtain

$$|\widehat{f}(z)| \leq m(K)^{1/2} \frac{C}{(2\pi)^{N/2}} \exp(H_K(\operatorname{Im} z)) t_0 \exp\left(-\frac{\lambda}{4} \omega(|P(z)|)\right).$$

In case $|P(z)| \leq 1$, the previous inequality is also true. To see this, recall that $\omega(|P(z)|) = 0$ and $\varphi^*(0) = 0$ and take $j = 0$ in (3.1). ■

We now introduce the following notation:

$$\begin{aligned} \mathcal{D}_{P,\omega}^\lambda(K) &= \{f \in \mathcal{C}^\infty(\mathbb{R}^N) : \operatorname{supp} f \subset K \text{ and } \|f\|_{K,\lambda} < \infty\}, \\ \mathcal{D}_{P,(\omega)}(K) &= \operatorname{proj}_{\lambda > 0} \mathcal{D}_{P,\omega}^\lambda(K) \quad \text{and} \quad \mathcal{D}_{P,\{\omega\}}(K) = \operatorname{ind}_{\lambda > 0} \mathcal{D}_{P,\omega}^\lambda(K). \end{aligned}$$

Let Ω be an open subset of \mathbb{R}^N and let ω be a weight function. We define the *test spaces of ultradifferentiable functions with respect to the iterates of the operator P* as

$$\mathcal{D}_{P,(\omega)}(\Omega) = \operatorname{ind}_{K \subset\subset \Omega} \operatorname{proj}_{\lambda > 0} \mathcal{D}_{P,\omega}^\lambda(K) \quad \text{and} \quad \mathcal{D}_{P,\{\omega\}}(\Omega) = \operatorname{ind}_{K \subset\subset \Omega} \operatorname{ind}_{\lambda > 0} \mathcal{D}_{P,\omega}^\lambda(K).$$

Given $\lambda > 0$, consider the seminorm

$$t_\lambda(f) := \left(\int_{\mathbb{R}^N} |\widehat{f}(\xi)|^2 \exp(\lambda \omega(|P(\xi)|)) d\xi \right)^{1/2}.$$

PROPOSITION 3.2. *Suppose P is hypoelliptic. Then the fundamental systems $\{\|\cdot\|_{K,\lambda}\}_{\lambda > 0}$ and $\{t_\lambda(\cdot)\}_{\lambda > 0}$ of seminorms on $\mathcal{D}_{P,(\omega)}(K)$ are equivalent.*

Proof. From Lemma 3.1(1) it is clear that

$$\|f\|_{K,\lambda} \leq \frac{1}{(2\pi)^{N/2}} t_{2\lambda}(f) \quad \forall \lambda > 0, \forall f \in \mathcal{D}_{P,(\omega)}(K).$$

In order to see the other inequality, we take $z = \xi \in \mathbb{R}^N$ and $C/(2\pi)^{N/2} = \|f\|_{K,\lambda/2}$ in Lemma 3.1(2) to obtain

$$|\widehat{f}(\xi)| \leq m(K)^{1/2} D \|f\|_{K,\lambda/2} \exp\left(-\frac{\lambda}{4}\omega(|P(\xi)|)\right).$$

Then

$$|\widehat{f}(\xi)|^2 \exp\left(\frac{\lambda}{4}\omega(|P(\xi)|)\right) \leq m(K) D^2 \|f\|_{K,\lambda/2}^2 \exp\left(-\frac{\lambda}{4}\omega(|P(\xi)|)\right).$$

Therefore,

$$t_{\lambda/4}(f) \leq m(K)^{1/2} D \|f\|_{K,\lambda/2} \left(\int_{\mathbb{R}^N} \exp\left(-\frac{\lambda}{4}\omega(|P(\xi)|)\right) d\xi \right)^{1/2}.$$

Now, we only have to check that $\exp(-(\lambda/4)\omega(|P(\xi)|))$ is integrable. Condition IIb of [H2, Theorem 11.1.3] asserts that there exist $D, d > 0$ such that

$$|P(\xi)| \geq D|\xi|^d \quad \text{if } |\xi| \text{ is large enough.}$$

This inequality and condition (γ) of the definition of weight functions allow one to prove that

$$\frac{\lambda}{4}\omega(|P(\xi)|) \geq \ln(1 + |\xi|^2)^N$$

if $|\xi|$ is large enough, and the conclusion follows. ■

Denote $B_A := \{x \in \mathbb{R}^N : |z| \leq A\}$.

THEOREM 3.3. *Let P be a hypoelliptic polynomial and ω a weight function. An entire function $F \in \mathcal{H}(\mathbb{C}^N)$ is the Fourier–Laplace transform of a function $f \in \mathcal{D}_{P,(\omega)}(B_A)$ if, and only if,*

$$|F(z)| \leq C e^{A|z|} \quad \forall z \in \mathbb{C}^N$$

for some constants $C, A > 0$ and moreover, for every $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |F(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

Proof. Let $f \in \mathcal{D}_{P,(\omega)}(\mathbb{R}^N)$ with $\text{supp } f \subset B_A$. By the classical Paley–Wiener theorem [H2, I, Theorem 7.3.1] there is a constant $C > 0$ such that $|\widehat{f}(z)| \leq C e^{A|z|}$ for all $z \in \mathbb{C}^N$. Proposition 3.2 shows that for each $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

Conversely, suppose that F is an entire function in \mathbb{C}^N satisfying the two conditions of the theorem. In particular $\int_{\mathbb{R}^N} |F(x)|^2 dx < \infty$. Thus,

[S, Theorem 4.9] gives a function $f \in \mathcal{L}^2(B_A)$ with $\text{supp } f \subset B_A$ such that $F(z) = \int_{B_A} f(x)e^{-ixz} dx$. Note that for each $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

By Proposition 3.2, in order to show that $f \in \mathcal{D}_{P,(\omega)}(\mathbb{R}^N)$ it suffices to check that we can take f in $\mathcal{C}^\infty(\mathbb{R}^N)$.

Proceeding as in Proposition 3.2, we find that the function $e^{-\frac{\lambda}{2}\omega(|P(x)|)}$ is in \mathcal{L}^2 , therefore Hölder’s inequality and the previous estimate imply $\widehat{f} \in \mathcal{L}^1$ and by the inversion formula (see [R, Theorem 9.14])

$$f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{f}(\xi)e^{ix\xi} d\xi \quad \text{a.e.}$$

In order to see that the function

$$g(x) = \int_{\mathbb{R}^N} \widehat{f}(\xi)e^{ix\xi} d\xi$$

is \mathcal{C}^∞ it is enough to show that for each $\alpha \in \mathbb{N}_0^N$, the function $\xi^\alpha \widehat{f}(\xi)e^{ix\xi}$ is integrable. Observe that

$$|\xi^{(\alpha)}| |\widehat{f}(\xi)| \leq \sqrt{N}^{|\alpha|} e^{\ln|\xi|^{|\alpha|}} |\widehat{f}(\xi)|.$$

If $|\xi|$ is large enough, condition IIb of [H2, II, Theorem 11.1.3] and condition (γ) of the definition of weight function imply

$$|\xi^{(\alpha)}| |\widehat{f}(\xi)| \leq \sqrt{N}^{|\alpha|} e^{\frac{\lambda}{4}\omega(|P(\xi)|)} |\widehat{f}(\xi)| = \sqrt{N}^{|\alpha|} |\widehat{f}(\xi)| e^{\frac{\lambda}{2}\omega(|P(\xi)|)} e^{-\frac{\lambda}{4}\omega(|P(\xi)|)},$$

which is integrable by the Hölder inequality. Thus, the derivative

$$g^{(\alpha)}(x) = \int_{\mathbb{R}^N} \xi^\alpha \widehat{f}(\xi)e^{ix\xi} d\xi$$

exists. ■

Analogously, one can prove the Roumieu case.

THEOREM 3.4. *Let P be a hypoelliptic polynomial and ω a weight function. An entire function $F \in \mathcal{H}(\mathbb{C}^N)$ is the Fourier–Laplace transform of a function $f \in \mathcal{D}_{P,\{\omega\}}(B_A)$ if, and only if,*

$$|F(z)| \leq Ce^{A|z|} \quad \forall z \in \mathbb{C}^N$$

for some constants $C, A > 0$, and for some $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |F(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

Using the classical Paley–Wiener theorem and Proposition 3.2 we obtain the following corollary.

COROLLARY 3.5. *P be a hypoelliptic polynomial and ω a weight function. Let K be a convex compact subset and $f \in \mathcal{D}_{P,(\omega)}(K)$. Then the Fourier–Laplace transform of f is an entire function and for every N there is a constant C_N such that*

$$|\widehat{f}(z)| \leq C_N(1 + |z|)^{-N} e^{H_K(\operatorname{Im} z)} \quad \forall z \in \mathbb{C}^N,$$

and moreover, for every $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

Conversely, every entire function F satisfying the last two inequalities is the Fourier–Laplace transform of a function in $\mathcal{D}_{P,(\omega)}(K)$.

Analogously, one can handle the Roumieu case.

COROLLARY 3.6. *Let P be a hypoelliptic polynomial and ω a weight function. Let K be a convex compact subset and $f \in \mathcal{D}_{P,\{\omega\}}(K)$. Then the Fourier–Laplace transform of f is an entire function and for every N there is a constant C_N such that*

$$|\widehat{f}(z)| \leq C_N(1 + |z|)^{-N} e^{H_K(\operatorname{Im} z)} \quad \forall z \in \mathbb{C}^N,$$

and moreover, for some $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x)|)) dx \right)^{1/2} < \infty.$$

Conversely, every entire function F satisfying the last two inequalities is the Fourier–Laplace transform of a function in $\mathcal{D}_{P,(\omega)}(K)$.

4. The nuclearity of $\mathcal{E}_{P,\omega}(\Omega)$ and $\mathcal{D}_{P,\omega}(\Omega)$. Let P be a hypoelliptic polynomial. Given $\lambda > 0$, we introduce the seminorm

$$s_\lambda(f) = \int_{\mathbb{R}^N} |\widehat{f}(\xi)| \exp(\lambda\omega(|P(\xi)|)) d\xi.$$

Given a compact subset K of Ω we define

$$\mathcal{D}_{1,P,(\omega)}(K) = \{f \in \mathcal{C}^\infty(\mathbb{R}^N) : \operatorname{supp} f \subset K \text{ and } \forall \lambda > 0, s_\lambda(f) < \infty\}.$$

Since $f \in \mathcal{L}^1$, its Fourier transform is bounded. Then

$$t_\lambda(f) \leq \|\widehat{f}\|_\infty^{1/2} (s_\lambda(f))^{1/2}.$$

Thus, $\mathcal{D}_{1,P,(\omega)}(K)$ is a subset of $\mathcal{D}_{P,(\omega)}(K)$. On the other hand, we write

$$\widehat{f}(\xi) \exp(\lambda\omega(|P(\xi)|)) = \widehat{f}(\xi) \exp(2\lambda\omega(|P(\xi)|)) \exp(-\lambda\omega(|P(\xi)|)).$$

The Hölder inequality and the fact that $\exp(-\lambda\omega(|P(\xi)|))$ is in \mathcal{L}^2 imply the continuous inclusion

$$\mathcal{D}_{P,(\omega)}(K) \hookrightarrow \mathcal{D}_{1,P,(\omega)}(K).$$

The inequality above shows that the identity

$$\mathcal{D}_{1,P,(\omega)}(K) = \mathcal{D}_{P,(\omega)}(K)$$

holds algebraically.

Now, we denote by τ_t the topology induced by the seminorms t_λ , and by τ_s the topology induced by the seminorms s_λ .

PROPOSITION 4.1. *Let P be a hypoelliptic polynomial. The identity map*

$$(\mathcal{D}_{P,(\omega)}(K), \tau_t) \rightarrow (\mathcal{D}_{P,(\omega)}(K), \tau_s)$$

is a homeomorphism.

Proof. We use the closed graph theorem. Since the metrizable space $(\mathcal{D}_{P,(\omega)}(K), \tau_t)$ is a closed subspace of the Fréchet space $\mathcal{E}_{P,(\omega)}(\mathbb{R}^N)$ we only need to prove that the space $(\mathcal{D}_{P,(\omega)}(K), \tau_s)$ is also complete. From the proof of Theorem 3.3, we infer that for each $f \in \mathcal{D}_{P,(\omega)}(K)$ its derivatives can be written as

$$f^{(\alpha)}(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \xi^\alpha \widehat{f}(\xi) e^{ix\xi} d\xi \quad \text{for all } \alpha \in \mathbb{N}_0^N$$

and we get positive constants $C, d > 0$ such that

$$\begin{aligned} |f^{(\alpha)}(x)| &\leq \frac{1}{(2\pi)^N} \sup_{\xi \neq 0} \exp(-\lambda\omega(|P(\xi)|) + |\alpha| \ln |\xi|) \int_{\mathbb{R}^N} |\widehat{f}(\xi)| e^{\lambda\omega(|P(\xi)|)} d\xi \\ &\leq \frac{C}{(2\pi)^N} \exp\left(\lambda\varphi^*\left(\frac{|\alpha|}{d\lambda}\right)\right) s_\lambda(f). \end{aligned}$$

As a consequence, the inclusion

$$(\mathcal{D}_{P,(\omega)}(K), \tau_s) \hookrightarrow \mathcal{D}(K)$$

is continuous. The completeness of $(\mathcal{D}_{P,(\omega)}(K), \tau_s)$ easily follows. ■

We can get a similar result for the space $\mathcal{E}_{P,(\omega)}(\Omega)$. Let P be a polynomial and let Ω be an open subset of \mathbb{R}^N . Consider the space

$$\mathcal{L}_P(\Omega) = \{f \in \mathcal{L}_{\text{loc}}^1(\Omega) : \forall j \in \mathbb{N}_0, P^j(D)f \in \mathcal{L}_{\text{loc}}^1(\Omega)\}.$$

Denote $\|f\|_{L,j} := \sup_{0 \leq k \leq j} \|P^k(D)f\|_{1,L}$ where $\|\cdot\|_{1,L}$ denotes the \mathcal{L}^1 -norm on the compact subset L , and endow $\mathcal{L}_P(\Omega)$ with the fundamental system of seminorms $\{\|\cdot\|_{L,j}\}_{L \subset\subset \Omega, j \in \mathbb{N}_0}$. Then $\mathcal{L}_P(\Omega)$ is a Fréchet space.

In the proof of our next lemma we use tools based on Hörmander's well known $\mathcal{B}_{p,k}$ spaces. We follow Chapters X and XI of [H2, II].

LEMMA 4.2. *If P is hypoelliptic, then $\mathcal{L}_P(\Omega) = \mathcal{C}^\infty(\Omega)$ as Fréchet spaces. As a consequence, for each $m \in \mathbb{N}_0$ and for each compact subset K in Ω there are a constant $C > 0$, a natural number $j \in \mathbb{N}_0$ and a compact subset*

L of Ω such that for all $f \in \mathcal{C}^\infty(\Omega)$,

$$(4.1) \quad \sup_{|\alpha| \leq m} \sup_{x \in K} |f^{(\alpha)}(x)| \leq C \sup_{0 \leq k \leq j} \|P^k(D)f\|_{1,L}.$$

Proof. Let $f \in \mathcal{L}_P(\Omega)$ and fix $j \in \mathbb{N}_0$. Since the Fourier transform of a function in \mathcal{L}^1 is bounded, we have $P^j(D)f \in \mathcal{L}_{\text{loc}}^1(\Omega) \subset \mathcal{B}_{\infty,1}^{\text{loc}}(\Omega)$ and in view of [H2, Theorem 11.1.8] we have $f \in \mathcal{B}_{\infty, \tilde{P}^j}^{\text{loc}}(\Omega)$. Hence, $f \in \bigcap_{j \in \mathbb{N}_0} \mathcal{B}_{\infty, \tilde{P}^j}^{\text{loc}}(\Omega) = \mathcal{C}^\infty(\Omega)$. The closed graph theorem implies that the inclusion $\mathcal{L}_P(\Omega) \hookrightarrow \mathcal{C}^\infty(\Omega)$ is continuous. ■

COROLLARY 4.3. *Let P be a hypoelliptic polynomial. On $\mathcal{E}_{P,\omega}(\Omega)$ we can replace the seminorms*

$$\|f\|_{K,\lambda} = \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp(-\lambda\varphi^*(j/\lambda))$$

by the seminorms

$$\|f\|_{K,\lambda}^p = \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{p,K} \exp(-\lambda\varphi^*(j/\lambda)), \quad p \geq 1,$$

and also by the seminorms

$$\|f\|_{K,\lambda}^\infty = \sup_{j \in \mathbb{N}_0} \sup_{x \in K} |P^j(D)f(x)| \exp(-\lambda\varphi^*(j/\lambda)).$$

Proof. Fix $1 \leq p < \infty$. For each compact subset K in Ω we will prove that the fundamental system of seminorms given by $\|\cdot\|_{K,\lambda}^p$ is equivalent to the system of seminorms given by $\|\cdot\|_{K,\lambda}^\infty$.

In view of the previous lemma, for each compact subset K in Ω there are a constant $C > 0$, a natural number $j \in \mathbb{N}_0$ and a compact subset L in Ω such that for all $f \in \mathcal{C}^\infty(\Omega)$,

$$\sup_{x \in K} |f(x)| \leq C \sup_{0 \leq k \leq j} \|P^k(D)f\|_{1,L}.$$

Fix $l \in \mathbb{N}_0$. Applying this inequality to the function $P^l(D)f$ we have

$$\sup_{x \in K} |P^l(D)f(x)| \leq C \sup_{0 \leq k \leq j} \|P^{k+l}(D)f\|_{1,L}$$

for all $l \in \mathbb{N}_0$ and for all $f \in \mathcal{C}^\infty(\Omega)$. Now, proceeding as in [JH, Lemma 2.3] we conclude that for each compact subset K in Ω and $\lambda > 0$ there is a compact subset L and positive constants $C' > 0$ and $\mu > 0$ depending on K and λ such that $\|f\|_{K,\lambda}^\infty \leq C' \|f\|_{L,\mu}^1$. Moreover, Hölder's inequality guarantees that $\|f\|_{K,\lambda}^1 \leq C'' \|f\|_{K,\lambda}^p$ for some positive constant $C'' > 0$.

Obviously, $\|f\|_{K,\lambda}^p \leq C''' \|f\|_{K,\lambda}^\infty$ for some positive constant $C''' > 0$. ■

Assume that P is hypoelliptic. Inspired by Proposition 4.1, Lemma 4.2 and Corollary 4.3 we study the nuclearity of the spaces $\mathcal{E}_{P,\omega}(\Omega)$ and $\mathcal{E}_{P,\{\omega\}}(\Omega)$ and the corresponding test spaces.

Recall that a projective limit of nuclear spaces is nuclear and that nuclearity is inherited by countable inductive limits. In order to see that an inductive limit $\text{ind}_n X_n$ is nuclear it suffices to prove that for all n there exists $m > n$ such that the inclusion $X_n \hookrightarrow X_m$ is absolutely summing. See Chapter 28 of [MV] and the books [J] and [Ko, II] for more details.

THEOREM 4.4. *If P is hypoelliptic, then the spaces $\mathcal{E}_{P,(\omega)}(\Omega)$ and $\mathcal{E}_{P,\{\omega\}}(\Omega)$ are nuclear.*

Proof. Beurling case. First, observe that [BMT, Lemma 1.4] allows us to describe the topology of $\mathcal{E}_{P,(\omega)}(\Omega)$ by the seminorms

$$\sum_{j \in \mathbb{N}_0} \int_K |P^j(D)f(x)| dx \exp(-\lambda\varphi^*(j/\lambda)).$$

Fix $\lambda > 0$ and a compact subset K . By the same lemma we have

$$\begin{aligned} & \sum_{j \in \mathbb{N}_0} \int_K |P^j(D)f(x)| dx \exp(-\lambda\varphi^*(j/\lambda)) \\ & \leq C \sum_{j \in \mathbb{N}_0} \int_K |P^j(D)f(x)| dx \exp\left(-L\lambda\varphi^*\left(\frac{j}{L\lambda}\right)\right) \exp(-j) \end{aligned}$$

for some positive constant $C > 0$. Define

$$\Delta_j : K \rightarrow (\mathcal{E}_{P,(\omega)}(\Omega))', \sigma(\mathcal{E}_{P,(\omega)}(\Omega)', \mathcal{E}_{P,(\omega)}(\Omega))$$

by

$$\Delta_j(x)[f] := P^j(D)f(x) \exp\left(-L\lambda\varphi^*\left(\frac{j}{L\lambda}\right)\right).$$

Then

$$|\Delta_j(x)[f]| \leq \|f\|_{K,L\lambda}^\infty.$$

Hence, $\Delta_j(x) \in \mathcal{E}_{P,(\omega)}(\Omega)'$ and Δ_j is a well defined and continuous map. Moreover $\Delta_j(K) \subseteq V^\circ$ where V is the absolutely convex zero neighborhood defined by

$$V := \{f \in \mathcal{E}_{P,(\omega)}(\Omega) : \|f\|_{K,L\lambda}^\infty \leq 1\}.$$

Now, consider the map

$$\mu_j : \mathcal{C}(V^\circ) \rightarrow \mathbb{R}, \quad \mu_j(g) := C \int_K g(\Delta_j(x)) \exp(-j) dx.$$

If $g : V^\circ \rightarrow \mathbb{R}$ is a continuous function on V° , it is clear that

$$|\mu_j(g)| \leq C \exp(-j)m(K) \sup_{f \in V^\circ} |g(f)|.$$

This fact implies that μ_j is a continuous linear map which is positive, i.e., $\mu_j(g) \geq 0$ whenever $g \geq 0$. So, μ_j defines a measure on (V°, σ^*) . We now

consider

$$\mu := \sum_{j \in \mathbb{N}_0} \mu_j,$$

which is a measure on (V°, σ^*) . Then

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} \int_K |P^j(D)f(x)| dx \exp(-\lambda\varphi^*(j/\lambda)) \\ \leq C \sum_{j \in \mathbb{N}_0} \int_K |\Delta_j(x)[f]| dx \exp(-j) = \int_{V^\circ} |y(f)| d\mu(y). \end{aligned}$$

and the nuclearity in the Beurling case follows.

Roumieu case. To see that

$$\mathcal{E}_{P,\{\omega\}}(\Omega) := \text{proj}_{K \subset \subset \Omega} \text{ind}_{\lambda > 0} \mathcal{E}_{P,\omega}^\lambda(K)$$

is nuclear it is enough to see that

$$\mathcal{E}_{P,\omega}^{1/n}(K) \hookrightarrow \mathcal{E}_{P,\omega}^{1/nL}(K)$$

is absolutely summing. $\mathcal{E}_{P,\omega}^{1/nL}(K)$ and $\mathcal{E}_{P,\omega}^{1/n}(K)$ are Banach spaces endowed with the norms $\|\cdot\|_{K,1/Ln}^\infty$ and $\|\cdot\|_{K,1/n}^\infty$, respectively. Again, [BMT, Lemma 1.4] gives constants $C, L > 0$ such that

$$\begin{aligned} \|f\|_{K,1/Ln}^\infty &\leq \sum_{j \in \mathbb{N}_0} \|P^j(D)f\|_{p,K} \exp\left(-\frac{1}{Ln}\varphi^*(Lnj)\right) \\ &\leq C \exp\left(\frac{1}{n}\right) \sum_{j \in \mathbb{N}_0} \|P^j(D)f\|_{p,K} \exp\left(-\frac{1}{n}\varphi^*(nj)\right) \exp(-j). \end{aligned}$$

Define

$$\Delta_j : K \rightarrow (\mathcal{E}_{P,\omega}^{1/n}(K))', \sigma(\mathcal{E}_{P,\omega}^{1/n}(K))', \mathcal{E}_{P,\omega}^{1/n}(K))$$

by

$$\Delta_j(x)[f] := P^j(D)f(x) \exp\left(-\frac{1}{n}\varphi^*(nj)\right).$$

Denote by U the unit ball of $\mathcal{E}_{P,\omega}^{1/n}(K)$. Proceeding as in the Beurling case we can define a measure μ on U° such that

$$\|f\|_{K,1/Ln}^\infty \leq \int_{U^\circ} |y(f)| d\mu(y). \quad \blacksquare$$

COROLLARY 4.5. *Let Ω be an open subset of \mathbb{R}^N . If P is a hypoelliptic polynomial, then the spaces $\mathcal{D}_{P,(\omega)}(\Omega)$ and $\mathcal{D}_{P,\{\omega\}}(\Omega)$ are nuclear.*

Proof. By Proposition 3.2, $\mathcal{D}_{P,(\omega)}(K)$ is a topological subspace of $\mathcal{E}_{P,(\omega)}(\Omega)$ and hence nuclear. Then the space $\mathcal{D}_{P,(\omega)}(\Omega) = \text{ind}_{K \subset \subset \Omega} \mathcal{D}_{P,(\omega)}(K)$ is also nuclear.

In the Roumieu case we consider $\mathcal{D}_{P,\{\omega\}}(\Omega)$ endowed with the seminorms s_λ and we prove that the inclusion

$$\mathcal{D}_{P,\omega}^{2\lambda}(K) \hookrightarrow \mathcal{D}_{P,\omega}^\lambda(K)$$

is absolutely summing. Let U be the unit ball of $\mathcal{D}_{P,\omega}^{2\lambda}(K)$ and define

$$\Delta : \mathbb{R}^N \rightarrow (\mathcal{D}_{P,\omega}^{2\lambda}(K))', \sigma(\mathcal{D}_{P,\omega}^{2\lambda}(K))', \mathcal{D}_{P,\omega}^{2\lambda}(K))$$

by

$$\Delta(\xi)[f] := C\widehat{f}(\xi) \exp(2\lambda\omega(|P(\xi)|)),$$

where the constant C is chosen in such a way that $\Delta(\mathbb{R}^N) \subseteq U^\circ$. Now, we consider the map

$$\mu : \mathcal{C}(U^\circ) \rightarrow \mathbb{R}, \quad \mu(g) := \int_{\mathbb{R}^N} g(\Delta(\xi)) \exp(-\lambda\omega(|P(\xi)|)) d\xi.$$

If $g : U^\circ \rightarrow \mathbb{R}$ is a continuous function, it is clear that

$$|\mu(g)| \leq \left(\int_{\mathbb{R}^N} \exp(-\lambda\omega(|P(\xi)|)) d\xi \right) \sup_{f \in U^\circ} |g(f)|.$$

This implies that μ is a continuous linear map which is positive, i.e., $\mu(g) \geq 0$ whenever $g \geq 0$. So, μ defines a measure on (U°, σ^*) and

$$\begin{aligned} s_\lambda(f) &= \int_{\mathbb{R}^N} |\widehat{f}(\xi)| \exp(2\lambda\omega(|P(\xi)|)) \exp(-\lambda\omega(|P(\xi)|)) d\xi \\ &= \int_{\mathbb{R}^N} |\Delta(\xi)[f]| \exp(-\lambda\omega(|P(\xi)|)) d\xi \leq \int_{U^\circ} |y(f)| d\mu(y). \quad \blacksquare \end{aligned}$$

To finish we give a sufficient condition for the test space $\mathcal{D}_{P,\omega}(\Omega)$ to be an algebra.

PROPOSITION 4.6. *Let P be a hypoelliptic polynomial and ω a weight function such that $\omega(|P(x+y)|) \leq K + K\omega(|P(x)|) + K\omega(|P(y)|)$ for some constant $K > 0$. Then $\mathcal{D}_{P,\omega}(\Omega)$ is an algebra.*

Proof. We consider the seminorms $s_\lambda(f) = \int_{\mathbb{R}^N} |\widehat{f}(x)| \exp(\lambda\omega(|P(x)|)) dx$. Note that

$$\begin{aligned} \int_{\mathbb{R}^N} |\widehat{f\hat{g}}(x)| \exp(\lambda\omega(|P(x)|)) dx &= \int_{\mathbb{R}^N} |\widehat{f} * \widehat{g}(x)| \exp(\lambda\omega(|P(x)|)) dx \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\widehat{f}(y)| |\widehat{g}(x-y)| \exp(\lambda\omega(|P(x)|)) dy dx. \end{aligned}$$

The hypothesis gives a positive constant $C > 0$ such that

$$s_\lambda(fg) \leq C s_{K\lambda}(f) s_{K\lambda}(g). \quad \blacksquare$$

EXAMPLE 4.7. Consider the hypoelliptic heat polynomial in two variables, $P(t, x) = it + x^2$, and the Gevrey weights $\omega(t) = t^a$ for $a \in]0, 1/2]$. Then $\mathcal{D}_{P, \omega}(\Omega)$ is an algebra.

Recall that $\mathcal{D}_{t^{2a}}(\Omega) \subset \mathcal{D}_{P, t^a}(\Omega)$, and therefore $\mathcal{D}_{P, t^a}(\Omega)$ is non-trivial. For $\omega(t) = t^{1/2}$, one can easily check

$$|P((x, t) + (y, u))|^{1/2} \leq K + K|P(x, t)|^{1/2} + K|P(y, u)|^{1/2}.$$

We set $X = P(x + y, t + u)$, $Y = P(x, t)$ and $Z = P(y, u)$. For $0 < a < 1/2$, we want to see

$$X^a \leq K(1 + Y^a + Z^a) \quad \text{for some } K > 0.$$

By the inequality above, we have $X^{1/2} \leq K(1 + Y^{1/2} + Z^{1/2})$ for all $X, Y, Z \geq 0$. Observe that $p := 1/2a > 1$. Since on \mathbb{R}^N all norms are equivalent, we have $\|\cdot\|_p \leq D\|\cdot\|_1$ for some $D > 0$. Then

$$(1 + (Y^a)^{1/2a} + (Z^a)^{1/2a})^{2a} \leq D(1 + Y^a + Z^a).$$

As a consequence,

$$X \leq K^2(1 + Y^{1/2} + Z^{1/2})^2 \leq K^2D(1 + Y^a + Z^a)^{1/a}$$

and then

$$X^a \leq K^{2a}D^a(1 + Y^a + Z^a).$$

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