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Geometric characterization of L_1-spaces

by

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Abstract. The paper is devoted to a description of all real strongly facially symmetric spaces which are isometrically isomorphic to L_1-spaces. We prove that if Z is a real neutral strongly facially symmetric space such that every maximal geometric tripotent from the dual space of Z is unitary, then the space Z is isometrically isomorphic to the space $L_1(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is an appropriate measure space having the direct sum property.

1. Introduction. One of the main problems in operator algebras is a geometric characterization of operator algebras and operator spaces. In this connection in papers of Y. Friedman and B. Russo the so-called facially symmetric spaces were introduced (see [4–9, 12]). In [8], the complete structure of atomic facially symmetric spaces was determined. More precisely, it was shown that an irreducible, neutral, strongly facially symmetric space is linearly isometric to the predual of one of the Cartan factors of types 1 to 6, provided that it satisfies some natural and physically significant axioms, four in number, which are known to hold in the preduals of all JBW^*-triples.

The project of classifying facially symmetric spaces was started in [7], where, using two of the pure state properties, denoted by STP and FE, geometric characterizations of complex Hilbert spaces and complex spin factors were given. The former is precisely a rank 1 JBW^*-triple and a special case of a Cartan factor of type 1, and the latter is the Cartan factor of type 4 and a special case of a JBW^*-triple of rank 2. The explicit structure of a spin factor naturally embedded in a facially symmetric space was then used in [8] to construct abstract generating sets and complete the classification in the atomic case. In [12] a geometric characterization of the dual ball of global JB^*-triples was given.

The present paper is devoted to a description of all real strongly facially symmetric spaces which are isometrically isomorphic to L_1-spaces.

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Using Kakutani's characterization of real L_1-spaces, we show that a neutral strongly facially symmetric space in which every maximal geometric tripotent is unitary, is isometrically isomorphic to an L_1-space. None of the extra axioms used in [7, 8, 12] are assumed.

2. Facially symmetric spaces. In this section we shall recall some basic facts and notation about facially symmetric spaces (see for details [4–8]).

Let Z be a real or complex normed space. Elements $x, y \in Z$ are *orthogonal*, notation $x \diamondsuit y$, if $\|x+y\| = \|x-y\| = \|x\| + \|y\|$. Subsets $S, T \subset Z$ are said to be *orthogonal*, notation $S \diamondsuit T$, if $x \diamondsuit y$ for all $(x, y) \in S \times T$. A *norm exposed face* of the unit ball Z_1 of Z is a non-empty set (necessarily $\neq Z_1$) of the form $F = F_u = \{x \in Z : u(x) = 1\}$, where $u \in Z^*$, $\|u\| = 1$. Recall that a *face* G of a convex set K is a non-empty convex subset of K such that if $\lambda y + (1-\lambda)z \in G$, where $y, z \in K$, $\lambda \in (0,1)$, then $y, z \in G$. In particular, an extreme point of K is a face of K. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, y\rangle = 0$ for all $y \in F_u^{\diamondsuit}$. Here, for any subset S, $S^{\diamondsuit}$ denotes the set of all elements orthogonal to each element of S.

A norm exposed face F_u in Z_1 is said to be a *symmetric face* if there is a linear isometric symmetry S_u of Z onto Z with $S_u^2 = I$ such that the fixed point set of S_u is $(\overline{\mathrm{sp}}\, F_u) \oplus F_u^{\diamondsuit}$.

Recall that a normed space Z is said to be *weakly facially symmetric* (WFS) if every norm exposed face in Z_1 is symmetric.

For each symmetric face F_u the contractive projections $P_k(F_u)$, $k = 0, 1, 2$, on Z are defined as follows. First $P_1(F_u) = (I - S_u)/2$ is the projection on the -1 eigenspace of S_u. Next define $P_2(F_u)$ and $P_0(F_u)$ as the projections of Z onto $\overline{\mathrm{sp}}\, F_u$ and $F_u^{\diamondsuit}$, respectively, so that $P_2(F_u) + P_0(F_u) = (I + S_u)/2$. A *geometric tripotent* is a projective unit u with the property that F_u is a symmetric face and $S_u^* u = u$ for a symmetry S_u corresponding to u. The projections $P_k(F_u)$ are called the *geometric Peirce projections*.

$\mathcal{GT}$ and $\mathcal{SF}$ denote the collections of geometric tripotents and symmetric faces respectively, and the map $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is a bijection [5, Proposition 1.6]. For each geometric tripotent u in the dual of a WFS space Z, we shall denote the geometric Peirce projections by $P_k(u) = P_k(F_u)$, $k = 0, 1, 2$. Two elements f and g of Z^* are orthogonal if one of them belongs to $P_2(u)^*(Z^*)$ and the other to $P_0(u)^*(Z^*)$ for some geometric tripotent u.

A contractive projection Q on a normed space Z is said to be *neutral* if for each $x \in Z$, $\|Q(x)\| = \|x\|$ implies $Q(x) = x$. A normed space Z is *neutral* if for every symmetric face F_u, the projection $P_2(F_u)$ is neutral.

A WFS space Z is *strongly facially symmetric* (SFS) if for every norm exposed face F_u in Z_1 and every $g \in Z^*$ with $\|g\| = 1$ and $F_u \subset F_g$, we have $S_u^* g = g$, where S_u denotes a symmetry associated with F_u.

The principal examples of neutral complex strongly facially symmetric spaces are preduals of complex JBW^*-triples, in particular, the preduals of von Neumann algebras (see [6]). In these cases, as shown in [6], geometric tripotents correspond to tripotents in a JBW^*-triple and to partial isometries in a von Neumann algebra.

In a neutral strongly facially symmetric space Z, every non-zero element has a polar decomposition [5, Theorem 4.3]: for non-zero $x \in Z$ there exists a unique geometric tripotent $v = v_x$ with $\langle v, x\rangle = \|x\|$ and $\langle v, x^{\diamondsuit}\rangle = 0$. If $x, y \in Z$, then $x \diamondsuit y$ if and only if $v_x \diamondsuit v_y$, as follows from [4, Corollary 1.3(b) and Lemma 2.1].

A partial ordering can be defined on the set of geometric tripotents as follows: if $u, v \in \mathcal{GT}$, then $u \le v$ if $F_u \subset F_v$, or equivalently, by [5, Lemma 4.2], $P_2(u)^* v = u$, or $v - u$ is either zero or a geometric tripotent orthogonal to u.

3. Main result. Henceforth "face" means "norm exposed face".

Let Z be a real neutral strongly facially symmetric space. A geometric tripotent $u \in \mathcal{GT}$ is said to be

- *maximal* if $P_0(u) = 0$;
- *unitary* if $P_2(u) = I$.

It is clear that any unitary geometric tripotent is maximal.

Notice that a geometric tripotent e is unitary if and only if the convex hull of the set $F_e \cup F_{-e}$ coincides with the unit ball Z_1, i.e.

$$Z_1 = \mathrm{co}\{F_e \cup F_{-e}\}. \tag{3.1}$$

Also note that property (3.1) is much stronger than the Jordan decomposition property of a face (see [12, Lemmata 2.3–2.6]). Recall that a face F_u has the *Jordan decomposition property* if its real span coincides with the geometric Peirce 2-space of the geometric tripotent u.

EXAMPLE 3.1. The space $\mathbb{R}^n$ with the norm

$$\|x\| = \sum_{i=1}^{n} |t_i|, \qquad x = (t_i) \in \mathbb{R}^n,$$

is a SFS space. If $e \in \mathbb{R}^n \cong (\mathbb{R}^n)^*$ is a maximal geometric tripotent then

$$e = (\varepsilon_1, \ldots, \varepsilon_n), \qquad \varepsilon_i \in \{-1, 1\},\ i \in \overline{1, n},$$

and in this case the face

$$F_e = \Big\{x \in \mathbb{R}^n : \sum_{i=1}^{n} \varepsilon_i t_i = 1,\ \varepsilon_i t_i \ge 0,\ i = \overline{1, n}\Big\}$$

satisfies (3.1).

More generally, consider a measure space (Ω, Σ, μ) with measure μ having the *direct sum property*, i.e. there is a family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$ with $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B of zero measure such that $A = \bigcup_{i\in J_0}(A \cap \Omega_i) \cup B$.

Let $L_1(\Omega, \Sigma, \mu)$ be the space of all real integrable functions on (Ω, Σ, μ). The space $L_1(\Omega, \Sigma, \mu)$ with the norm

$$\|f\| = \int_\Omega |f(t)|\,d\mu(t), \qquad f \in L_1(\Omega, \Sigma, \mu),$$

is a SFS space. If $e \in L^\infty(\Omega, \Sigma, \mu) \cong L_1(\Omega, \Sigma, \mu)^*$ is a maximal geometric tripotent then

$$e = \tilde{\chi}_A - \tilde{\chi}_{\Omega\setminus A} \quad \text{for some } A \in \Sigma,$$

where $\tilde{\chi}_A$ is the class containing the indicator function of the set $A \in \Sigma$. Then the face

$$F_e = \Big\{f \in L_1(\Omega, \Sigma, \mu) : \|f\| = 1, \int_\Omega e(t)f(t)\,d\mu(t) = 1\Big\}$$

satisfies (3.1).

The next result is the main result of the paper, giving a description of all strongly facially symmetric spaces which are isometrically isomorphic to L_1-spaces.

THEOREM 3.2. *Let Z be a real neutral strongly facially symmetric space such that every maximal geometric tripotent from Z^* is unitary. Then there exists a measure space (Ω, Σ, μ) with measure μ having the direct sum property such that the space Z is isometrically isomorphic to the space $L_1(\Omega, \Sigma, \mu)$.*

For the proof we need several lemmata.

Let $u, v \in \mathcal{GT}$. If $F_u \cap F_v \neq \emptyset$ then by $u \wedge v$ we denote the unique geometric tripotent such that $F_{u\wedge v} = F_u \cap F_v$, otherwise we set $u \wedge v = 0$.

LEMMA 3.3. *Let $e \in \mathcal{GT}$ be unitary and let $v \in \mathcal{GT}$. Then $F_v \cap F_e \neq \emptyset$ or $F_{-v} \cap F_e \neq \emptyset$.*

Proof. Let $x \in F_v$. By equality (3.1) we obtain

$$x = ty + (1-t)z$$

for some $y, -z \in F_e$ and $0 \le t \le 1$.

If $t = 1$ or $t = 0$ then $x = y$ or $x = z$, respectively. Hence $x \in F_v \cap F_e$ or $-x \in F_{-v} \cap F_e$.

Let $0 < t < 1$. Since F_v is a face, $y, z \in F_v$. Therefore $F_v \cap F_e \neq \emptyset$ and $F_{-v} \cap F_e \neq \emptyset$. ∎

LEMMA 3.4. *Let $e \in \mathcal{GT}$ be unitary. Then for every $u \in \mathcal{GT}$ there exist mutually orthogonal geometric tripotents $u_1, u_2 \le e$ such that $u = u_1 - u_2$.*

Proof. Put

$$u_1 = u \wedge e, \qquad u_2 = (-u) \wedge e.$$

Let us prove that

$$u_1 \diamondsuit u_2, \qquad u = u_1 - u_2.$$

Let $x_1 \in F_{u_1}$ and $x_2 \in F_{u_2}$. Then

$$x_1, x_2 \in F_e, \qquad x_1, -x_2 \in F_u,$$

and therefore

$$\frac{x_1 + x_2}{2} \in F_e, \qquad \frac{x_1 - x_2}{2} \in F_u.$$

Thus

$$\left\| \frac{x_1 + x_2}{2} \right\| = 1, \qquad \left\| \frac{x_1 - x_2}{2} \right\| = 1,$$

and

$$\|x_1 + x_2\| = \|x_1 - x_2\| = 2 = \|x_1\| + \|x_2\|.$$

Hence $x_1 \diamondsuit x_2$, and therefore $u_1 \diamondsuit u_2$.

Now suppose that $v = u - u_1 + u_2 \neq 0$. By Lemma 3.3 we know that $F_v \cap F_e \neq \emptyset$ or $F_{-v} \cap F_e \neq \emptyset$. Without loss of generality it can be assumed that $F_v \cap F_e \neq \emptyset$. Thus there exists an element $x \in Z_1$ such that

$$\langle v, x \rangle = \langle e, x \rangle = 1.$$

Since $v \leq u$, we have $\langle u, x \rangle = 1$. Thus $x \in F_u \cap F_e$, i.e. $x \in F_{u_1}$ or $\langle u_1, x \rangle = 1$. Since $u_1 \diamondsuit u_2$, we have $\langle u_2, x \rangle = 0$. Hence

$$\langle v, x \rangle = \langle u, x \rangle - \langle u_1, x \rangle + \langle u_2, x \rangle = 0,$$

a contradiction. ∎

Lemma 3.5. *Let u, w be orthogonal geometric tripotents. Then $u + w$ is maximal if and only if $u - w$ is maximal.*

Proof. Let $u + w$ be maximal. Suppose that $u - w$ is not maximal. Then there exists a maximal geometric tripotent e such that $e > u - w$. Set $w_1 = e - u + w$. Then $w_1 \diamondsuit u$ and $w_1 \diamondsuit w$. Therefore $u + w < u + w + w_1$. This contradicts the maximality of $u + w$. ∎

Recall that a face F of a convex set K is called a *split face* if there exists a face G, called *complementary* to F, such that $F \cap G = \emptyset$ and K is the direct convex sum $F \oplus_c G$, i.e. any element $x \in K$ can be uniquely represented in the form $x = ty + (1 - t)z$, where $t \in [0, 1]$, $y \in F$, $z \in G$ (see e.g. [1, p. 420], [2]).

Lemma 3.6. *Let u, w be orthogonal geometric tripotents. If $u + w$ is maximal then*

$$F_{u+w} = F_u \oplus_c F_w. \tag{3.2}$$

Proof. First we shall show that

$$F_{u+w} = \mathrm{co}\{F_u \cup F_w\}.$$

It suffices to show that

$$F_{u+w} \subseteq \mathrm{co}\{F_u \cup F_w\}.$$

By Lemma 3.5 the geometric tripotent $u - w$ is maximal. Therefore the face F_{u-w} satisfies equality (3.1), i.e.

$$Z_1 = \mathrm{co}\{F_{u-w} \cup F_{w-u}\}.$$

Thus every element $x \in F_{u+w}$ has the form

$$x = ty + (1-t)z$$

for some $y, -z \in F_{u-w}$ and $0 \leq t \leq 1$.

Consider the following three cases.

Case 1. If $t = 0$ then $x \in F_{u+w} \cap F_{w-u} = F_w$.

Case 2. If $t = 1$ then $x \in F_{u+w} \cap F_{u-w} = F_u$.

Case 3. If $0 < t < 1$, then applying the geometric tripotent $u + w$ to the equality $x = ty + (1-t)z$ we obtain

$$tu(y) + tw(y) + (1-t)u(z) + (1-t)w(z) = 1. \tag{3.3}$$

Since $y \in F_{u-w}$ and $z \in F_{w-u}$ we see that

$$u(y) - w(y) = 1, \qquad w(z) - u(z) = 1.$$

Thus

$$tu(y) - tw(y) - (1-t)u(z) + (1-t)w(z) = 1. \tag{3.4}$$

Summing (3.3) and (3.4) we get

$$tu(y) + (1-t)w(z) = 1.$$

Since $|u(y)| \leq 1$ and $|w(z)| \leq 1$ the last equality implies that

$$u(y) = w(z) = 1.$$

This means that $y \in F_u$ and $z \in F_w$. Therefore

$$x = ty + (1-t)z \in \mathrm{co}\{F_u \cup F_w\}.$$

Consequently, $F_{u+w} = \mathrm{co}\{F_u \cup F_w\}$. Taking into account that $F_u \diamondsuit F_w$ we get $F_{u+w} = F_u \oplus_c F_w$. ∎

Let u be an arbitrary geometric tripotent and let e be a maximal geometric tripotent such that $u \leq e$. First we shall show that

$$Z = \overline{\mathrm{sp}}\, F_u \oplus \overline{\mathrm{sp}}\, F_w,$$

where $w = e - u$. Using equalities (3.1) and (3.2) we obtain

$$\begin{aligned} Z &= \operatorname{sp} Z_1 = \operatorname{sp}\{\operatorname{co}\{F_e \cup F_{-e}\}\} \\ &= \operatorname{sp} F_e = \operatorname{sp}\{F_u \oplus_c F_w\} = \operatorname{sp} F_u \oplus \operatorname{sp} F_w. \end{aligned}$$

From $\operatorname{sp} F_u \diamondsuit \operatorname{sp} F_w$ it follows that $\overline{\operatorname{sp}} F_u \diamondsuit \overline{\operatorname{sp}} F_w$, and therefore

$$Z = \overline{\operatorname{sp}} F_u \oplus \overline{\operatorname{sp}} F_w.$$

This implies that

$$P_2(u) + P_2(w) = I.$$

Since $P_1(u)P_0(u) = 0$ and $P_2(w) = P_0(u)P_2(w)$ (see [5, Corollary 3.4]) we obtain $P_1(u)P_2(w) = 0$. Therefore

$$P_1(u) = P_1(u)I = P_1(u)[P_2(u) + P_2(w)] = 0.$$

So we have

LEMMA 3.7. *For every $u \in \mathcal{GT}$ the projection $P_1(u)$ is zero.*

For orthogonal geometric tripotents v_1, v_2 we have

$$P_2(v_1 + v_2) = P_2(v_1) + P_2(v_2). \tag{3.5}$$

Indeed, by [5, Lemma 1.8] we have

$$P_0(v_1 + v_2) = P_0(v_1)P_0(v_2).$$

Using the last equality and taking into account the equalities $P_1(v_1) = P_1(v_2) = P_1(v_1 + v_2) = 0$, together with Corollary 3.4 of [5], we get

$$\begin{aligned} P_2(v_1 + v_2) &= I - P_0(v_1 + v_2) = I^2 - P_0(v_1 + v_2) \\ &= (P_2(v_1) + P_0(v_1))(P_2(v_2) + P_0(v_2)) - P_0(v_1)P_0(v_2) \\ &= P_2(v_1) + P_2(v_2) + P_0(v_1)P_0(v_2) - P_0(v_1)P_0(v_2) \\ &= P_2(v_1) + P_2(v_2). \end{aligned}$$

Now we fix a unitary $e \in \mathcal{GT}$.

On the space Z we define an order (depending on e) by the following rule:

$$x \geq y \Leftrightarrow x - y \in \mathbb{R}^+ F_e. \tag{3.6}$$

LEMMA 3.8. *Z is a partially ordered linear space, i.e.*

(i) $x \leq x$;
(ii) $x \leq y$, $y \leq z \Rightarrow x \leq z$;
(iii) $x \leq y$, $y \leq x \Rightarrow x = y$;
(iv) $x \leq y \Rightarrow x + z \leq y + z$;
(v) $x \geq 0$, $\lambda \geq 0 \Rightarrow \lambda x \geq 0$.

Proof. The properties (i), (iv) and (v) are trivial.

To prove (ii), let $x \leq y$ and $y \leq z$. Then $y - x, z - y \in \mathbb{R}^+ F_e$. Thus $z - x \in \mathbb{R}^+ F_e$, i.e. $x \leq z$.

For (iii), let $x \leq y$, $y \leq x$. Then $y - x = \alpha a$ and $x - y = \beta b$ for some $\alpha, \beta \geq 0$ and $a, b \in F_e$. Therefore $\alpha a + \beta b = 0$. Applying to this equality the geometric tripotent e we obtain $\alpha + \beta = 0$. Thus $\alpha = \beta = 0$, i.e. $x = y$. ∎

Remark 3.9. Note that if $v \leq e$ then [12, Lemma 2.4] implies that

$$P_k(v)(F_e) \subseteq F_e, \qquad k = 0, 2. \tag{3.7}$$

Lemma 3.10. *Let $a, b, x, y \geq 0$ with $a \diamondsuit b$. If $a - b = x - y$ then*

$$x - a = y - b \geq 0;$$

if in addition $x \diamondsuit y$, then $x = a$ and $y = b$.

Proof. Let v_a be the smallest geometric tripotent such that $v_a(a) = \|a\|$ (polar decomposition). Since $a \geq 0$ it follows that $v_a \leq e$. Applying the projection $P_2(v_a)$ to the equality $a - b = x - y$ we obtain

$$P_2(v_a)(x) - P_2(v_a)(y) = P_2(v_a)(a - b) = P_2(v_a)(a) = a.$$

Using (3.7) we get

$$P_2(v_a)(x) - a = P_2(v_a)(y) \in \mathbb{R}^+ F_e,$$

and therefore

$$\begin{aligned} x - a &= P_2(v_a)(x) + P_0(v_a)(x) - a \\ &= [P_2(v_a)(x) - a] + P_0(v_a)(x) \in \mathbb{R}^+ F_e, \end{aligned}$$

i.e. $x \geq a$.

Now suppose that $x \diamondsuit y$. Then as shown above, $x \geq a$ and $a \geq x$. Thus $x = a$ and $y = b$. ∎

Lemma 3.11. *For $x \in Z$ the following conditions are equivalent:*

(i) $x \geq 0$;
(ii) $\|x\| = \langle e, x \rangle$.

Proof. Take $x \geq 0$, i.e. $x = \alpha y$ for some $\alpha \geq 0$ and $y \in F_e$. Then

$$\|x\| = \|\alpha y\| = \alpha \|y\| = \alpha = \alpha \langle e, y \rangle = \langle e, x \rangle.$$

Conversely, if $\|x\| = \langle e, x \rangle, x \neq 0$, then $x/\|x\| \in F_e$, i.e. $x \geq 0$. ∎

Lemma 3.12. *Every element $x \in Z$ can be uniquely represented as*

$$x = x_+ - x_-,$$

where $x_+, x_- \geq 0$ and $x_+ \diamondsuit x_-$.

Proof. Take the smallest geometric tripotent $v_x \in \mathcal{GT}$ such that $v_x(x) = \|x\|$. By Lemma 3.4 there exist mutually orthogonal geometric tripotents $v_1, v_2 \leq e$ such that $v_x = v_1 - v_2$. Put

$$x_+ = P_2(v_1)(x), \qquad x_- = -P_2(v_2)(x).$$

By the proof of [5, Theorem 4.3(d)] we get $\langle v_1, x\rangle = \|P_2(v_1)(x)\|$, and therefore

$$\begin{aligned}\langle e, x_+\rangle &= \langle e, P_2(v_1)(x)\rangle = \langle P_2^*(v_1)e, x\rangle \\ &= \langle v_1, x\rangle = \|P_2(v_1)(x)\| = \|x_+\|.\end{aligned}$$

This means that $x_+ \geq 0$. Similarly $x_- \geq 0$. Further using equality (3.5) we find that $x = x_+ - x_-$ and $x_+ \diamondsuit x_-$. Uniqueness follows from Lemma 3.10. ∎

LEMMA 3.13. *Z is a lattice, i.e. for any $x, y \in Z$ there exist*

$$x \vee y,\ x \wedge y \in Z.$$

Proof. By Lemma 3.12 there exist mutually orthogonal elements $a, b \geq 0$ such that $x - y = a - b$. Then

$$x \vee y = \frac{x+y+a+b}{2}, \tag{3.8}$$

$$x \wedge y = \frac{x+y-a-b}{2}. \tag{3.9}$$

Indeed,

$$x \vee y - x = \frac{x+y+a+b}{2} - x = \frac{y-x+a+b}{2} = b \geq 0$$

and

$$x \vee y - y = \frac{x+y+a+b}{2} - y = \frac{x-y+a+b}{2} = a \geq 0.$$

Now let $x, y \leq z$, where $z \in Z$. Denote

$$x_1 = z - x \geq 0, \qquad y_1 = z - y \geq 0.$$

Thus $x - y = y_1 - x_1$. Therefore $y_1 - x_1 = a - b$. Lemma 3.10 implies that

$$y_1 - a = x_1 - b \geq 0.$$

Further

$$\begin{aligned} z - x \vee y &= \frac{x+y+x_1+y_1}{2} - \frac{x+y+a+b}{2} \\ &= \frac{x_1+y_1-a-b}{2} = y_1 - a \geq 0.\end{aligned}$$

This means that

$$x \vee y = \frac{x+y+a+b}{2}.$$

In the same way we can prove equality (3.9). ∎

A Banach lattice X is said to be an *abstract L-space* if

$$\|x+y\| = \|x\| + \|y\|$$

for all $x, y \in X$ with $x \wedge y = 0$ (see [11, p. 14] and [10]).

Lemma 3.14. *Z is an abstract L-space.*

Proof. First we show that

- $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$;
- $\|x\| = \| \, |x| \, \|$,

where $|x| = x_+ + x_-$ is the absolute value of x.

Let $0 \leq x \leq y$. Then

$$\|x\| = \langle e, x \rangle \leq \langle e, y \rangle = \|y\|.$$

Further

$$\| \, |x| \, \| = \|x_+ + x_-\| = [x_+ \diamondsuit x_-] = \|x_+ - x_-\| = \|x\|.$$

Hence Z is a Banach lattice.

For $x, y \geq 0$, using Lemma 3.11 we obtain

$$\|x + y\| = \langle e, x + y \rangle = \langle e, x \rangle + \langle e, y \rangle = \|x\| + \|y\|.$$

This means that Z is an abstract L-space. ∎

Now Theorem 3.2 follows from Lemma 3.14 and [11, Theorem 1.b.2].

Remark 3.15. The following observations were kindly suggested by the referee, to whom the authors are deeply indebted.

Theorem 3.2 fails for complex spaces. Indeed, by [6, Theorem 2.11] for any finite von Neumann algebra its predual is a neutral strongly facially symmetric space in which every maximal geometric tripotent is unitary. However, that predual is not isometric to an L_1-space, for example for the algebra $B(H)$ of all bounded linear operators on the finite-dimensional Hilbert space H of dimension at least 2.

The predual of a real JBW^*-triple is a neutral weakly facially symmetric space (see [3, Theorem 5.5] and [6, Theorem 3.1]) which is not strongly facially symmetric. The strong facial symmetry of the predual of a complex von Neumann algebra depends on the field being complex (see the proof of Corollary 2.9 in [6]). Indeed, if the predual of a non-commutative real von Neumann algebra were a strongly facially symmetric space, this would contradict Theorem 3.2 above.

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