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Borel parts of the spectrum of an operator and of the operator algebra of a separable Hilbert space

by

Piotr Niemiec (Kraków)

Abstract. For a linear operator T in a Banach space let $\sigma_p(T)$ denote the point spectrum of T, let $\sigma_{p,n}(T)$ for finite $n > 0$ be the set of all $\lambda \in \sigma_p(T)$ such that $\dim \ker(T - \lambda) = n$ and let $\sigma_{p,\infty}(T)$ be the set of all $\lambda \in \sigma_p(T)$ for which $\ker(T-\lambda)$ is infinite-dimensional. It is shown that $\sigma_p(T)$ is $\mathcal{F}_\sigma$, $\sigma_{p,\infty}(T)$ is $\mathcal{F}_{\sigma\delta}$ and for each finite n the set $\sigma_{p,n}(T)$ is the intersection of an $\mathcal{F}_\sigma$ set and a $\mathcal{G}_\delta$ set provided T is closable and the domain of T is separable and weakly σ-compact. For closed densely defined operators in a separable Hilbert space $\mathcal{H}$ a more detailed decomposition of the spectra is obtained and the algebra of all bounded linear operators on $\mathcal{H}$ is decomposed into Borel parts. In particular, it is shown that the set of all closed range operators on $\mathcal{H}$ is Borel.

1. Introduction. Dixmier and Foiaş [3] and (independently) Nikol'skaya [13] showed that the point spectrum of a bounded operator acting on a separable reflexive Banach space is $\mathcal{F}_\sigma$. In both the papers the authors showed that any bounded $\mathcal{F}_\sigma$ subset of the complex plane coincides with the point spectrum of a certain bounded operator acting on a separable Hilbert space. Later Kaufman [8, 9] proved that a necessary and sufficient condition for a subset of a complex plane to be the point spectrum of a bounded operator in some separable complex Banach space is that it be analytic (in the sense of Suslin) and bounded. All these results were extended by Smolyanov and Shkarin who studied another part of the spectrum of a closed operator in a Hilbert space [15] as well as in an arbitrary topological vector space [16]. In particular, they showed that if the graph of an operator T in a Hausdorff TVS X is the union of a countable family of metrizable compact sets, then the point spectrum of T and the sets $\bigcup_{k=n}^{\infty} \sigma_{p,k}(T)$ for natural n (see Abstract) are all $\mathcal{F}_\sigma$. In the present paper we propose another result in this fashion: if X is a Banach space, then in the above theorem the assumption about the graph of T may be replaced by the domain of T being separable

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and weakly σ-compact (see the proof of 3.2 below). Although our method is quite similar to that of Smolyanov and Shkarin, it seems that the two theorems are 'incomparable' (i.e. none of them implies the other for Banach spaces). As is shown in 2.7, our result applies not only to closed operators. In the case of Hilbert space operators, we introduce a more detailed Borel decomposition of the spectra (than in [15]) as well as of the whole algebra of bounded operators (see 4.7 and 4.5, respectively), related to the closedness of ranges.

Since the celebrated paper by Cowen and Douglas [2], the point spectra of bounded linear operators have been widely investigated and those operators which have these spectra rich have attracted a growing interest: see e.g. [11], [20], [6] or [7] and references there.

Notation. In this paper Banach spaces are real or complex and they need not be separable, and $\mathbb{K}$ is the field of real or complex numbers. By an *operator* in a Banach space X we mean any linear function from a linear subspace of X into X, and such an operator is *on* X if its domain is the whole space. For an operator T in X, we let $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}}(T)$ denote the domain, the kernel, the range and the closure of the range of T, respectively. For a scalar $\lambda \in \mathbb{K}$, $T - \lambda$ stands for the operator given by $\mathcal{D}(T - \lambda) = \mathcal{D}(T)$ and $(T - \lambda)(x) = Tx - \lambda x$ ($x \in \mathcal{D}(T)$). The *point spectrum* $\sigma_p(T)$ of T is the set of all eigenvalues of T; that is, $\sigma_p(T)$ consists of all scalars $\lambda \in \mathbb{K}$ such that $\mathcal{N}(T - \lambda)$ is nonzero. The operator T is *closed* iff its graph $\Gamma(T) := \{(x, Tx) \colon x \in \mathcal{D}(T)\}$ is a closed subset of $X \times X$. We call T *closable* iff the norm closure of $\Gamma(T)$ in $X \times X$ is the graph of some operator in X. Whenever E and F are Banach spaces, $\mathcal{B}(E, F)$ stands for the space of all bounded operators from the whole space E to F. A subset of a topological space is said to be *σ-compact* if it is the union of a countable family of compact subsets of the space. A *weakly σ-compact* subset of a Banach space is a set which is σ-compact with respect to the weak topology of the space. A *Borel* subset of a topological space is a member of the σ-algebra generated by all open sets in the space. Adapting the notation proposed by A. H. Stone [17], we call a subset of a topological space (*of type*) $\mathcal{F} \cap \mathcal{G}$ if it is the intersection of an open and a closed set. Similarly, any set which is the intersection of a $\mathcal{G}_\delta$ set and an $\mathcal{F}_\sigma$ set will be said to be $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$.

2. Operators with weakly σ-compact domains. For an operator T in X, a subset K of X and a nonnegative real constant M, let

$$\Lambda_T(K, M) = \{w \in \mathbb{K} \colon \mathcal{N}(T - w) \cap K \neq \emptyset \text{ and } |w| \leq M\}.$$

Notice that $\Lambda_T(K, M)$ consists of all $w \in \mathbb{K}$ with $|w| \leq M$ provided $0 \in K$. The main tool of this section is the following

Lemma 2.1. *If T is a closable operator in X and K is a weakly compact subset of $\mathcal{D}(T)$, then the set $\Lambda_T(K,M)$ is compact for every $M \geq 0$.*

Proof. We may and do assume that $0 \notin K$. Let W be the set of all $x \in K$ for which there is a (unique) $\lambda(x) \in \mathbb{K}$ such that $Tx = \lambda(x)x$ and $|\lambda(x)| \leq M$. Thus we have obtained a function $\lambda\colon W \to \mathbb{K}$. Since $\lambda(W) = \Lambda_T(K,M)$, it suffices to show that W is weakly compact and λ is continuous when W is equipped with the weak topology.

Let $\mathcal{X} = (x_\sigma)_{\sigma\in\Sigma}$ be a net in W which is weakly convergent to some $x \in K$. We need to show that $x \in W$ and $\lim_{\sigma\in\Sigma}\lambda(x_\sigma) = \lambda(x)$. If $(x_\tau)_{\tau\in\Sigma'}$ is a subnet of $\mathcal{X}$ such that $\lim_{\tau\in\Sigma'}\lambda(x_\tau) = w \in \mathbb{K}$, then $(x_\tau, Tx_\tau)_{\tau\in\Sigma'}$ is weakly convergent to (x, wx). Since the norm closure of $\Gamma(T)$ is weakly closed (and is the graph of some operator), we infer that $Tx = wx$ and thus $x \in W$ and $\lambda(x) = w$. Since $\lambda(W)$ is bounded and any convergent subnet of $(\lambda(x_\sigma))_{\sigma\in\Sigma}$ has the same limit, the net converges to $\lambda(x)$ and we are done. ∎

With the use of the foregoing result we now easily prove

Proposition 2.2. *If T is a closable operator in X such that $\mathcal{D}(T)\setminus\{0\}$ is weakly σ-compact, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

Proof. Write $\mathcal{D}(T)\setminus\{0\} = \bigcup_{n=1}^{\infty} K_n$ with each K_n weakly compact, note that $\lambda \in \sigma_p(T)$ iff $\lambda \in \Lambda_T(K_n, m)$ for some natural n and m and apply Lemma 2.1. ∎

For applications of the above result, we need the next well known fact. For the reader's convenience, we give its short proof.

Lemma 2.3. *Every weakly compact subset of X which is separable in the norm topology is weakly metrizable.*

Proof. Let K be a separable weakly compact subset of X. Then $(K - K)\setminus\{0\}$ is separable as well and thus there is a sequence of linear functionals $f_n\colon X \to \mathbb{K}$ of norm 1 such that for every nonzero $z \in K - K$ there is n with $f_n(z) \neq 0$. Observe that then the family $\{f_n\}_{n\in\mathbb{N}}$ separates the points of K. Finally, since K is weakly compact, the formula $K \ni x \mapsto (f_n(x))_{n\in\mathbb{N}} \in \Delta^{\mathbb{N}}$ with $\Delta = \{w \in \mathbb{K}\colon |w| \leq 1\}$ defines a topological embedding of K (equipped with the weak topology) into the compact metrizable space $\Delta^{\mathbb{N}}$. ∎

Now we have

Proposition 2.4. *If $\mathcal{D}$ is a linear subspace of X, then $\mathcal{D}\setminus\{0\}$ is weakly σ-compact iff $\mathcal{D}$ is separable and weakly σ-compact as well.*

Proof. First assume that $\mathcal{D}$ is separable and $\mathcal{D} = \bigcup_{n=1}^{\infty} K_n$ with each K_n weakly compact. By 2.3, K_n is weakly metrizable and thus $L_n := K_n\setminus\{0\}$ is an $\mathcal{F}_\sigma$ subset of K_n with respect to the weak topology. So, $\mathcal{D}\setminus\{0\} = \bigcup_{n=1}^{\infty} L_n$ and each L_n is weakly σ-compact.

Conversely, if $\mathcal{D} \setminus \{0\}$ is weakly σ-compact, so is $\mathcal{D}$, and $\{0\}$ is weakly $\mathcal{G}_\delta$ in $\mathcal{D}$. Thus, by the definition of the weak topology, there are sequences $(f_n)_{n=1}^{\infty}$ and $(\varepsilon_n)_{n=1}^{\infty}$ of continuous linear functionals on X and of positive real numbers (respectively) such that $\{0\} = \mathcal{D} \cap \{x \in X : |f_n(x)| < \varepsilon_n,\ n = 1, 2, \ldots\}$. In particular, $\bigcap_{n=1}^{\infty} \mathcal{N}(f_n) \cap \mathcal{D} = \{0\}$ and therefore the function $\psi : \mathcal{D} \ni x \mapsto (f_n(x))_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ is one-to-one. What is more, ψ is continuous with respect to the weak topology on $\mathcal{D}$ and the product topology on $\mathbb{K}^{\mathbb{N}}$. So, ψ restricted to any weakly compact subset K of $\mathcal{D}$ is a topological embedding and therefore every such K is weakly separable. We conclude that $\mathcal{D}$ itself is weakly separable, being weakly σ-compact. Finally, if A is a countable weakly dense subset of $\mathcal{D}$ and $\mathcal{E}$ is the norm closure of the linear span of A, then $\mathcal{E}$ is separable and weakly closed. This implies that $\mathcal{D} \subset \mathcal{E}$ and we are done. ∎

It follows from 2.4 that 2.2 is equivalent to

Theorem 2.5. *If T is a closable operator in X whose domain is separable and weakly σ-compact, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

By Baire's theorem, a Banach space is weakly σ-compact iff it is reflexive. Thus 2.5 applies mainly to reflexive spaces. For example:

Corollary 2.6. *If T is a closable operator in X whose domain is separable and is the image of a reflexive Banach space under a bounded operator, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

Example 2.7. The inclusion of $L^\infty[0,1]$ into $L^2[0,1]$, being the dual operator of the inclusion of $L^2[0,1]$ into $L^1[0,1]$, is continuous with respect to the weak* topology of the domain and the weak topology in $L^2[0,1]$. This implies that $L^\infty[0,1]$, considered as a subspace of $L^2[0,1]$, is weakly σ-compact. However, there is no closed operator in $L^2[0,1]$ whose domain is $L^\infty[0,1]$ (compare with the Remarks on page 257 in [5]). This example shows that 2.5 can have quite natural applications also for nonclosed operators and that it is more general than 2.6 which does not apply here, since there is no bounded operator on a reflexive Banach space into $L^2[0,1]$ whose image is $L^\infty[0,1]$ (again by the Remarks on page 257 in [5]).

3. Decomposition of the point spectrum. Let T be an operator in X. For $n \geq 1$ let $\sigma_{p,n}(T)$ be the set of all $\lambda \in \mathbb{K}$ such that $\dim \mathcal{N}(T-\lambda) = n$ and let $\sigma_{p,\infty}(T) = \sigma_p(T) \setminus \bigcup_{n=1}^{\infty} \sigma_{p,n}(T)$. Our aim is to show that all sets defined above are Borel provided T is closable and has separable and weakly σ-compact domain. To do this, we need the following well-known result (cf. e.g. the proof of Proposition 1 in [16, §2]).

LEMMA 3.1. *If V is a T_2 topological vector space, then for each $n \geq 2$ the set $F[n]$ of all $(x_1, \ldots, x_n) \in V^n$ such that $x_1, \ldots, x_n$ are linearly dependent is closed in the product topology of V^n.*

The main result of this section is the following result which may be seen as a counterpart of Proposition 2 in [16, §2].

THEOREM 3.2. *If T is a closable operator in X whose domain is separable and weakly σ-compact, then $\sigma_{p,n}(T)$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite n and $\sigma_{p,\infty}(T)$ is $\mathcal{F}_{\sigma\delta}$.*

Proof. First fix a finite $N > 0$. Let $F[1] = \{0\} \subset X$ and $F[N]$ be as in the statement of 3.1 for $N > 1$. By 3.1, $F[N]$ is weakly closed in X^N. Write $\mathcal{D}(T) = \bigcup_{n=1}^{\infty} K_n$ with each K_n weakly compact. By 2.3, all K_n's are weakly metrizable and hence $(\prod_{j=1}^{N} K_{n_j}) \setminus F[N]$ is $\mathcal{F}_\sigma$ in $\prod_{j=1}^{N} K_{n_j}$ for any $n_1, \ldots, n_N$ when each K_n is equipped with the weak topology. So, $\mathcal{D}(T)^N \setminus F[N]$ may be written in the form

$$\mathcal{D}(T)^N \setminus F[N] = \bigcup_{n=1}^{\infty} L_n \tag{3.1}$$

where each L_n is a weakly compact subset of X^N. Put

$$S\colon \mathcal{D}(T)^N \ni (x_1, \ldots, x_N) \mapsto (Tx_1, \ldots, Tx_N) \in X^N$$

and observe that S is a closable operator in X^N. Thanks to 2.1, the set $\Lambda_S(L_n, m)$ is compact for all natural n and m. So, $G_N := \bigcup_{n,m} \Lambda_S(L_n, m)$ is $\mathcal{F}_\sigma$. But G_N is the set of all $\lambda \in \sigma_p(T)$ such that $\mathcal{N}(T - \lambda)$ is at least N-dimensional, by (3.1).

Now observe that for finite n, $\sigma_{p,n}(T) = G_n \setminus G_{n+1}$ and thus $\sigma_{p,n}(T)$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$. Finally, since $\sigma_{p,\infty}(T) = \bigcap_{n=1}^{\infty} G_n$, $\sigma_{p,\infty}(T)$ is $\mathcal{F}_{\sigma\delta}$. ∎

4. Decomposition of the spectrum: Hilbert space. From now on, $\mathcal{H}$ is a complex separable infinite-dimensional Hilbert space and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$. Every subset of $\mathcal{B}(\mathcal{H})$ which is Borel with respect to, respectively, the weak operator, the strong operator or the norm topology is called, respectively, *WOT-Borel*, *SOT-Borel* and briefly *Borel*. Similarly, if $f\colon S \to \mathcal{B}(\mathcal{H})$ with $S \subset \mathcal{B}(\mathcal{H})$ is such that S and the inverse image of any WOT-Borel (respectively SOT-Borel; Borel) set is WOT-Borel (SOT-Borel; Borel), then f is said to be *WOT-Borel* (*SOT-Borel*; *Borel*). A fundamental result in this area says that every WOT-Borel set is SOT-Borel and conversely (see e.g. [4]). Thus the same property holds for WOT-Borel and SOT-Borel functions. Therefore we shall only speak of WOT-Borel and Borel sets and functions. Whenever the classes $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F} \cap \mathcal{G}$, etc., appear, they are understood with respect to the norm topology.

We are interested in the decomposition of $\mathcal{B}(\mathcal{H})$ into Borel sets each of which collects operators of a similar type. For $n = 0, 1, \ldots$ let $\Sigma_n(\mathcal{H})$ be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{R}(A) = n$. Further, for $n, m = 0, 1, \ldots, \infty$ let $\Sigma^1_{n,m}(\mathcal{H})$ and $\Sigma^0_{n,m}(\mathcal{H})$ consist of all $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{N}(A) = n$, $\dim \mathcal{R}(A)^\perp = m$, $\dim \overline{\mathcal{R}}(A) = \infty$ and, respectively, $\mathcal{R}(A)$ is closed or not. Notice that all above sets of operators form a countable decomposition of $\mathcal{B}(\mathcal{H})$, denoted by $\Sigma(\mathcal{H})$, into nonempty and pairwise disjoint sets. We want to show that each of them is Borel in $\mathcal{B}(\mathcal{H})$. To do this, we need

Lemma 4.1. *For each $n = 0, 1, \ldots, \infty$, let $\Sigma_{n,*}(\mathcal{H})$ be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{N}(A) = n$. Then $\Sigma_{0,*}(\mathcal{H})$ is $\mathcal{G}_\delta$, $\Sigma_{n,*}(\mathcal{H})$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite $n > 0$ and $\Sigma_{\infty,*}(\mathcal{H})$ is $\mathcal{F}_{\sigma\delta}$ in $\mathcal{B}(\mathcal{H})$.*

Proof. We mimic the proof of 3.2. For fixed finite $N > 0$ let $F[N]$ be defined as there. Write $\mathcal{H}^N \setminus F[N] = \bigcup_{n=1}^\infty L_n$ with each L_n weakly compact in $\mathcal{H}^N$. For $T \in \mathcal{B}(\mathcal{H})$ let $T^{\times N} \in \mathcal{B}(\mathcal{H}^N)$ denote the operator $\mathcal{H}^N \ni (x_1, \ldots, x_N) \mapsto (Tx_1, \ldots, Tx_N) \in \mathcal{H}^N$. Since L_n is bounded, the function $\psi\colon \mathcal{B}(\mathcal{H}) \times L_n \ni (T, x) \mapsto T^{\times N} x \in \mathcal{H}^N$ is continuous when $\mathcal{B}(\mathcal{H})$ is considered with the norm topology and L_n and $\mathcal{H}^N$ with the weak one. Hence $\psi^{-1}(\{0\})$ is closed in $\mathcal{B}(\mathcal{H}) \times L_n$ in the product of these topologies. Finally, since L_n is weakly compact, the set $F_n := p(\psi^{-1}(\{0\}))$ is closed in the norm topology of $\mathcal{B}(\mathcal{H})$ where $p\colon \mathcal{B}(\mathcal{H}) \times L_n \to \mathcal{B}(\mathcal{H})$ is the projection onto the first factor. Thus $G_N := \bigcup_{n=1}^\infty F_n$ is $\mathcal{F}_\sigma$. Note that G_N coincides with the set of all $A \in \mathcal{B}(\mathcal{H})$ for which $\dim \mathcal{N}(A) \geq N$.

Now we have $\Sigma_{n,*}(\mathcal{H}) = G_n \setminus G_{n+1}$ for finite $n > 0$, $\Sigma_{\infty,*}(\mathcal{H}) = \bigcap_{n=1}^\infty G_n$ and $\Sigma_{0,*}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \setminus G_1$, which clearly finishes the proof. ∎

Now let $\mathcal{CR}(\mathcal{H})$ be the set of all closed range operators of $\mathcal{B}(\mathcal{H})$. Our next purpose is to show that $\mathcal{CR}(\mathcal{H})$ is WOT-Borel (and hence Borel as well). This is however not so simple. Firstly, the maps $\mathcal{B}(\mathcal{H}) \ni A \mapsto A^* \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \ni (A, B) \mapsto AB \in \mathcal{B}(\mathcal{H})$ are WOT-Borel (cf. [4]), since the first of them is WOT-continuous and the second is SOT-continuous on bounded sets. Secondly, closed range operators may be characterized in the space $\mathcal{B}(\mathcal{H})$ by means of Moore–Penrose inverses in the following way:

Proposition 4.2. *An operator $A \in \mathcal{B}(\mathcal{H})$ has closed range iff there is $B \in \mathcal{B}(\mathcal{H})$ such that $ABA = A$, $BAB = B$ and AB and BA are selfadjoint. What is more, the operator B is uniquely determined by these properties and if A has closed range, then $B = A^\dagger := (A|_{\mathcal{R}(A^*)})^{-1} P$ where P is the orthogonal projection onto the range of A.*

The operator $A^\dagger$ appearing in the statement of 4.2 is the Moore–Penrose inverse of an operator $A \in \mathcal{CR}(\mathcal{H})$. For a proof of 4.2, see e.g. [14].

In the proof of the next result we use the fact that if Y and Z are separable complete metric spaces, B is a Borel subset of Y and $f\colon B \to Z$ is a continuous one-to-one function, then $f(B)$ is a Borel subset of Z (see [10, Theorem XIII.1.9] or [18, Corollary A.7], or [19, Theorem A.25] for a more general result).

THEOREM 4.3. *The set $\mathcal{CR}(\mathcal{H})$ is WOT-Borel.*

Proof. For $r > 0$ let B_r be the closed ball in $\mathcal{B}(\mathcal{H})$ with center at 0 and of radius r equipped with the weak operator topology. Let $\psi_r\colon B_r^2 \ni (T,S) \mapsto (TST - T, STS - S, S^*T^* - TS, T^*S^* - ST) \in B_s^4$ where $s := 2(r+1)^3$. By the remark preceding the statement of the theorem, ψ_r is WOT-Borel. So, the set $C_r := \psi_r^{-1}(\{(0,0,0,0)\})$ is WOT-Borel in B_r as well. By 4.2, $C_r = \{(A, A^\dagger)\colon A \in \mathcal{CR}(\mathcal{H}), A, A^\dagger \in B_r\}$. So, the projection p_r of C_r onto the first factor is one-to-one. But p_r is WOT-continuous and B_r and B_s are compact metrizable spaces. We infer that $E_r := p_r(C_r)$ is WOT-Borel in B_s. Finally, the observation that B_s is WOT-Borel in $\mathcal{B}(\mathcal{H})$ and $\mathcal{CR}(\mathcal{H}) = \bigcup_{n=1}^\infty E_n$ finishes the proof. ∎

PROBLEM 4.4. *Of which additive or multiplicative class, in the hierarchy of WOT-Borel sets in $\mathcal{B}(\mathcal{H})$, is the set $\mathcal{CR}(\mathcal{H})$?*

Now we are ready to prove the following

THEOREM 4.5. *For each $k, n, m = 0, 1, \ldots, \infty$ with finite k:*

(a) *$\Sigma_k(\mathcal{H})$ is $\mathcal{F} \cap \mathcal{G}$,*
(b) *$\Sigma_{n,m}^1(\mathcal{H})$ is $\mathcal{F} \cap \mathcal{G}$ (respectively open) provided m or n is finite (respectively $m = 0$ or $n = 0$),*
(c) *$\Sigma_{0,0}^0(\mathcal{H})$ is $\mathcal{G}_\delta$; $\Sigma_{n,m}^0(\mathcal{H})$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite n and m; $\Sigma_{n,m}^0(\mathcal{H})$ is $\mathcal{F}_{\sigma\delta}$ if either n or m is infinite (and the other is finite),*
(d) *$\Sigma_{\infty,\infty}^1(\mathcal{H})$ and $\Sigma_{\infty,\infty}^0(\mathcal{H})$ are Borel.*

Proof. Clearly, for each finite k the set of all finite rank operators $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{R}(A) \le k$ is closed and therefore $\Sigma_k(\mathcal{H})$ is $\mathcal{F}\cap\mathcal{G}$. Further, for each $n = 0, 1, \ldots, \infty$ let $\Sigma_{n,*}(\mathcal{H})$ be as in 4.1 and let $\Sigma_{*,n}(\mathcal{H}) = \{A^*\colon A \in \Sigma_{n,*}(\mathcal{H})\}$. Observe that $\Sigma_{*,n}(\mathcal{H})$ is of the same Borel class as $\Sigma_{n,*}(\mathcal{H})$.

Suppose that n or m is finite. With no loss of generality, we may assume that $n \le m$. Put $k = n - m \in \mathbb{Z} \cup \{-\infty\}$. The set $F(k)$ of all semi-Fredholm operators of index k is open in $\mathcal{B}(\mathcal{H})$ (see e.g. Proposition XI.2.4 and Theorem XI.3.2 in [1]). It may also be shown that for each integer $l \ge 0$ the set $F_l(k)$ of all $A \in F(k)$ such that $\dim \mathcal{N}(A) \le l$ is open in $\mathcal{B}(\mathcal{H})$ as well (see e.g. [12, Proposition 5.3]). Now the relation $\Sigma_{n,m}^1(\mathcal{H}) = F_n(k) \setminus F_{n-1}(k)$ (with $F_{-1}(k) = \emptyset$) shows (b). Further, since $\Sigma_{n,m}^0(\mathcal{H}) = \Sigma_{n,*}(\mathcal{H}) \cap \Sigma_{*,m}(\mathcal{H}) \setminus F(k)$, we infer from 4.1 the assertion of (c).

Finally, $\Sigma_{\infty,\infty}^1(\mathcal{H}) = \Sigma_{\infty,*}(\mathcal{H}) \cap \Sigma_{*,\infty}(\mathcal{H}) \cap \mathcal{CR}(\mathcal{H}) \setminus \bigcup_{n=0}^\infty \Sigma_n(\mathcal{H})$. So, this set is Borel by 4.3. Since $\Sigma_{\infty,\infty}^0(\mathcal{H})$ is the complement in $\mathcal{B}(\mathcal{H})$ of the union of all other members of $\Sigma(\mathcal{H})$, it is Borel as well. ∎

PROBLEM 4.6. *To which additive or multiplicative class, in the hierarchy of Borel sets in $\mathcal{B}(\mathcal{H})$, do the sets $\Sigma^1_{\infty,\infty}(\mathcal{H})$ and $\Sigma^0_{\infty,\infty}(\mathcal{H})$ belong?*

Now let T be a closed densely defined operator in $\mathcal{H}$. We denote by $\sigma(T)$ the spectrum of T. That is, a complex number λ does **not** belong to $\sigma(T)$ iff $\mathcal{N}(T-\lambda) = \{0\}$, $\mathcal{R}(T-\lambda) = \mathcal{H}$ and $(T-\lambda)^{-1}$ is bounded (the last condition may be omitted by the Closed Graph Theorem). We decompose the complex plane into parts corresponding to the members of $\Sigma(\mathcal{H})$:

- $\sigma^f(T)$ is the set of all $z \in \mathbb{C}$ such that $\mathcal{R}(T-z)$ is finite-dimensional,
- for $n, m = 0, 1, \ldots, \infty$ let $\sigma^1_{n,m}(T)$ and $\sigma^0_{n,m}(T)$ be the sets consisting of all $z \in \mathbb{C}$ for which $\dim \mathcal{N}(T-z) = n$, $\dim \mathcal{R}(T-z)^\perp = m$, $\dim \overline{\mathcal{R}}(T-z) = \infty$ and, respectively, $\mathcal{R}(T-z)$ is closed or not.

Notice that $\mathbb{C} \setminus \sigma(T) = \sigma^1_{0,0}(T)$ is the resolvent set of T and that the sets defined above are pairwise disjoint and cover the complex plane. The collection of all of them is denoted by $\Sigma(T)$. We say that the sets $\sigma^1_{n,m}(T)$ and $\sigma^0_{n,m}(T)$ correspond to, respectively, $\Sigma^1_{n,m}(\mathcal{H})$ and $\Sigma^0_{n,m}(\mathcal{H})$.

PROPOSITION 4.7. *For every closed densely defined operator T in $\mathcal{H}$, $\Sigma(T)$ consists of Borel subsets of $\mathbb{C}$. What is more,* $\operatorname{card} \sigma^f(T) \leq 1$ *and each member of $\Sigma(T)$ different from $\sigma^f(T)$ is of the same Borel class as the corresponding member of $\Sigma(\mathcal{H})$.*

Proof. First observe that if $z \in \sigma^f(T)$, then $\mathcal{D}(T) = \mathcal{H}$ and T is bounded (since $\mathcal{N}(T-z)$ is closed). We conclude that indeed $\operatorname{card} \sigma^f(T) \leq 1$. Further, since T is closed, there is an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = \mathcal{D}(T)$ (e.g. $A := (|T|+1)^{-1}$; see also [5, Theorem 1.1]). Put $C = TA \in \mathcal{B}(\mathcal{H})$. Notice that for each $z \in \mathbb{C}$, we have $\mathcal{R}(C - zA) = \mathcal{R}(T-z)$ and $\mathcal{N}(C-zA) = A^{-1}(\mathcal{N}(T-z))$, and thus $\dim \mathcal{N}(T-z) = \dim \mathcal{N}(C-zA)$. This implies that $z \in \sigma^j_{n,m}(T)$ iff $C - zA \in \Sigma^j_{n,m}(\mathcal{H})$. So, the continuity of the function $\mathbb{C} \ni z \mapsto C - zA \in \mathcal{B}(\mathcal{H})$ finishes the proof. ∎

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Piotr Niemiec
Instytut Matematyki
Wydział Matematyki i Informatyki
Uniwersytet Jagielloński
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: piotr.niemiec@uj.edu.pl

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