Necessary and sufficient Tauberian conditions for the logarithmic summability of functions and sequences

by

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Dedicated to Professor Wiesław Żelazko on his 80th birthday

Abstract. Let $s : [1, \infty) \to \mathbb{C}$ be a locally Lebesgue integrable function. We say that s is summable $(L, 1)$ if there exists some $A \in \mathbb{C}$ such that

$$(*) \qquad \lim_{t \to \infty} \tau(t) = A, \quad \text{where} \quad \tau(t) := \frac{1}{\log t} \int_1^t \frac{s(u)}{u} \, du.$$

It is clear that if the ordinary limit $s(t) \to A$ exists, then also $\tau(t) \to A$ as $t \to \infty$. We present sufficient conditions, which are also necessary, in order that the converse implication hold true. As corollaries, we obtain so-called Tauberian theorems which are analogous to those known in the case of summability $(C, 1)$. For example, if the function s is slowly oscillating, by which we mean that for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$|s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq t^\lambda,$$

then the converse implication holds true: the ordinary convergence $\lim_{t \to \infty} s(t) = A$ follows from $(*)$.

We also present necessary and sufficient Tauberian conditions under which the ordinary convergence of a numerical sequence (s_k) follows from its logarithmic summability. Furthermore, we give a more transparent proof of an earlier Tauberian theorem due to Kwee.

1. Introduction: Summability $(C, 1)$ and $(L, 1)$ of functions. Let $s : [0, \infty) \to \mathbb{C}$ be a function Lebesgue integrable on every bounded interval $[0, t]$, $t > 0$, in symbols $s \in L_{\text{loc}}[0, \infty)$. We recall (see, e.g., [2, p. 11]) that s is said to be *Cesàro summable of first order*, briefly: summable $(C, 1)$, if

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there exists some $A \in \mathbb{C}$ such that

(1.1) $\displaystyle\lim_{t\to\infty} \sigma(t) = A,$ where $\displaystyle\sigma(t) := \frac{1}{t}\int_0^t s(u)\,du, \quad t > 0.$

Clearly, if the ordinary limit

(1.2) $$\lim_{t\to\infty} s(t) = A$$

exists, then (1.1) holds. The converse implication holds true only under some supplementary, so-called Tauberian condition(s).

We note that the left endpoint of the definition domain of s is irrelevant in (1.1). That is, given any $a > 0$, the existence of the limit in (1.1) is equivalent to

$$\lim_{t\to\infty} \frac{1}{t}\int_a^t s(u)\,du = A.$$

For the Cesàro summability of order α, where $\alpha \geq 0$ is a real number, briefly: summability (C, α), we refer to [7, p. 26]. The case $\alpha = 0$ is ordinary convergence.

Next, let $s : [1, \infty) \to \mathbb{C}$ be such that $s \in L_{\mathrm{loc}}[1, \infty)$. Motivated by the concept of logarithmic (sometimes also called harmonic) summability of a numerical sequence (see, e.g., [5]), the function s is said to be *logarithmic summable of first order*, briefly: summable $(L, 1)$, if there exists some $A \in \mathbb{C}$ such that

(1.3) $\displaystyle\lim_{t\to\infty} \tau(t) = A,$ where $\displaystyle\tau(t) := \frac{1}{\log t}\int_1^t \frac{s(u)}{u}\,du, \quad t > 1,$

where the logarithm is to the natural base e.

In Section 4, we will prove that summability $(C, 1)$ of a function implies its summability $(L, 1)$ to the same limit, but the converse implication is not true in general.

We note that a complex-valued function $s \in L_{\mathrm{loc}}[e, \infty)$ is said to be *logarithmic summable of order* 2, briefly: summable $(L, 2)$, if there exists some $A \in \mathbb{C}$ such that

(1.4) $\displaystyle\lim_{t\to\infty} \tau_2(t) = A$ where $\displaystyle\tau_2(t) := \frac{1}{\log\log t}\int_e^t \frac{s(u)}{u\log u}\,du, \quad t > e.$

We also note that in the particular cases when

(1.5)   (i) $\displaystyle s(u) := \int_0^u f(x)\,dx,\ u > 0,$ or   (ii) $\displaystyle s(u) := \int_1^u f(x)\,dx,\ u > 1,$

where f is a locally integrable function on $[0, \infty)$ or $[1, \infty)$, respectively, the above summability methods may be applied to assign a value to the following

integrals, respectively:

$$(1.6) \qquad \text{(i)} \int_0^\infty f(x)\,dx \quad \text{or} \quad \text{(ii)} \int_1^\infty f(x)\,dx.$$

If the finite limit in (1.2) exists, then the improper integrals $\int_0^{\to\infty} f(x)\,dx$ and $\int_1^{\to\infty} f(x)\,dx$ exist, respectively. In the case when the finite limit in (1.1) exists, the integral in (1.6)(i) is said to be summable $(C,1)$; while in the case when only the finite limit in (1.3) exists, the integral in (1.6)(ii) is said to be summable $(L,1)$.

2. Main results. In our first new result we characterize the converse implication when the ordinary limit of a real-valued function at ∞ follows from its summability $(L,1)$.

THEOREM 2.1. *If a real-valued function* $s \in L_{\mathrm{loc}}[1,\infty)$ *is summable* $(L,1)$ *to some* $A \in \mathbb{R}$, *then the ordinary limit* (1.2) *exists if and only if*

$$(2.1) \qquad \limsup_{\lambda\to 1+} \liminf_{t\to\infty} \frac{1}{(\lambda-1)\log t} \int_t^{t^\lambda} \frac{s(u)-s(t)}{u}\,du \geq 0$$

and

$$(2.2) \qquad \limsup_{\lambda\to 1-} \liminf_{t\to\infty} \frac{1}{(1-\lambda)\log t} \int_{t^\lambda}^{t} \frac{s(t)-s(u)}{u}\,du \geq 0.$$

Motivated by the definition of 'slow decrease' with respect to summability $(C,1)$ (see, e.g., [2, pp. 124–125; and cf. our Remark 2.1 below]), we say that a function $s : [1,\infty) \to \mathbb{R}$ is *slowly decreasing with respect to summability* $(L,1)$ if for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(2.3) \qquad s(u) - s(t) \geq -\varepsilon \quad \text{whenever} \quad t_0 \leq t < u \leq t^\lambda.$$

It is easy to check that a function s is slowly decreasing with respect to summability $(L,1)$ if and only if

$$(2.4) \qquad \lim_{\lambda\to 1+} \liminf_{t\to\infty} \inf_{t<u\leq t^\lambda} (s(u)-s(t)) \geq 0.$$

Since the auxiliary function

$$a(\lambda) := \liminf_{t\to\infty} \inf_{t<u\leq t^\lambda} (s(u)-s(t))$$

is evidently decreasing in λ on the infinite interval $(1,\infty)$, the right limit in (2.4) exists and $\lim_{\lambda\to 1+}$ can be replaced by $\sup_{\lambda>1}$.

It is clear that if a function $s \in L_{\mathrm{loc}}[1,\infty)$ is slowly decreasing with respect to summability $(L,1)$, then conditions (2.1) and (2.2) are trivially

satisfied. Thus, the next corollary is an immediate consequence of Theorem 2.1.

COROLLARY 2.2. *Suppose a real-valued function $s \in L_{\mathrm{loc}}[1, \infty)$ is slowly decreasing with respect to summability $(L, 1)$. If s is summable $(L, 1)$ to some $A \subset \mathbb{R}$, then the ordinary limit (1.2) also exists.*

Historically, the term 'slow decrease' was introduced by Schmidt [6] in the case of the summability $(C, 1)$ of sequences of real numbers.

In our second new result we characterize the converse implication when the ordinary convergence of a complex-valued function follows from its summability $(L, 1)$.

THEOREM 2.3. *If a complex-valued function $s \in L_{\mathrm{loc}}[1, \infty)$ is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (1.2) exists if and only if*

$$(2.5) \qquad \liminf_{\lambda \to 1+} \limsup_{t \to \infty} \left| \frac{1}{(\lambda - 1) \log t} \int_{t}^{t^{\lambda}} \frac{s(u) - s(t)}{u} \, du \right| = 0.$$

Motivated by the definition of 'slow oscillation' with respect to summability $(C, 1)$ of numerical sequences introduced by Hardy [1] (see also in [2, pp. 124–125]), we say that a function $s : [1, \infty) \to \mathbb{C}$ is *slowly oscillating with respect to summability* $(L, 1)$ if for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(2.6) \qquad |s(u) - s(t)| \le \varepsilon \quad \text{whenever} \quad t_0 \le t < u \le t^{\lambda}.$$

It is easy to check that a function s is slowly oscillating with respect to summability $(L, 1)$ if and only if

$$(2.7) \qquad \lim_{\lambda \to 1+} \limsup_{t \to \infty} \sup_{t < u \le t^{\lambda}} |s(u) - s(t)| = 0.$$

It is clear that if a function $s \in L_{\mathrm{loc}}[1, \infty)$ is slowly oscillating with respect to summability $(L, 1)$, then condition (2.5) is trivially satisfied. Thus, the next corollary is an immediate consequence of Theorem 2.3.

COROLLARY 2.4. *Suppose a complex-valued function $s \in L_{\mathrm{loc}}[1, \infty)$ is slowly oscillating with respect to summability $(L, 1)$. If s is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (1.2) also exists.*

REMARK 2.1. According to Hardy's definition (see [2, pp. 124–125]), a function $s : (0, \infty) \to \mathbb{C}$ is said to be *slowly oscillating* if

$$(2.8) \qquad \lim(s(u) - s(t)) = 0 \quad \text{whenever} \quad u > t \to \infty \quad \text{and} \quad u/t \to 1;$$

and a function $s : (0, \infty) \to \mathbb{R}$ is said to be *slowly decreasing* if

$$(2.9) \qquad \liminf(s(u) - s(t)) \ge 0 \quad \text{under the same circumstances.}$$

We claim that definition (2.8) is equivalent to the following: for every $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon) > 0$ and $\lambda = \lambda(\varepsilon) > 1$ such that

(2.10) $\qquad |s(u) - s(t)| \leq \varepsilon \quad$ whenever $\quad t_0 \leq t < u \leq \lambda t.$

The implication (2.8)$\Rightarrow$(2.10) is trivial. To justify the converse, let $\lambda > 1$ be arbitrarily close to 1 and set $\varepsilon := \log \lambda$. Then by (2.10), we have

$$|s(u) - s(t)| \leq \varepsilon \quad \text{whenever} \quad u > t \geq t_0 \quad \text{and} \quad 0 < \log \frac{u}{t} \leq \log \lambda = \varepsilon.$$

Now, the claimed equivalence is obvious.

It is worth considering the special case (1.6)(ii), where $f \in L_{\mathrm{loc}}[1, \infty)$. If f is a real-valued function and

(2.11) $\qquad x(\log x) f(x) \geq -C \quad$ for almost every $x > x_0,$

where $C > 0$ and $x_0 \geq 1$ are constants, then s defined in (1.5)(ii) is slowly decreasing with respect to summability $(L, 1)$, and Corollary 2.2 applies. Likewise, if f is a complex-valued function and

(2.12) $\qquad x(\log x) |f(x)| \leq C \quad$ for almost every $x > x_0,$

where $C > 0$ and $x_0 \geq 1$ are constants, then s is slowly oscillating with respect to summability $(L, 1)$, and Corollary 2.4 applies.

Condition (2.11) is called a *one-sided Tauberian condition*, while (2.12) is called a *two-sided Tauberian condition* with respect to summability $(L, 1)$. These terms go back to Landau [4] with respect to summability $(C, 1)$ of sequences of real numbers, as well as to Hardy [1] (see also [2, p. 149]) with respect to summability $(C, 1)$ of sequences of complex numbers.

We note that theorems containing appropriate additional conditions such as (2.11), (2.12), etc. are called 'Tauberian', after A. Tauber, who first proved one of the simplest theorems of this kind; and these supplementary conditions are called 'Tauberian conditions'.

3. Proofs of Theorems 2.1 and 2.3. The following two representations of $s(t) - \tau(t)$ will be of vital importance in our proofs below.

LEMMA 3.1.

(i) *If* $\lambda > 1$ *and* $t > 1$, *then*

(3.1) $\quad s(t) - \tau(t) = \dfrac{\lambda}{\lambda - 1} (\tau(t^\lambda) - \tau(t)) - \dfrac{1}{(\lambda - 1) \log t} \displaystyle\int\limits_t^{t^\lambda} \dfrac{s(u) - s(t)}{u} \, du.$

(ii) *If* $0 < \lambda < 1$ *and* $t > 1$, *then*

(3.2) $\quad s(t) - \tau(t) = \dfrac{\lambda}{1 - \lambda} (\tau(t) - \tau(t^\lambda)) + \dfrac{1}{(1 - \lambda) \log t} \displaystyle\int\limits_{t^\lambda}^t \dfrac{s(t) - s(u)}{u} \, du.$

Proof. (i) By definition in (1.3), we have

$$\tau(t^\lambda) - \tau(t) = \frac{1}{\lambda \log t} \int_1^{t^\lambda} \frac{s(u)}{u} \, du - \frac{1}{\log t} \int_1^t \frac{s(u)}{u} \, du$$

$$= \frac{1-\lambda}{\lambda \log t} \int_1^t \frac{s(u)}{u} \, du + \frac{1}{\lambda \log t} \int_t^{t^\lambda} \frac{s(u)}{u} \, du$$

$$= -\frac{\lambda - 1}{\lambda} \tau(t) + \frac{1}{\lambda \log t} \int_t^{t^\lambda} \frac{s(u)}{u} \, du.$$

Multiplying both sides by $\lambda/(\lambda - 1)$ gives

$$\frac{\lambda}{\lambda - 1}(\tau(t^\lambda) - \tau(t)) = -\tau(t) + \frac{1}{(\lambda - 1)\log t} \int_t^{t^\lambda} \frac{s(u)}{u} \, du$$

$$= s(t) - \tau(t) + \frac{1}{(\lambda - 1)\log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} \, du,$$

whence (3.1) follows.

(ii) The proof of (3.2) is analogous to that of (3.1). ∎

Proof of Theorem 2.1. Necessity. Suppose that (1.2) is satisfied. By (1.2) and (1.3),

(3.3)
$$\text{(i)} \quad \lim_{t \to \infty} (s(t) - \tau(t)) = 0,$$
$$\text{(ii)} \quad \lim_{t \to \infty} (\tau(t^\lambda) - \tau(t)) = 0$$

for each fixed $\lambda > 1$. By (3.1) and (3.3), we conclude that

(3.4)
$$\lim_{t \to \infty} \frac{1}{\log t} \int_t^{t^\lambda} \frac{s(u) - s(t)}{u} \, du = 0$$

for every $\lambda > 1$. This proves (2.1) even in a stronger form.

An analogous argument yields (2.2) for every $0 < \lambda < 1$, also in a stronger form.

Sufficiency. Suppose that (2.1) and (2.2) are satisfied. By (2.1), there exists a sequence $\lambda_j \downarrow 1$ such that

(3.5)
$$\lim_{j \to \infty} \liminf_{t \to \infty} \frac{1}{(\lambda_j - 1)\log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u} \, du \geq 0.$$

By (1.3), (3.1) and (3.5), we conclude that

$$(3.6) \qquad \limsup_{t\to\infty}(s(t) - \tau(t)) \le \lim_{j\to\infty} \limsup_{t\to\infty} \frac{\lambda_j}{\lambda_j - 1}(\tau(t^{\lambda_j}) - \tau(t))$$

$$+ \lim_{j\to\infty} \limsup_{t\to\infty} \left(-\frac{1}{(\lambda_j - 1)\log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u}\, du\right)$$

$$= -\lim_{j\to\infty} \liminf_{t\to\infty} \frac{1}{(\lambda_j - 1)\log t} \int_t^{t^{\lambda_j}} \frac{s(u) - s(t)}{u}\, du \le 0.$$

By (2.2), there exists a sequence $0 < \lambda_k \uparrow 1$ such that

$$(3.7) \qquad \lim_{k\to\infty} \liminf_{t\to\infty} \frac{1}{(1 - \lambda_k)\log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u}\, du \ge 0.$$

By (1.3), (3.2) and (3.7), we conclude that

$$(3.8) \qquad \liminf_{t\to\infty}(s(t) - \tau(t)) \ge \lim_{k\to\infty} \liminf_{t\to\infty} \frac{\lambda_k}{1 - \lambda_k}(\tau(t) - \tau(t^{\lambda_k}))$$

$$+ \lim_{k\to\infty} \liminf_{t\to\infty} \frac{1}{(1 - \lambda_k)\log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u}\, du$$

$$= \lim_{k\to\infty} \liminf_{t\to\infty} \frac{1}{(1 - \lambda_k)\log t} \int_{t^{\lambda_k}}^t \frac{s(t) - s(u)}{u}\, du \ge 0.$$

Combining (3.6) and (3.8) yields (3.3)(i), and a fortiori (1.2), due to summability $(L, 1)$ of the function s.

The proof of Theorem 2.1 is complete. ∎

Proof of Theorem 2.3. It also hinges on Lemma 3.1 and runs along similar lines to the proof of Theorem 2.1. The details are left to the reader. ∎

4. Inclusions. We will prove that summability $(L, 1)$ is more effective than summability $(C, 1)$.

THEOREM 4.1. *If a complex-valued function $s \in L_{\mathrm{loc}}[1, \infty)$ is summable $(C, 1)$ to some $A \in \mathbb{C}$, then it is also summable $(L, 1)$ to the same A. The converse implication is not true in general.*

Proof. (i) First, let $t := m$, where $m = 2, 3, \ldots$. By definition in (1.4) and applying the Second Mean-Value Theorem, we get

$$(4.1) \quad \tau(m) \log m = \sum_{k=1}^{m-1} \int_{k}^{k+1} \frac{s(u)}{u} \, du$$

$$= \sum_{k=1}^{m-1} \left(\frac{1}{k} \int_{k}^{\xi_k} s(u) \, du + \frac{1}{k+1} \int_{\xi_k}^{k+1} s(u) \, du \right)$$

$$= \int_{1}^{\xi_1} s(u) \, du + \sum_{k=1}^{m-2} \frac{1}{k+1} \int_{\xi_k}^{\xi_{k+1}} s(u) \, du + \frac{1}{m} \int_{\xi_{m-1}}^{m} s(u) \, du,$$

where

$$(4.2) \qquad\qquad k < \xi_k < k+1 \quad \text{for } k = 1, \ldots, m-1.$$

By the definition in (1.1), we may write

$$(4.3) \qquad\qquad \int_{\xi}^{\eta} s(u) \, du = \eta\sigma(\eta) - \xi\sigma(\xi), \quad 0 < \xi < \eta.$$

Making use of this equality, from (4.1) it follows that

$$\tau(m) \log m = -(\xi_1\sigma(\xi_1) - \sigma(1))$$

$$+ \sum_{k=1}^{m-2} \frac{1}{k+1}(\xi_{k+1}\sigma(\xi_{k+1}) - \xi_k\sigma(\xi_k)) + \frac{1}{m}(m\sigma(m) - \xi_{m-1}\sigma(\xi_{m-1}))$$

$$= -\sigma(t) + \sum_{k=1}^{m-1} \frac{1}{k(k+1)}\xi_k\sigma(\xi_k) + \sigma(m) - \sigma(1),$$

whence

$$(4.4) \qquad \tau(m) = \frac{1}{\log m} \sum_{k=1}^{m-1} \frac{\xi_k}{k(k+1)}\sigma(\xi_k) + \frac{1}{\log m}(\sigma(m) - \sigma(1)).$$

We will apply Toeplitz' theorem on the summability of numerical sequences (see, e.g., [8, p. 74]) in the case of (4.4) with the infinite triangular matrix

$$\left(a_{m,k} := \frac{1}{\log m} \frac{\xi_k}{k(k+1)}, \quad k = 1, \ldots, m-1; \; m = 2, 3, \ldots \right).$$

By (4.2), we have

$$\frac{1}{\log m} \sum_{k=1}^{m-1} \frac{1}{k+1} < \sum_{k=1}^{m-1} a_{m,k} < \frac{1}{\log m} \sum_{k=1}^{m-1} \frac{1}{k}, \quad m = 2, 3, \ldots,$$

whence

$$\lim_{m \to \infty} \sum_{k=1}^{m-1} a_{m,k} = 1.$$

It is also clear that

$$0 < a_{m,k} < \frac{1}{k \log m} \to 0 \quad \text{as } m \to \infty \text{ for } k = 1, 2, \ldots.$$

Thus, the sufficient conditions in Toeplitz' theorem are satisfied, and we conclude that the limit in (1.3) holds for the particular choice $t = m \in \mathbb{N}$.

(ii) Second, given any real number $t > 3$, let $m := [t]$, the integer part of t. We use (4.3) and the Second Mean-Value Theorem again to obtain

$$(4.5) \quad \tau(t) \log t - \tau(m) \log m = \int_m^t \frac{s(u)}{u} \, du$$

$$= \frac{1}{m} \int_m^\xi s(u) \, du - \frac{1}{t} \int_\xi^t s(u) \, du$$

$$= \left(\frac{1}{m} - \frac{1}{t} \right) \xi \sigma(\xi) - \sigma(m) + \sigma(t), \quad m < \xi < t.$$

By (1.1) and (4.5), we get

$$|\tau(t) \log t - \tau(m) \log m| = \frac{t - m}{mt} \xi |\sigma(\xi)| + |\sigma(t) - \sigma(m)|$$

$$\leq \frac{1}{m} |\sigma(\xi)| + |\sigma(t) - \sigma(m)| \to 0 \quad \text{as } m \to \infty.$$

Hence

$$\lim_{t \to \infty} \tau(t) = \lim_{t \to \infty} \tau(m) \frac{\log m}{\log t} = A, \quad \text{where } m = [t].$$

(iii) Third, to see that the converse implication is not true in general, we consider the function s defined by

$$s(t) := \begin{cases} me^{2^m} & \text{if } t \in [e^{2^m}, e^{2^m} + 1], \ m = 1, 2, \ldots, \\ 0 & \text{otherwise on } [1, \infty). \end{cases}$$

We claim that this function s is not summable $(C, 1)$ to any finite number A. To see this, we recall that if we had (1.1), then for any number $a > 0$ we would have

$$\frac{1}{t} \int_t^{t+a} s(u) \, du = \frac{t + a}{t} \sigma(t + a) - \sigma(t) \to 0 \quad \text{as } t \to \infty.$$

But for $t := e^{2^m}$ and $a := 1$, we have

$$\frac{1}{t} \int_t^{t+1} s(u) \, du = e^{-2^m} \int_t^{t+1} me^{2^m} \, du = m \nrightarrow 0 \quad \text{as } m \to \infty.$$

Consequently, for this function s the limit (1.1) does not exist for any finite number A.

On the other hand, if t is such that

$$e^{2^{m-1}} \le t < e^{2^m}, \quad m = 1, 2, \dots,$$

then we estimate as follows:

$$0 \le \tau(t) \le \frac{1}{2^{m-1}} \sum_{k=1}^{m-1} \int_{e^{2^{k-1}}}^{e^{2^k}} \frac{s(u)}{u} \, du$$

$$\le \frac{1}{2^{m-1}} \sum_{k=1}^{m-1} k \to 0 \quad \text{as } t \to \infty.$$

This proves that the limit in (1.3) exists with $A = 0$.

The proof of Theorem 4.1 is complete. ∎

REMARK 4.1. We note that summability $(L, 2)$ is more effective than summability $(L, 1)$. This can be proved in an analogous way to Theorem 4.1 above. We refer to [5, p. 382], where an analogous result is proved for the logarithmic mean $\tau_2(n)$ of second order of a numerical sequence (s_k) (see also (5.2) below).

5. Summability $(L, 1)$ of numerical sequences. The above methods of summability are the nondiscrete version of the methods of logarithmic summability of numerical sequences $(s_k) = (s_k : k = 1, 2, \dots)$ of complex numbers. We recall that a sequence (s_k) is said to be *logarithmic summable of order* 1 (see [5], where the term 'harmonic summable of order 1' was used), briefly: summable $(L, 1)$, if there exists some $A \in \mathbb{C}$ such that

$$(5.1) \qquad \lim_{n \to \infty} \frac{1}{\ell_n} \sum_{k=1}^n \frac{s_k}{k} = A, \quad \text{where} \quad \ell_n := \sum_{k=1}^n \frac{1}{k} \sim \log n,$$

where for two sequences (a_n) and (b_n) of positive numbers we write $a_n \sim b_n$ if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

We note that the sequence (s_k) is said to be *logarithmic summable of order* 2 (see also [5]), briefly: summable $(L, 2)$, if there exists some $A \in \mathbb{C}$ such that

$$(5.2) \quad \lim_{n \to \infty} \tau_2(n) := \frac{1}{\ell_n(2)} \sum_{k=1}^n \frac{s_k}{k \ell_k}, \quad \text{where} \quad \ell_n(2) := \sum_{k=1}^n \frac{1}{k \ell_k} \sim \log \log n.$$

It is clear that if the ordinary limit

$$(5.3) \qquad \qquad \lim_{n \to \infty} s_n = A$$

exists, then the limit in (5.1) also exists with the same A. Even more is true (see, e.g., [5, p. 376]): If a sequence (s_k) is such that the finite limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = A$$

exists, then the limit in (5.1) also exists with the same A. The converse is not true in general.

We note that if the finite limit in (5.1) exists, then the limit in (5.2) also exists with the same A (see also in [5, p. 382]. Again, the converse is not true in general.

Now, the discrete analogue of Theorem 2.1 reads as follows.

THEOREM 5.1. *If a sequence (s_k) of real numbers is summable $(L, 1)$ to some $A \in \mathbb{R}$, then the ordinary limit (5.3) exists if and only if*

$$(5.4) \qquad \limsup_{\lambda\to 1+} \liminf_{n\to\infty} \frac{1}{([n^\lambda] - n)\ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k} \geq 0$$

and

$$(5.5) \qquad \limsup_{\lambda\to 1-} \liminf_{n\to\infty} \frac{1}{(n - [n^\lambda])\ell_n} \sum_{k=[n^\lambda]+1}^{n} \frac{s_n - s_k}{k} \geq 0,$$

where $[\,\cdot\,]$ denotes integer part, and ℓ_n is defined in (5.1).

Analogously to (2.3), we say that a sequence (s_k) of real numbers is *slowly decreasing with respect to summability $(L, 1)$* if for every $\varepsilon > 0$ there exist a natural number $n_0 = n_0(\varepsilon)$ and a real number $\lambda = \lambda(e) > 1$ such that

$$(5.6) \qquad s_k - s_n \geq -\varepsilon \quad \text{whenever} \quad n_0 \leq n < k \leq n^\lambda.$$

It is easy to check (cf. (2.4)) that a sequence (s_k) is slowly decreasing with respect to summability $(L, 1)$ if and only if

$$(5.7) \qquad \lim_{\lambda\to 1+} \liminf_{n\to\infty} \min_{n < k \leq n^\lambda} (s_k - s_n) \geq 0.$$

Clearly, if a sequence (s_k) is slowly decreasing with respect to summability $(L, 1)$, then both conditions (5.4) and (5.5) are satisfied. Thus, the next corollary is an immediate consequence of Theorem 5.1.

COROLLARY 5.2. *Suppose a sequence (s_k) of real numbers is slowly decreasing with respect to summability $(L, 1)$. If (s_k) is summable $(L, 1)$ to some $A \in \mathbb{R}$, then the ordinary limit (5.3) also exists.*

Corollary 5.2 was earlier proved by Kwee [3, Lemma 3] in a different way. We note that the definition of slow decrease of a sequence (s_k) in [3] is formally different from the definitions given in (5.6) and (5.7) above.

REMARK 5.1. According to Kwee's definition in [3, a condition in both Theorem A and Lemma 3], a sequence (s_k) of real numbers is said to be *slowly decreasing* if

$$(5.8) \quad \liminf(s_k - s_n) \geq 0 \quad \text{whenever} \quad k > n \to \infty \quad \text{and} \quad \frac{\log k}{\log n} \to 1.$$

We claim that definition (5.8) is equivalent to the one in (5.6) (as well as to (5.7)). The implication (5.8)$\Rightarrow$(5.6) is trivial. To justify the converse, let $\lambda > 1$ be arbitrarily close to 1 and set $\varepsilon := \log \lambda$. By (5.8), there exists $n_0 = n_0(\lambda) > 1$ such that

$$s_k - s_n \geq -\varepsilon \quad \text{whenever} \quad k > n \geq n_0 \quad \text{and} \quad 0 < \log \frac{\log k}{\log n} \leq \log \lambda = \varepsilon.$$

Now, the claimed equivalence is obvious.

Next, the discrete analogue of Theorem 2.3 reads as follows:

THEOREM 5.3. *If a sequence (s_n) of complex numbers is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (5.3) exists if and only if*

$$(5.9) \qquad \lim_{\lambda \to 1+} \limsup_{t \to \infty} \left| \frac{1}{([n^\lambda] - n)\ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k} \right| = 0.$$

Analogously to (2.6), we say that a sequence (s_k) of complex numbers is *slowly oscillating with respect to summability $(L, 1)$* if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$ such that

$$(5.10) \qquad |s_k - s_n| \leq \varepsilon \quad \text{whenever} \quad n_0 \leq n < k \leq n^\lambda.$$

It is easy to check that a sequence (s_k) is slowly oscillating with respect to summability $(L, 1)$ if and only if

$$(5.11) \qquad \lim_{\lambda \to 1+} \limsup_{n \to \infty} \max_{n < k \leq n^\lambda} |s_k - s_n| = 0.$$

REMARK 5.2. The concept of slow oscillation with respect to summability $(L, 1)$ is not defined in [3]. However, analogously to (5.8), a sequence (s_k) of complex numbers may be called *slowly oscillating* if

$$(5.12) \quad \lim(s_k - s_n) = 0 \quad \text{whenever} \quad k > n \to \infty \quad \text{and} \quad \frac{\log k}{\log n} \to 1.$$

A reasoning similar to the one in Remark 5.1 gives that the definitions (5.12) and (5.10) (as well as (5.11)) are equivalent.

It is clear that if a sequence (s_k) is slowly oscillating with respect to $(L, 1)$, then condition (5.11) is satisfied. Thus, the next corollary is an immediate consequence of Theorem 5.3.

COROLLARY 5.4. *Suppose a sequence (s_k) of complex numbers is slowly oscillating with respect to summability $(L, 1)$. If (s_k) is summable $(L, 1)$ to some $A \in \mathbb{C}$, then the ordinary limit (5.3) also exists.*

The proofs of Theorems 5.1 and 5.3 run along similar lines to those of Theorems 2.1 and 2.3, respectively; the key ingredient is provided by the following

LEMMA 5.5.

(i) *For all $\lambda > 1$ and large enough n, namely when $[n^\lambda] > n$, we have the representation*

$$s_n - \tau_n = \frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]} - \ell_n}(\tau_{[n^\lambda]} - \tau_n) - \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}.$$

(ii) *For all $0 < \lambda < 1$ and large enough n, namely when $n > [n^\lambda]$, we have*

$$s_n - \tau_n = \frac{\ell_{[n^\lambda]}}{\ell_n - \ell_{[n^\lambda]}}(\tau_n - \tau_{[n^\lambda]}) + \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^{n} \frac{s_n - s_k}{k}.$$

Proof. Performing steps analogous to those in the proof of Lemma 3.1 yields the above representations. ∎

References

[1] G. H. Hardy, *Theorems relating to the summability and convergence of slowly oscillating sequences*, Proc. London Math. Soc. (2) 8 (1910), 301–320.
[2] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.
[3] B. Kwee, *A Tauberian theorem for the logarithmic method of summation*, Proc. Cambridge Philos. Soc. 63 (1967), 401–405.
[4] E. L. Landau, *Über die Bedeutung einiger neuen Grenzwertsätze der Herren Hardy und Axer*, Prace Mat.-Fiz. 21 (1910), 97–177.
[5] F. Móricz, *On the harmonic averages of numerical sequences*, Arch. Math. (Basel) 86 (2006), 375–384.
[6] R. Schmidt, *Über divergente Folgen und Mittelbildungen*, Math. Z. 22 (1925), 89–152.
[7] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1937.
[8] A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge Univ. Press, 1959.

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