

Order isomorphisms on function spaces

by

DENNY H. LEUNG (Singapore) and LEI LI (Tianjin)

Abstract. The classical theorems of Banach and Stone (1932, 1937), Gelfand and Kolmogorov (1939) and Kaplansky (1947) show that a compact Hausdorff space X is uniquely determined by the linear isometric structure, the algebraic structure, and the lattice structure, respectively, of the space $C(X)$. In this paper, it is shown that for rather general subspaces $A(X)$ and $A(Y)$ of $C(X)$ and $C(Y)$, respectively, any linear bijection $T : A(X) \rightarrow A(Y)$ such that $f \geq 0$ if and only if $Tf \geq 0$ gives rise to a homeomorphism $h : X \rightarrow Y$ with which T can be represented as a weighted composition operator. The three classical results mentioned above can be derived as corollaries. Generalizations to noncompact spaces and other function spaces such as spaces of Lipschitz functions and differentiable functions are presented.

1. Introduction. A well established area of research seeks to determine the relationship between the structure of a set and the properties of certain function spaces defined on it. A closely related question is to determine the general form of an operator mapping between various function spaces. In the case of spaces of continuous functions, a vast literature exists. A good summary of the classical results is the text of Gillman and Jerison [GJ]. From the classical period, three results in this area stand out, namely the theorems of Banach and Stone, Gelfand and Kolmogorov, and Kaplansky (see Corollary 2.8 below). When X and Y are compact Hausdorff spaces, these results determine the precise forms of the norm isometries, algebra isomorphisms and vector lattice isomorphisms between $C(X)$ and $C(Y)$ respectively. In particular, the existence of any one of these three types of operators leads to homeomorphism between X and Y . More recent results of Banach–Stone type are found in [A1, A2, CS2, GaJ1, JV1, JV2], for example.

Kadison [Kad] showed that a linear order isomorphism T between two C^* -algebras which maps the identity to the identity is a C^* -isomorphism, i.e., T satisfies $T(x^2) = (Tx)^2$ for all x . In the commutative case, it follows that for any compact Hausdorff spaces X and Y , any linear order isomor-

2010 *Mathematics Subject Classification*: Primary 46E50; Secondary 46E15, 47B65.

Key words and phrases: order isomorphisms, realcompact spaces, adequate subspaces.

phism from $C(X)$ onto $C(Y)$ that maps the constant function 1 to the constant function 1 is an algebra isomorphism. The theorem of Gelfand–Kolmogorov then implies that T is a composition operator. Order isomorphisms, even nonlinear ones, have been studied by various authors, see for instance [CS1, CS2, CS3, GaJ1, GaJ2, JV1, JV2]. These results concern order isomorphisms between specific function spaces. Moreover, the spaces considered are all either lattices or algebras of functions. One of the aims of the present paper is to provide a unified treatment of linear order isomorphisms within a general framework. In particular, our results apply to all unital function lattices that separate points from closed sets and to many function algebras.

For quite general subspaces $A(X)$ and $A(Y)$ of $C(X)$ and $C(Y)$, respectively, where X and Y are compact Hausdorff spaces, Theorem 2.1 in §2 determines the precise form of a linear order isomorphism $T : A(X) \rightarrow A(Y)$ and shows that the existence of such a map leads to homeomorphism of X and Y . The classical results cited above can all be subsumed under this theorem. We also provide an example of a space to which the theorem applies which is neither a lattice nor an algebra.

In §3, we apply a Stone–Čech like compactification procedure to extend Theorem 2.1 to noncompact spaces. As a result, existence of a linear order isomorphism $T : A(X) \rightarrow A(Y)$ gives rise to a homeomorphism between some compactifications of X and Y , respectively. In §4, we show that in certain circumstances, the homeomorphism obtained restricts to a homeomorphism between X and Y .

Because much of the paper is concerned with maps preserving order, we only consider *real* vector spaces.

2. Order isomorphisms of spaces of continuous functions on compact Hausdorff spaces. Let X be a topological space and let $A(X)$ be a vector subspace of $C(X)$. $A(X)$ is said to *separate points from closed sets* if given $x \in X$ and a closed set F in X not containing x , there exists $f \in A(X)$ such that $f(x) = 1$ and $f(F) \subseteq \{0\}$. If, in addition, f can be chosen to have values in $[0, 1]$, then we say that $A(X)$ *precisely separates points from closed sets*. It is clear that any sublattice of $C(X)$ that separates points from closed sets does so precisely. Let Y be a topological space and let $A(Y)$ be a vector subspace of $C(Y)$. A linear bijection $T : A(X) \rightarrow A(Y)$ is an *order isomorphism* if $f \geq 0$ if and only if $Tf \geq 0$. The aim of this section is to prove the following theorem.

THEOREM 2.1. *Let X, Y be compact Hausdorff spaces and let $A(X)$ and $A(Y)$ be subspaces of $C(X)$ and $C(Y)$, respectively, that contain the constant functions and precisely separate points from closed sets. If $T : A(X) \rightarrow A(Y)$*

is a linear order isomorphism, then there is a homeomorphism $h : X \rightarrow Y$ such that $Tf = T1_X \cdot f \circ h^{-1}$ for all $f \in A(X)$.

The proof is divided into a number of steps listed below (Propositions 2.2 to 2.6). If $f \in A(X)$ or $A(Y)$, let $Z(f) = \{f = 0\}$.

PROPOSITION 2.2. For any $x_0 \in X$, let

$$\mathcal{Z}_{x_0} = \{Z(Tf) : f \in A(X), f \geq 0, f(x_0) = 0\}.$$

Then \mathcal{Z}_{x_0} has the finite intersection property.

Proof. Suppose that $f_i \in A(X)$, $f_i \geq 0$, and $f_i(x_0) = 0$, $1 \leq i \leq n$. Let $f = \sum_{i=1}^n f_i$. Then $f \in A(X)$, $f \geq 0$, and $f(x_0) = 0$. In particular, $Tf \geq 0$. If $Z(Tf) = \emptyset$, that is, $(Tf)(y) > 0$ for all $y \in Y$, then there exists $\varepsilon > 0$ such that $Tf - \varepsilon T1_X \geq 0$. Thus $f - \varepsilon 1_X \geq 0$, which is manifestly untrue. Let $y_0 \in Y$ be such that $Tf(y_0) = 0$. Then $\sum_{i=1}^n Tf_i(y_0) = 0$. Since $Tf_i \geq 0$, $Tf_i(y_0) = 0$, $1 \leq i \leq n$. Thus $y_0 \in \bigcap_{i=1}^n Z(Tf_i)$. ■

Define \mathcal{Z}_{y_0} similarly for $y_0 \in Y$, using the operator T^{-1} in place of T . By Proposition 2.2, $\bigcap \mathcal{Z}_{x_0}$ and $\bigcap \mathcal{Z}_{y_0}$ are nonempty for all $x_0 \in X$, $y_0 \in Y$.

PROPOSITION 2.3. Let $x_0 \in X$ and $y_0 \in Y$. Then $y_0 \in \bigcap \mathcal{Z}_{x_0}$ if and only if $x_0 \in \bigcap \mathcal{Z}_{y_0}$.

Proof. Suppose that $y_0 \in \bigcap \mathcal{Z}_{x_0}$ but $x_0 \notin \bigcap \mathcal{Z}_{y_0}$. Choose $x_1 \in \bigcap \mathcal{Z}_{y_0}$. Then $x_1 \neq x_0$. There exists $f \in A(X)$, $\text{range } f \subseteq [0, 1]$, such that $f(x_1) = 1$ and $f(x_0) = 0$. Since $y_0 \in \bigcap \mathcal{Z}_{x_0}$, $Tf(y_0) = 0$. As T is an order isomorphism, $Tf \geq 0$. Then $x_1 \in \bigcap \mathcal{Z}_{y_0}$ implies that $f(x_1) = T^{-1}(Tf)(x_1) = 0$, contrary to the choice of f . The “if” part of the proposition follows by symmetry. ■

PROPOSITION 2.4. $\bigcap \mathcal{Z}_{x_0}$ contains exactly one point.

Proof. It has already been observed that $\bigcap \mathcal{Z}_{x_0}$ is nonempty. Suppose that there are distinct points y_1 and y_2 in $\bigcap \mathcal{Z}_{x_0}$. Choose $g \in A(Y)$ such that $\text{range } g \subseteq [0, 1]$, $g(y_1) = 0$ and $g(y_2) = 1$. By Proposition 2.3, $x_0 \in \bigcap \mathcal{Z}_{y_1}$. Hence $T^{-1}g(x_0) = 0$. Since $y_2 \in \bigcap \mathcal{Z}_{x_0}$, $T(T^{-1}g)(y_2) = 0$. Thus $g(y_2) = 0$, yielding a contradiction. ■

PROPOSITION 2.5. $T1_X(y) > 0$ and $T^{-1}1_Y(x) > 0$ for all $x \in X$ and all $y \in Y$.

Proof. Suppose that there exists $y_0 \in Y$ such that $T1_X(y_0) = 0$. For any $f \in A(X)$, there exists $0 \leq c \in \mathbb{R}$ such that $-c1_X \leq f \leq c1_X$. Then $-cT1_X(y_0) \leq Tf(y_0) \leq cT1_X(y_0)$ for all $f \in A(X)$. Hence $Tf(y_0) = 0$ for all $f \in A(X)$. This is a contradiction since T maps onto $A(Y)$, and $A(Y)$ contains all constant functions. ■

Define $h : X \rightarrow Y$ by $h(x_0) = y_0$, where $\{y_0\} = \bigcap \mathcal{Z}_{x_0}$.

PROPOSITION 2.6. *The map h is a homeomorphism from X onto Y so that $Tf = T1_X \cdot f \circ h^{-1}$ for all $f \in A(X)$ and all $y \in Y$.*

Proof. The injectivity of h follows from Proposition 2.3. If $y \in Y$, let $\{x\} = \bigcap \mathcal{Z}_y$. By Proposition 2.3, $y \in \bigcap \mathcal{Z}_x$. Thus $h(x) = y$. This shows that h is surjective.

Suppose that $x_0 \in X$ and $y_0 = h(x_0)$. Let $f \in A(X)$ and let $m = \min\{f(x) : x \in X\}$. Given $\varepsilon > 0$, let U be an open neighborhood of x_0 so that $f(x) > f(x_0) - \varepsilon$ for all $x \in U$. There exists $g_1 \in A(X)$ such that $\text{range } g_1 \subseteq [0, 1]$, $g_1(x_0) = 1$ and $g_1(x) = 0$ for all $x \notin U$. Since $1_X - g_1 \geq 0$, $(1_X - g_1)(x_0) = 0$ and $y_0 \in \bigcap \mathcal{Z}_{x_0}$, we see that $T(1_X - g_1)(y_0) = 0$. Hence $(Tg_1)(y_0) = (T1_X)(y_0)$. Now

$$f - m1_X - (f(x_0) - m - \varepsilon)g_1 \geq 0.$$

Thus

$$(Tf)(y_0) \geq m(T1_X)(y_0) + (f(x_0) - m - \varepsilon)(Tg_1)(y_0),$$

that is, $(Tf)(y_0) \geq (T1_X)(y_0)(f(x_0) - \varepsilon)$. As $\varepsilon > 0$ is arbitrary, $Tf(y_0) \geq (T1_X)(y_0)f(x_0)$. Applying the argument to $-f$ yields the reverse inequality. Thus $Tf(y_0) = (T1_X)(y_0)f(x_0)$.

It remains to show that h is a homeomorphism. Let $x_0 \in X$ and $y_0 = h(x_0)$. Suppose that V is an open neighborhood of y_0 in Y . There exists $g \in A(Y)$ such that $g(y_0) = 1$, $\text{range } g \subseteq [0, 1]$, and $g = 0$ outside V . Since $g \geq 0$, $T^{-1}g \geq 0$. If $T^{-1}g(x_0) = 0$, then $y_0 \in Z(T(T^{-1}g)) = Z(g)$, contrary to the choice of g . Thus $T^{-1}g(x_0) > 0$. Therefore, the set $U = \{T^{-1}g > 0\}$ is an open neighborhood of $x_0 \in X$. Suppose that $x \in U$. By the previous paragraph,

$$g(h(x)) = T(T^{-1}g)(h(x)) = (T1_X)(h(x))T^{-1}g(x) > 0.$$

Hence $h(x) \in V$. This proves that h is continuous. Since h is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. ■

REMARK. Applying Theorem 2.1 to the map T^{-1} gives a homeomorphism $k : Y \rightarrow X$ such that $T^{-1}g = T^{-1}1_Y \cdot g \circ k^{-1}$. Because of Proposition 2.3, k must be h^{-1} .

The classical theorems of Banach [B] and Stone [S], Gelfand and Kolmogorov [GK] and Kaplansky [K] show that a compact Hausdorff space X is uniquely determined by the linear isometric structure, the algebraic structure, and the lattice structure, respectively, of the space $C(X)$. These results can be subsumed under Theorem 2.1. As usual, $C(X)$ and $C(Y)$ are endowed with their respective supremum norms.

LEMMA 2.7. *Let X and Y be compact Hausdorff spaces. If $T : C(X) \rightarrow C(Y)$ is an onto linear isometry, then $|T1_X| = 1_Y$ and, for any $f \in C(X)$, $f \geq 0$ if and only if $Tf \cdot T1_X \geq 0$.*

Proof. Since T is an isometry, $\|T1_X\| = 1$. Suppose that there exists $y_0 \in Y$ such that $|T1_X(y_0)| < 1$. There exists a neighborhood V of y_0 and $a > 0$ such that $|T1_X(y)| < 1 - a$ for all $y \in V$. Choose $g \in C(Y)$ such that $g(y_0) = 1$, $\text{range } g \subseteq [0, 1]$ and $g = 0$ outside V . Then $\|1_X + aT^{-1}g\| = \|T1_X + ag\| \leq 1$. Thus $1 + aT^{-1}g(x) \leq 1$ for all $x \in X$. Therefore, $T^{-1}g \leq 0$. As $\|T^{-1}g\| = \|g\| = 1$, there must be some $x_0 \in X$ where $T^{-1}g(x_0) = -1$. Then

$$2 = |(1_X - T^{-1}g)(x_0)| \leq \|1_X - T^{-1}g\| \leq \|1_X\| + \|T^{-1}g\| = 2.$$

Hence $\|T1_X - g\| = \|1_X - T^{-1}g\| = 2$. However, if $y \in V$, then

$$|(T1_X - g)(y)| \leq |T1_X(y)| + \|g\| < 1 - a + 1 < 2.$$

On the other hand, if $y \notin V$, then $|(T1_X - g)(y)| = |T1_X(y)| \leq \|1_X\| = 1$. This proves that $\|T1_X - g\| < 2$, contrary to the above. So, $|T1_X| = 1_Y$.

Given $f \in C(X)$, $f \geq 0$ if and only if $\|f - \|f\|1_X\| \leq \|f\|$. Since T is an isometry, this is equivalent to $\|Tf - \|Tf\|T1_X\| \leq \|Tf\|$. By the above, $|T1_X| = 1_Y$. Thus the final inequality holds if and only if $Tf \cdot T1_X \geq 0$. ■

A linear bijection $T : C(X) \rightarrow C(Y)$ is

- a *lattice isomorphism* if $|Tf| = T|f|$ for all $f \in C(X)$;
- an *algebra isomorphism* if $T1_X = 1_Y$ and $T(fg) = Tf \cdot Tg$ for all $f, g \in C(X)$.

COROLLARY 2.8. *Let X and Y be compact Hausdorff spaces and let $T : C(X) \rightarrow C(Y)$ be a linear bijection.*

- (a) (Banach and Stone) *If T is an isometry, then there is a homeomorphism $h : X \rightarrow Y$ and a function $g \in C(Y)$, $|g| = 1_Y$, such that $Tf = g \cdot f \circ h^{-1}$ for all $f \in C(X)$.*
- (b) (Kaplansky) *If T is a lattice isomorphism, then there is a homeomorphism $h : X \rightarrow Y$ and a function $g \in C(Y)$, $g(y) > 0$ for all $y \in Y$, such that $Tf = g \cdot f \circ h^{-1}$ for all $f \in C(X)$.*
- (c) (Gelfand and Kolmogorov) *If T is an algebra isomorphism, then there is a homeomorphism $h : X \rightarrow Y$ such that $Tf = f \circ h^{-1}$ for all $f \in C(X)$.*

Proof. Let $g = T1_X$. For case (a), it follows from Lemma 2.7 that $|g| = 1_Y$ and that the operator $\tilde{T} : C(X) \rightarrow C(Y)$ given by $\tilde{T}(f) = Tf/g$ is a linear order isomorphism. By Theorem 2.1, there exists a homeomorphism $h : X \rightarrow Y$ such that $\tilde{T}f = \tilde{T}1_X \cdot f \circ h^{-1}$. It follows that $Tf = g \cdot f \circ h^{-1}$. For cases (b) and (c), it is clear that T is a linear order isomorphism. Proposition 2.5 gives $g(y) > 0$ for all $y \in Y$. Moreover, $g = 1_Y$ for case (c). The representation of T is seen immediately from Theorem 2.1. ■

REMARK. Lemma 2.7 may be extended to linear isometries $T : C(X, \mathbb{C}) \rightarrow C(Y, \mathbb{C})$. As a result, Corollary 2.8(a) for complex isometries may also be derived in a similar manner.

EXAMPLE. There is a subspace \mathcal{X} of $C[0, 1]$ that contains the constants and precisely separates points from closed sets, so that \mathcal{X} is neither a sublattice nor a subalgebra of $C[0, 1]$.

Define $\theta : \mathbb{R} \rightarrow \mathbb{R}$ by $\theta(t) = \sin(\pi t/2)$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(t) = 0, \theta(t), 1$ if $t \leq 0, 0 < t < 1, t \geq 1$, respectively. Denote by t the identity function $t \mapsto t$ on $[0, 1]$ or \mathbb{R} , as the case may be. If \mathcal{F} is a set of real-valued functions defined on $[0, 1]$ or \mathbb{R} , let $g \circ \mathcal{F} = \{g \circ f : f \in \mathcal{F}\}$. Denote by \mathcal{X}_1 and Σ_1 the span of the functions 1 and t in $C[0, 1]$ and $C(\mathbb{R})$, respectively. Let $\mathcal{X}_{n+1} = \text{span}\{\mathcal{X}_n \cup g \circ \mathcal{X}_n\}$ and $\Sigma_{n+1} = \text{span}\{\Sigma_n \cup \theta \circ \Sigma_n\}$. Then set $\mathcal{X} = \bigcup_n \mathcal{X}_n$ and $\Sigma = \bigcup_n \Sigma_n$. It is easy to see that \mathcal{X} is a subspace of $C[0, 1]$ and that Σ is a subspace of $C(\mathbb{R})$. Moreover, Σ consists of real-analytic functions on \mathbb{R} . We claim that \mathcal{X} has the desired properties.

First observe that for any $a < b$ in $[0, 1]$, the linear function f on $[0, 1]$ such that $f(a) = 0$ and $f(b) = 1$ belongs to \mathcal{X}_1 . Hence $g \circ f \in \mathcal{X}_2 \subseteq \mathcal{X}$. Furthermore, $g \circ f$ has the property that $g \circ f(t) = 0$ if $t \leq a$, $g \circ f(t) = 1$ if $t \geq b$ and $0 < g \circ f(t) < 1$ if $a < t < b$. Taking differences of two such functions shows that \mathcal{X} separates points from closed sets in $[0, 1]$. By construction, $g \circ f \in \mathcal{X}$ for any $f \in \mathcal{X}$. Thus \mathcal{X} satisfies conditions (a) and (b) of Definition 3.1 below. Since \mathcal{X} consists of bounded functions, condition (c) is also satisfied. It follows from Lemma 3.3 that \mathcal{X} precisely separates points from closed sets in $[0, 1]$.

LEMMA 2.9. *If $f \in \mathcal{X}_n$ and I is a nondegenerate interval in $[0, 1]$, then there is a nondegenerate interval $J \subseteq I$ and a function $u \in \Sigma_n$ such that $f = u$ on J .*

Proof. Induct on n . The case $n = 1$ is trivial. Assume that the result holds for some n and let $f \in \mathcal{X}_{n+1}$. Then $f = f_0 + \sum_{i=1}^m c_i g \circ f_i$, where $f_i \in \mathcal{X}_n$, $0 \leq i \leq m$. By the inductive assumption, there exist a nondegenerate interval $I_0 \subseteq I$ and $u_0 \in \Sigma_n$ such that $f_0 = u_0$ on I_0 . If $f_1(I_0) \cap (0, 1) = \emptyset$, then $g \circ f_1(I_0) \subseteq \{0, 1\}$. By continuity of $g \circ f_1$ and connectedness of I_0 , $g \circ f_1$ takes a constant value, say c_1 ($= 0$ or 1) on I_0 . In this case, let $I_1 = I_0$ and $u_1 = c_1$. Then $g \circ f_1 = c_1 = \theta \circ u_1$ on I_1 . Otherwise, $f_1(I_0) \cap (0, 1) \neq \emptyset$. There is a nondegenerate interval $I'_0 \subseteq I_0$ such that $f_1(I'_0) \subseteq (0, 1)$. Then $g \circ f_1 = \theta \circ f_1$ on I'_0 . By the inductive assumption, there exist a nondegenerate interval $I_1 \subseteq I'_0$ and $u_1 \in \Sigma_n$ such that $f_1 = u_1$ on I_1 . Then $g \circ f_1 = \theta \circ f_1 = \theta \circ u_1$ on I_1 . Continue choosing nondegenerate intervals $I_0 \supseteq I_1 \supseteq \dots \supseteq I_m$, $u_0, \dots, u_m \in \Sigma_n$ such that $g \circ f_i = \theta \circ u_i$ on I_i , $1 \leq i \leq m$, and $f_0 = u_0$ on I_0 . Then $f = u_0 + \sum_{i=1}^m c_i \theta \circ u_i$ on I_m and the latter function belongs to Σ_{n+1} . ■

LEMMA 2.10. *If f is a real-analytic function on \mathbb{R} and $f|_{[0,1]} \in \mathcal{X}$, then $f \in \Sigma$.*

Proof. Let $f_0 = f|_{[0,1]}$. By Lemma 2.9, there is a nondegenerate interval J and $u \in \Sigma$ such that $f_0 = u$ on J . Since both f and u are real-analytic on \mathbb{R} , $f = u$ on \mathbb{R} . ■

We can now verify the remaining properties of \mathcal{X} stated above, namely that \mathcal{X} is neither a sublattice nor a subalgebra of $C[0, 1]$. If \mathcal{X} is an algebra, then $t^2 \in \mathcal{X}$. By Lemma 2.10, t^2 (as a function on \mathbb{R}) belongs to Σ . We show that this is impossible by showing that $\lim_{t \rightarrow \infty} f(t)/t^2 = 0$ for any $f \in \Sigma$. Indeed, the statement holds for any $f \in \Sigma_1$. Inductively, any $f \in \Sigma_{n+1}$ can be written as $f = f_0 + \sum_{i=1}^m c_i \theta \circ f_i$, where $f_i \in \Sigma_n$, $0 \leq i \leq m$. By induction, $\lim_{t \rightarrow \infty} f_0(t)/t^2 = 0$. Since $0 \leq \theta \circ f_i \leq 1$, the induction is complete.

Finally, suppose that \mathcal{X} is a lattice. Then $t \wedge 1/2 \in \mathcal{X}$. Say it belongs to \mathcal{X}_n . It is easy to check via induction that

$$\mathcal{X}_n = \mathcal{X}_1 + \text{span}(g \circ \mathcal{X}_1) + \cdots + \text{span}(g \circ \mathcal{X}_{n-1}).$$

Then we can write $f = f_0 + g \circ f_1 + \cdots + g \circ f_{n-1}$, where $f_0 \in \mathcal{X}_1$, $f_i \in \mathcal{X}_i$, $1 \leq i \leq n - 1$. Say $f_0 = a + bt$. By Lemma 2.9, there exist nondegenerate intervals $J_1 \subseteq [0, 1/2]$, $J_2 \subseteq [1/2, 1]$ and functions $u_i, v_i \in \Sigma$, $1 \leq i \leq n - 1$, such that

$$f = \begin{cases} a + bt + \sum_{i=1}^{n-1} \theta \circ u_i & \text{on } J_1, \\ a + bt + \sum_{i=1}^{n-1} \theta \circ v_i & \text{on } J_2. \end{cases}$$

Then $t = a + bt + h_1$ on J_1 and $1/2 = a + bt + h_2$ on J_2 , where h_1 and h_2 are bounded real-analytic functions. Thus these equations hold on \mathbb{R} . Dividing both equations by t and taking limits as $t \rightarrow \infty$ gives $1 = b$ and $0 = b$, which is absurd.

3. Order isomorphisms on completely regular spaces. In this section, we employ the method of compactification to extend results in the previous section to more general function spaces. Let X be a (Hausdorff) completely regular space and let $A(X)$ be a subspace of $C(X)$ that separates points from closed sets. Denote by \mathbb{R}_∞ the interval $[-\infty, \infty]$ with the order topology. The map $i : X \rightarrow \mathbb{R}_\infty^{A(X)}$, $i(x)(\varphi) = \varphi(x)$, is a homeomorphic embedding. Let $\mathcal{A}X$ be the closure of $i(X)$ in $\mathbb{R}_\infty^{A(X)}$. Then $\mathcal{A}X$ is a compact Hausdorff space. Identify X with $i(X)$ and regard X as a subspace of $\mathcal{A}X$. For each $f \in A(X)$, there is a unique continuous extension $\hat{f} : \mathcal{A}X \rightarrow \mathbb{R}_\infty$ given by $\hat{f}(x) = x(f)$. When $A(X) = C(X)$, $\mathcal{A}X$ coincides with the Stone–Cech compactification βX . In some versions of this type of compactification, one embeds \mathbb{R} into the one-point compactification $\mathbb{R} \cup \{\infty\}$. We prefer to use the compactification $[-\infty, \infty]$ instead since this space is ordered. Let

$g : \mathbb{R} \rightarrow \mathbb{R}$ be a function. We say that $A(X)$ is g -invariant if $g \circ f \in A(X)$ for all $f \in A(X)$.

DEFINITION 3.1. We will say that a vector subspace $A(X)$ of $C(X)$ is *adequate* if:

- (a) $A(X)$ separates points from closed sets and contains the constant functions.
- (b) There exists a continuous nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, with $g(t) = 0$ if $t \leq 0$ and $g(t) = 1$ if $t \geq 1$, such that $A(X)$ is g -invariant.
- (c) The positive cone $A(X)_+$ generates $A(X)$, i.e., every $f \in A(X)$ can be written as $f_1 - f_2$, where f_1 and f_2 are nonnegative functions in $A(X)$.

Observe that conditions (b) and (c) hold if $A(X)$ is a sublattice of $C(X)$. Indeed, condition (c) is obvious for a sublattice. Take $g(t)$ to be 0, t or 1 for $t \leq 0$, $0 < t < 1$ or $t \geq 1$, respectively. Then $A(X)$ is g -invariant if $A(X)$ is a sublattice of $C(X)$. Also, condition (c) holds if $A(X)$ consists of bounded functions and contains the constants. Indeed, in this case, any $f \in A(X)$ can be written as $c1_X - (c1_X - f)$, where $c \geq 0$ and $c1_X \geq f$.

LEMMA 3.2. *Let X be a completely regular space and let $A(X)$ be an adequate subspace of $C(X)$. Given $x_0 \in \mathcal{A}X$, $f \in A(X)$, and a neighborhood U of $x_0(f)$ in $[-\infty, \infty]$, set $V = \{x \in \mathcal{A}X : x(f) \in U\}$. Then there exists $h \in A(X)$ with $0 \leq h \leq 1_X$ and $x_0(h) = 0$ such that $W = \{x \in \mathcal{A}X : x(h) < 1\} \subseteq V$.*

Proof. Let g be the function given in Definition 3.1. Define $g(\infty) = 1$ and $g(-\infty) = 0$. Then $x(g \circ f) = g(x(f))$ for all $x \in \mathcal{A}X$ and all $f \in A(X)$. First consider the case where $x_0(f) = a \in \mathbb{R}$. Choose $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq U$. Set $f_1 = \varepsilon^{-1}(f - a1_X) \in A(X)$. Then $x_0(f_1) = 0$. If $x \in \mathcal{A}X$ and $|x(f_1)| < 1$, then $|x(f) - a| = \varepsilon|x(f_1)| < \varepsilon$. Thus $W' = \{x \in \mathcal{A}X : |x(f_1)| < 1\} \subseteq V$. Set $h = 1_X + g \circ f_1 - g \circ (f_1 + 1_X) \in A(X)$. Using the observation above, we see that for any $x \in \mathcal{A}X$, $x(h) = 1 + g(x(f_1)) - g(x(f_1) + 1)$. It is easy to check that $0 \leq h \leq 1_X$, $x_0(h) = 0$, and $W \subseteq W' \subseteq V$.

If $x_0(f) = +\infty$ or $-\infty$, the proof is similar. Assume the former. Choose $0 < m \in \mathbb{R}$ such that $(m, \infty] \subseteq U$ and define $h = 1_X - g \circ (f - (m + 1)1_X)$. We omit the verification that h satisfies the requirements. ■

Denote by $C_b(X)$ the subspace of $C(X)$ consisting of the bounded functions. Let $A_b(X) = A(X) \cap C_b(X)$ and $\hat{A}_b(X) = \{\hat{f} : f \in A_b(X)\}$. The map $f \mapsto \hat{f}$ is a bijection from $A_b(X)$ onto the subspace $\hat{A}_b(X)$ of $C(\mathcal{A}X)$. Since $A_b(X)$ contains the constant functions on X , $\hat{A}_b(X)$ contains the constant functions on $\mathcal{A}X$.

LEMMA 3.3. *Let X be a completely regular space. If $A(X)$ is an adequate subspace of $C(X)$, then $\hat{A}_b(X)$ precisely separates points from closed sets in $\mathcal{A}X$.*

Proof. Let $x_0 \in \mathcal{A}X$ and F be a closed subset of $\mathcal{A}X$ not containing x_0 . Then $V = \mathcal{A}X \setminus F$ is an open neighborhood of x_0 . Choose $f_1, \dots, f_n \in A(X)$ and open neighborhoods U_i of $x_0(f_i)$ in $[-\infty, \infty]$ such that $\bigcap_{i=1}^n V_i \subseteq V$, where $V_i = \{x \in \mathcal{A}X : x(f_i) \in U_i\}$. By Lemma 3.2, there exist $h_i, 1 \leq i \leq n$, such that $0 \leq h_i \leq 1_X$, $x_0(h_i) = 0$, and $W_i = \{x \in \mathcal{A}X : x(h_i) < 1\} \subseteq V_i$. Set $h = 1_X - g \circ \sum_{i=1}^n h_i \in A(X)$. Then $0 \leq h \leq 1_X$ and $\hat{h}(x_0) = x_0(h) = 1$. If $x \in \mathcal{A}X$, $x \notin V$, then there exists j such that $x \notin V_j$, and hence $x \notin W_j$. Thus $x(h_j) \geq 1$ and so $x(\sum_{i=1}^n h_i) \geq 1$. Therefore, $\hat{h}(x) = x(h) = 0$. ■

THEOREM 3.4. *Let X and Y be completely regular spaces. Suppose that $A(X)$ and $A(Y)$ are adequate subspaces of $C(X)$ and $C(Y)$ respectively. If $T : A(X) \rightarrow A(Y)$ is a linear order isomorphism such that $T(A_b(X)) = A_b(Y)$, then there exists a homeomorphism $h : \mathcal{A}X \rightarrow \mathcal{A}Y$ such that for all $x \in X$, $y \in Y$, $f \in A(X)$ and $g \in A(Y)$, $\hat{f}(h^{-1}(y))$, $\hat{g}(h(x)) \in \mathbb{R}$, $Tf = T1_X \cdot \hat{f} \circ h^{-1}|_Y$ and $T^{-1}g = T^{-1}1_Y \cdot \hat{g} \circ h|_X$.*

Proof. It is clear that T induces a linear order isomorphism $\hat{T} : \hat{A}_b(X) \rightarrow \hat{A}_b(Y)$ given by $\hat{T}\hat{f} = (Tf)^\wedge$. The spaces $\hat{A}_b(X)$ and $\hat{A}_b(Y)$ are subspaces of $C(\mathcal{A}X)$ and $C(\mathcal{A}Y)$, respectively. By Lemma 3.3, $\hat{A}_b(X)$ and $\hat{A}_b(Y)$ precisely separate points from closed sets. Therefore, by Theorem 2.1 and the Remark following Proposition 2.6, there is a homeomorphism $h : \mathcal{A}X \rightarrow \mathcal{A}Y$ such that $\hat{T}\hat{f} = (T1_X)^\wedge \cdot \hat{f} \circ h^{-1}$ for all $\hat{f} \in \hat{A}_b(X)$ and $\hat{T}^{-1}\hat{g} = (T^{-1}1_Y)^\wedge \cdot \hat{g} \circ h$ for all $\hat{g} \in \hat{A}_b(Y)$. Since $T^{-1}1_Y$ is bounded, we get in particular

$$(3.1) \quad 1_{\mathcal{A}Y} = \hat{T}(T^{-1}1_Y)^\wedge = (T1_X)^\wedge \cdot (T^{-1}1_Y)^\wedge \circ h^{-1}.$$

Let $f \in A(X)$ with $f \geq 0$. Suppose that $x_0 \in \mathcal{A}X$ and $y_0 = h(x_0) \in Y$. Note that $\hat{f}(x_0) \geq 0$. If $\hat{f}(x_0) = 0$, then $Tf(y_0) \geq (T1_X)(y_0)\hat{f}(x_0)$. Otherwise, for all $a \in \mathbb{R}$ with $0 \leq a < \hat{f}(x_0)$, there exists an open neighborhood U of x_0 in $\mathcal{A}X$ such that $f(x) > a$ for all $x \in U \cap X$. By Lemma 3.3, there exists $g \in A_b(X)$ such that $\text{range } \hat{g} \subseteq [0, 1]$, $\hat{g}(x_0) = 1$ and $\hat{g} = 0$ outside U . Note that $(Tg)(y_0) = (T1_X)(y_0)\hat{g}(x_0) = (T1_X)(y_0)$ since $g \in A_b(X)$ and $y_0 \in Y$. As $f - ag \geq 0$, $T(f - ag) \geq 0$. Thus $(Tf)(y_0) \geq a(Tg)(y_0) = a(T1_X)(y_0)$. This shows that $(Tf)(y_0) \geq (T1_X)(y_0)\hat{f}(x_0)$. Since $T1_X(y_0) = (T1_X)^\wedge(y_0) > 0$ by Proposition 2.5, we see that, in particular, $\hat{f}(x_0) \in \mathbb{R}$. In other words, $\hat{f}(h^{-1}(y)) \in \mathbb{R}$ and $(Tf)(y) \geq (T1_X)(y)\hat{f}(h^{-1}(y))$ for all $y \in Y$. By symmetry and equation (3.1), we also get, for all $x \in X$,

$$\begin{aligned} (T1_X)^\wedge(h(x)) \cdot f(x) &= (T1_X)^\wedge(h(x)) \cdot (T^{-1}(Tf))(x) \\ &\geq (T1_X)^\wedge(h(x)) \cdot (T^{-1}1_Y)(x) \cdot (Tf)^\wedge(h(x)) = (Tf)^\wedge(h(x)). \end{aligned}$$

Given $y \in Y$, let (x_α) be a net in X so that $(h(x_\alpha))$ converges to y . Applying the preceding calculation to x_α and taking limit gives $T1_X(y) \cdot \hat{f}(h^{-1}(y)) \geq (Tf)(y)$. Thus $(Tf)(y) = (T1_X)(y)\hat{f}(h^{-1}(y))$ for all $y \in Y$.

For a general $f \in A(X)$, write $f = f_1 - f_2$, where $0 \leq f_1, f_2 \in A(X)$. If $y \in Y$, $\hat{f}_1(h^{-1}(y))$ and $\hat{f}_2(h^{-1}(y)) \in \mathbb{R}$. By the previous paragraph,

$$\begin{aligned} Tf(y) &= Tf_1(y) - Tf_2(y) = T1_X(y) \cdot (\hat{f}_1(h^{-1}(y)) - \hat{f}_2(h^{-1}(y))) \\ &= T1_X(y) \cdot \hat{f}(h^{-1}(y)). \end{aligned}$$

The formula for $T^{-1}g$ follows by the same argument. ■

REMARK. If $T : A(X) \rightarrow A(Y)$ is a linear order isomorphism such that there exists $0 < c < 1$ so that $c1_Y \leq T1_X \leq c^{-1}1_Y$, then $T(A_b(X)) = A_b(Y)$. This holds in particular if $T1_X = 1_Y$.

Proof. In fact, if $f \in A_b(X)$, then there exists $0 < M < \infty$ such that $-M1_X \leq f \leq M1_X$. Then

$$-Mc^{-1}1_Y \leq -MT1_X \leq Tf \leq MT1_X \leq Mc^{-1}1_Y.$$

Since the condition $c1_Y \leq T1_X \leq c^{-1}1_Y$ is equivalent to $c1_X \leq T^{-1}1_Y \leq c^{-1}1_X$, the other direction follows by symmetry. ■

Theorem 3.4 applies to all function spaces that are commonly considered in the context of order isomorphisms. Given a function space $A(X)$, let $A^{\text{loc}}(X)$ be the space of all real-valued functions f on X such that for every $x_0 \in X$, there are a neighborhood U of x_0 and a function $g \in A(X)$ such that, $f = g$ on the set U . The space $A_b^{\text{loc}}(X)$ is the subspace of all bounded functions in $A^{\text{loc}}(X)$.

PROPOSITION 3.5.

- (a) *Let X be a completely regular space and let $A(X)$ be a sublattice of $C(X)$ that separates points from closed sets and contains the constant functions. Then $A(X)$ is adequate.*
- (b) *If $A(X)$ is adequate, then so are $A_b(X)$, $A^{\text{loc}}(X)$ and $A_b^{\text{loc}}(X)$.*
- (c) *Let X be an open set in a Banach space E and let $p \in \mathbb{N} \cup \{\infty\}$. Suppose that E supports a C^p bump function, i.e., there exists $\varphi \in C^p(E)$ such that $\varphi(0) \neq 0$ and that φ has bounded support. Then $C^p(X)$ is adequate.*

Proof. For part (a), see the remark following Definition 3.1. Part (b) is clear. For part (c), take $g : \mathbb{R} \rightarrow \mathbb{R}$ to be a nondecreasing C^∞ function such that $g(t) = 0$ if $t \leq 0$ and $g(t) = 1$ if $t \geq 1$. Then $C^p(X)$ and $C_b^p(X)$ are g -invariant. ■

Let (X, d) be a metric space. The space of *Lipschitz functions* on X , $\text{Lip}(X)$, consists of all real-valued functions f on X such that

$$\sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\} < \infty.$$

The space $\text{lip}(X)$ of *little Lipschitz functions* consists of all $f \in \text{Lip}(X)$ such that

$$\limsup_{t \rightarrow 0} \left\{ \frac{|f(x) - f(y)|}{t} : 0 < d(x, y) < t \right\} = 0.$$

The space of uniformly continuous functions on X is denoted by $\text{UC}(X)$. Observe that $\text{Lip}(X)$, $\text{lip}(X)$ and $\text{UC}(X)$ are sublattices of $C(X)$ that contain the constant functions. $\text{Lip}(X)$ and $\text{UC}(X)$ always separate points from closed sets and hence are adequate by Proposition 3.5. If $\text{lip}(X)$ separates points from closed sets, then it is also adequate. This occurs in particular if there exists $0 < \alpha < 1$ and a metric D on X so that $d = D^\alpha$. By Proposition 3.5(b), the local, bounded, and bounded local versions of these spaces are also adequate.

An obvious question with regard to Theorem 3.4 is under what circumstances would the homeomorphism h map X onto Y . A simple example shows that this may not always be the case. In fact, for any completely regular space X , $C_b(X)$ is linearly order isomorphic to $C(\beta X)$. Thus, if X and Y are nonhomeomorphic spaces with homeomorphic Stone-Ćech compactifications, then $C_b(X)$ is linearly order isomorphic to $C_b(Y)$ under a linear isomorphism that preserves bounded functions.

4. Refinements. The purpose of this section is to refine Theorem 3.4 by showing that, in many situations, the homeomorphism h maps X onto Y . We will also show that in some cases, one may remove the condition that T preserves bounded functions. First we look at a classical situation. A completely regular space X is said to be *realcompact* if, given $x \in \beta X \setminus X$, there exists $f \in C(X)$ whose extension \hat{f} takes an infinite value at x .

THEOREM 4.1. *Let X and Y be realcompact spaces and let $T : C(X) \rightarrow C(Y)$ be a linear order isomorphism. Then there exists a homeomorphism $\theta : X \rightarrow Y$ such that $Tf = T1_X \cdot f \circ \theta^{-1}$ and $T^{-1}g = T^{-1}1_Y \cdot g \circ \theta$ for all $f \in C(X)$ and all $g \in C(Y)$.*

Proof. Let $u = 1_X + T^{-1}1_Y$. Observe that $u \geq 1_X$ and $Tu \geq 1_Y$. Define $S : C(X) \rightarrow C(Y)$ by $Sf = T(uf)/Tu$ for all $f \in C(X)$. It is easy to check that S is a linear order isomorphism such that $S1_X = 1_Y$. Let $h : \beta X \rightarrow \beta Y$ be the homeomorphism obtained by applying Theorem 3.4 to the map S . Since $\hat{f}(h^{-1}(y)) \in \mathbb{R}$ for all $f \in C(X)$ and $y \in Y$, and X is realcompact, we find that $h^{-1}(y) \in X$ for all $y \in Y$. Similarly, $h(x) \in Y$ for all $x \in X$. Let θ be the restriction of h to X . Then θ is a homeomorphism from X onto Y . By Theorem 3.4 again, and $Sf = f \circ \theta^{-1}$ for all $f \in C(X)$. In particular,

$(1/u) \circ \theta^{-1} = S(1/u) = T1_X/Tu$. Hence, for all $f \in C(X)$,

$$\begin{aligned} Tf &= Tu \cdot S\left(\frac{f}{u}\right) = Tu \cdot \frac{f}{u} \circ \theta^{-1} = Tu \cdot (f \circ \theta^{-1}) \cdot \left(\frac{1}{u} \circ \theta^{-1}\right) \\ &= Tu \cdot (f \circ \theta^{-1}) \cdot \frac{T1_X}{Tu} = T1_X \cdot (f \circ \theta^{-1}). \end{aligned}$$

Similarly, $T^{-1}g = T^{-1}1_Y \cdot g \circ \theta$ for all $g \in C(Y)$. ■

A representation of nonlinear order isomorphisms between spaces of continuous functions on compact Hausdorff spaces has been obtained by F. Cabello Sánchez [CS1].

For the remainder of this section, we consider metric spaces X and Y .

PROPOSITION 4.2. *Let X and Y be metric spaces and let $A(X)$ and $A(Y)$ be adequate subspaces of $C(X)$ and $C(Y)$, respectively. If $h : \mathcal{A}X \rightarrow \mathcal{A}Y$ is a homeomorphism and $x_0 \in X$, then there is a sequence (y_n) in Y that converges to $h(x_0)$ in $\mathcal{A}Y$.*

Proof. In the first instance, suppose that x_0 is an isolated point in X . Since $A(X)$ separates points from closed sets, the characteristic function $\chi_{\{x_0\}}$ belongs to $A(X)$. Let $U = \{x \in \mathcal{A}X : x(\chi_{\{x_0\}}) > 0\}$. Then U is an open neighborhood of x_0 in $\mathcal{A}X$. We claim that U contains only x_0 . To see this, suppose that $x \in \mathcal{A}X \setminus \{x_0\}$. Choose a net (x_α) in X that converges to x . Since $\chi_{\{x_0\}}(x_\alpha) \in \{0, 1\}$ and $\lim_\alpha \chi_{\{x_0\}}(x_\alpha) = x(\chi_{\{x_0\}})$, the latter value is either 0 or 1. If it is 1, then there exists α_0 such that $\chi_{\{x_0\}}(x_\alpha) = 1$ for all $\alpha \geq \alpha_0$. Then $x_\alpha = x_0$ for all $\alpha \geq \alpha_0$. As a result, $x = x_0$, yielding a contradiction. Thus $x(\chi_{\{x_0\}}) = 0$, i.e., $x \notin U$, as claimed. By the claim, x_0 is an isolated point in $\mathcal{A}X$. Hence $h(x_0)$ is an isolated point in $\mathcal{A}Y$. But since Y is dense in $\mathcal{A}Y$, $h(x_0)$ cannot be in $\mathcal{A}Y \setminus Y$. So $h(x_0) \in Y$ and the conclusion of the proposition is obvious.

Now suppose that x_0 is not an isolated point in X . Fix a pairwise distinct sequence of points (x_n) in X that converges to x_0 , and a strictly positive null sequence (ε_n) in \mathbb{R} . Choose $f_n \in A(X)$ such that $f_n(x_n) = 1$ and $f_n = 0$ outside $B(x_n, \varepsilon_n)$. Set $U_n = \{x \in \mathcal{A}X : x(f_n) > 0\}$. Then U_n is an open neighborhood of x_n in $\mathcal{A}X$. Thus $h(U_n)$ is a nonempty open set in $\mathcal{A}Y$ and hence $h(U_n) \cap Y \neq \emptyset$. Pick $y_n \in h(U_n) \cap Y$ and let $z_n = h^{-1}(y_n)$. Take any $f \in A(X)$. In particular, f is continuous on X . For any $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - f(x_0)| < \varepsilon$ if $x \in B(x_0, \delta)$. Observe that U_n is open in $\mathcal{A}X$ and hence $U_n \subseteq \overline{U_n \cap X}$. Then $z_n \in U_n \subseteq \overline{U_n \cap X}$. By continuity of \hat{f} , $z_n(f) = \hat{f}(z_n) \in \overline{\hat{f}(U_n \cap X)}$. There exists n_0 such that $B(x_n, \varepsilon_n) \subseteq B(x_0, \delta)$ for all $n \geq n_0$. Hence $U_n \cap X \subseteq B(x_0, \delta)$ for all $n \geq n_0$. Therefore, $|z_n(f) - f(x_0)| \leq \varepsilon$ for all $n \geq n_0$. This proves that $\lim z_n(f) = f(x_0)$. As $f \in A(X)$ is arbitrary, $z_n \rightarrow x_0$ in $\mathcal{A}X$. Thus $y_n = h(z_n) \rightarrow h(x_0)$ in $\mathcal{A}Y$, as desired. ■

A set of points S in a metric space is *separated* if there exists $\varepsilon > 0$ such that $d(x_1, x_2) \geq \varepsilon$ whenever x_1 and x_2 are distinct points in S .

COROLLARY 4.3. *Let X and Y be metric spaces and let $A(X)$ and $A(Y)$ be adequate subspaces of $C(X)$ and $C(Y)$ respectively. Assume that*

- (a) $A(Y) = A^{\text{loc}}(Y)$ or $A_b^{\text{loc}}(Y)$, or
- (b) Y is complete and, for any separated sequence (y_n) in Y , there exists $g \in A(Y)$ such that $g(y_{2n}) = 1$ and $g(y_{2n-1}) = 0$ for all n .

If $h : AX \rightarrow AY$ is a homeomorphism and $x_0 \in X$, then $h(x_0) \in Y$.

Proof. (a) By Proposition 4.2, there is a sequence (y_n) in Y that converges to $h(x_0)$. If (y_n) has a subsequence that converges in Y , then we are done. Assume that (y_n) has no subsequence that converges in Y . We may then further assume that there is a strictly positive real sequence (ε_n) so that $B(y_m, 2\varepsilon_m) \cap B(y_n, 2\varepsilon_n) = \emptyset$ if $m \neq n$. By Lemma 3.3, there exists $g_n \in A(Y)$ so that $g_n(y_n) = 1$, $g_n = 0$ outside $B(y_n, \varepsilon_n)$, and $0 \leq g_n \leq 1_Y$. Let g be the pointwise sum $\sum g_{2n}$. Take $y \in Y$. If $y \in B(y_n, 2\varepsilon_n)$ for some n , then there exists $\varepsilon > 0$ such that $B(y, \varepsilon)$ does not intersect $B(y_m, \varepsilon_m)$ for any $m \neq n$. Suppose $y \notin B(y_n, 2\varepsilon_n)$ for any n . If, for all $\varepsilon > 0$, $B(y, \varepsilon)$ intersects $B(y_n, \varepsilon_n)$ for at least two n , then for any $\varepsilon > 0$, $B(y, \varepsilon)$ intersects infinitely many $B(y_n, \varepsilon_n)$. This implies that (y_n) has a convergent subsequence, contrary to the choice of (y_n) . This establishes that there exists $\varepsilon > 0$ so that $B(y, \varepsilon)$ intersects at most one $B(y_n, \varepsilon_n)$. Thus g is a bounded function in $A^{\text{loc}}(Y)$. By the assumption, $g \in A(Y)$. In particular, $(g(y_n))$ converges to $\hat{g}(h(x_0))$. However, this is impossible since $g(y_n) = 1$ if n is even and $g(y_n) = 0$ if n is odd.

(b) By Proposition 4.2, there is a sequence (y_n) in Y that converges to $h(x_0)$. If (y_n) has a Cauchy subsequence, then we are done. Otherwise, we may assume that (y_n) is separated. By the assumption, there exists $g \in A(Y)$ so that $g(y_{2n}) = 1$ and $g(y_{2n-1}) = 0$ for all n . But this is impossible since $(g(y_n))$ converges to $\hat{g}(h(x_0))$. ■

The next theorem unifies many results concerning unital order isomorphisms on most types of commonly considered function spaces defined on metric spaces.

THEOREM 4.4. *Let X and Y be metric spaces. Assume that $A(X)$ is an adequate subspace of $C(X)$ and that either*

- (1) $A(X) = A^{\text{loc}}(X)$ or $A_b^{\text{loc}}(X)$, or
- (2) X is complete and, for any separated sequence (x_n) in X , there exists $f \in A(X)$ such that $f(x_{2n}) = 1$ and $f(x_{2n-1}) = 0$ for all n .

Assume the same for $A(Y)$. If $T : A(X) \rightarrow A(Y)$ is a linear order isomorphism such that $T(A_b(X)) = A_b(Y)$, then there is a homeomorphism

$h : X \rightarrow Y$ such that $Tf = T1_X \cdot f \circ h^{-1}$ and $T^{-1}g = T1_Y \cdot g \circ h$ for all $f \in A(X)$ and $g \in A(Y)$.

Proof. Apply Theorem 3.4 and Corollary 4.3(a) or (b) as the case may be. ■

Theorem 4.4 applies if $A(X)$ (and $A(Y)$) is of one of the following types:

- $\text{Lip}(X), \text{Lip}_b(X), \text{UC}(X), \text{UC}_b(X)$, where X is a complete metric;
- $\text{lip}(X), \text{lip}_b(X)$, where X is a complete metric and $\text{lip}(X)$ satisfies condition (2) in Theorem 4.4;
- $\text{Lip}^{\text{loc}}(X), \text{Lip}_b^{\text{loc}}(X), \text{UC}^{\text{loc}}(X), \text{UC}_b^{\text{loc}}(X)$, where X is a metric (not necessarily complete);
- $\text{lip}^{\text{loc}}(X), \text{lip}_b^{\text{loc}}(X)$, where X is metric, not necessarily complete, and $\text{lip}^{\text{loc}}(X)$ separates points from closed sets;
- $C^p(X), C_b^p(X)$, where X is an open set in a Banach space E that supports a C^p bump function.

Finally, we show that the condition that T preserves bounded functions may be removed in certain cases. The idea is to use the “division trick” that has been employed in the proof of Theorem 4.1.

THEOREM 4.5. *Let X and Y be metric spaces. Assume that either*

- (1) $A(X)$ is a sublattice of $C(X)$ that separates points from closed sets and contains the constants, and that either
 - (a) $A(X) = A^{\text{loc}}(X)$ or $A_b^{\text{loc}}(X)$, or
 - (b) X is complete and $A(X) = \text{Lip}(X)$ or $\text{Lip}_b(X)$, or
 - (c) $A(X) = \text{lip}(X)$ or $\text{lip}_b(X)$, where X is complete with metric d such that $d = D^\alpha$ for some metric D on X and $0 < \alpha < 1$; or
- (2) $A(X) = C^p(X)$ or $C_b^p(X)$, where X is an open set in a Banach space E that supports a C^p bump function.

Assume the same for $A(Y)$. If $T : A(X) \rightarrow A(Y)$ is a linear order isomorphism, then there is a homeomorphism $h : X \rightarrow Y$ such that $Tf = T1_X \cdot f \circ h^{-1}$ and $T^{-1}g = T1_Y \cdot g \circ h$ for all $f \in A(X)$ and $g \in A(Y)$.

Proof. Let $u = 1_X + T^{-1}1_Y$. Then $u \in A(X)$ and $u \geq 1_X, Tu \geq 1_Y$. Define

$$F(X) = \left\{ \frac{f}{u} : f \in A(X) \right\} \quad \text{and} \quad F(Y) = \left\{ \frac{g}{Tu} : g \in A(Y) \right\}.$$

If $A(X)$ is a sublattice of $C(X)$ that separates points from closed sets, then $F(X)$ is a sublattice of $C(X)$ that contains the constants and separates points from closed sets. If $A(X) = C^p(X)$, then $F(X) = C^p(X)$. If $A(X) = C_b^p(X)$, then u is bounded and $\geq 1_X$. Hence $F(X) = C_b^p(X)$. Thus, in all

cases, $F(X)$ is an adequate subspace of $C(X)$. The same considerations apply to $F(Y)$. Define a map $S : F(X) \rightarrow F(Y)$ by $Sf = T(uf)/Tu$ for all $f \in F(X)$. It is easy to check that S is a linear isomorphism with $S1_X = 1_Y$. From the remark following Theorem 3.4, we see that $S(F_b(X)) = F_b(Y)$. Denote the $F(X)$ - and $F(Y)$ -compactifications of X and Y by $\mathcal{F}X$ and $\mathcal{F}Y$, respectively. By Theorem 3.4, there is a homeomorphism $h : \mathcal{F}X \rightarrow \mathcal{F}Y$ such that $Sf = \hat{f} \circ h^{-1}|_Y$ for all $f \in F(X)$. Thus

$$(4.1) \quad Tf(y) = Tu(y) \cdot (f/u) \circ h^{-1}(y) \quad \text{for all } f \in A(X) \text{ and all } y \in Y.$$

If h is a homeomorphism from X onto Y , then it follows from (4.1) that $T1_X(y) = Tu(y)/u(h^{-1}(y))$ and hence $Tf(y) = T1_X(y) \cdot f(h^{-1}(y))$. The result for $T^{-1}g$ can be obtained similarly. By symmetry, it remains to show that $h(X) \subseteq Y$.

If $A(Y) = A^{\text{loc}}(Y)$ or $A_b^{\text{loc}}(Y)$ (including the cases where $A(Y) = C^p(Y)$ or $C_b^p(Y)$), then $F(Y) = F^{\text{loc}}(Y)$ or $F_b^{\text{loc}}(Y)$ (note that in the latter case Tu is bounded and also bounded away from 0). So by Corollary 4.3, $h(X) \subseteq Y$.

Finally, we consider case (1)(b) (for case (1)(c), a similar argument works using the metric D). Suppose that Y is complete and $A(Y) = \text{Lip}(Y)$ or $\text{Lip}_b(Y)$. Let $x_0 \in X$ and $y_0 = h(x_0)$. By Proposition 4.2, there exists a sequence (y_n) in Y that converges to y_0 in $\mathcal{F}Y$. If (y_n) has a Cauchy subsequence in Y , then $y_0 \in Y$. Otherwise, by using a subsequence, we may assume that (y_n) is separated. In the first instance, suppose that $(Tu(y_n))$ has a bounded subsequence. Then we may assume without loss of generality that $(Tu(y_n))$ converges to a real number a . Applying (4.1) with $y = y_n$ and taking limit, we see that $\lim Tf(y_n) = a \cdot f(x_0)/u(x_0) \in \mathbb{R}$ for any $Tf \in A(Y)$. But since (y_n) is a separated sequence in Y and $\text{Lip}_b(Y) \subseteq A(Y)$, this is not true. Hence it must be that $(Tu(y_n))$ diverges to ∞ . By choosing a subsequence if necessary, we may assume that $Tu(y_m) > 4Tu(y_n)$ if $m < n$. Note that there is a constant $C > 0$ such that $d(y_m, y_n) \geq C|Tu(y_m) - Tu(y_n)|$. Then

$$d(y_m, y_n) \geq \frac{C}{2}(Tu(y_m) + Tu(y_n)) \quad \text{if } m < n.$$

Hence the balls $B(y_n, \frac{C}{2}Tu(y_n))$ are pairwise disjoint. Furthermore, as Tu is unbounded, $A(Y) = \text{Lip}(Y)$. In this case, it is easy to construct a function g in $\text{Lip}(Y)$ such that $g(y_{2n-1}) = \frac{C}{2}Tu(y_{2n-1})$ and $g(y_{2n}) = 0$. However, it follows from (4.1) that $\lim Tf(y_n)/Tu(y_n) = f(x_0)/u(x_0)$ for any $f \in A(X)$. Taking $f = T^{-1}g$ yields a contradiction. ■

Acknowledgements. The authors wish to thank the referee whose astute comments on the first submission of this paper prompted much further reflection and great improvements in both the content and exposition of the paper.

Research of the first author was partially supported by AcRF project no. R-146-000-157-112, and the second author was supported in part by NSF of China (11301285, 11371201).

References

- [A1] J. Araujo, *Separating maps and linear isometries between some spaces of continuous functions*, J. Math. Anal. Appl. 226 (1998), 23–39.
- [A2] J. Araujo, *Realcompactness and Banach–Stone theorems*, Bull. Belg. Math. Soc. Simon Stevin 10 (2003), 247–258.
- [B] S. Banach, *Théorie des Opérations Linéaires*, Monograf. Mat. 1, Warszawa, 1932; reprint, Chelsea, New York, 1963.
- [CS1] F. Cabello Sánchez, *Homomorphisms on lattices of continuous functions*, Positivity 12 (2008), 341–362.
- [CS2] F. Cabello Sánchez and J. Cabello Sánchez, *Some preserver problems on algebras of smooth functions*, Ark. Mat. 48 (2010), 289–300.
- [CS3] F. Cabello Sánchez and J. Cabello Sánchez, *Nonlinear isomorphisms of lattices of Lipschitz functions*, Houston J. Math. 37 (2011), 181–202.
- [GK] I. Gelfand and A. Kolmogoroff, *On rings of continuous functions on topological spaces*, Dokl. Akad. Nauk. SSSR (C. R. Acad. Sci. USSR) 22 (1939), 11–15.
- [GaJ1] M. I. Garrido and J. A. Jaramillo, *A Banach–Stone theorem for uniformly continuous functions*, Monatsh. Math. 131 (2000), 189–192.
- [GaJ2] M. I. Garrido and J. A. Jaramillo, *Homomorphisms on function lattices*, Monatsh. Math. 141 (2004), 127–146.
- [GJ] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.
- [JV1] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Order isomorphisms of little Lipschitz algebras*, Houston J. Math. 34 (2008), 1185–1195.
- [JV2] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Homomorphisms on real-valued little Lipschitz function spaces*, Topology Appl. 156 (2009), 2908–2913.
- [Kad] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math. 56 (1952), 494–503.
- [K] I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Soc. 53 (1947), 617–623.
- [S] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), 375–481.

Denny H. Leung
 Department of Mathematics
 National University of Singapore
 Singapore 119076
 E-mail: matlhh@nus.edu.sg

Lei Li
 School of Mathematical Sciences and LPMC
 Nankai University
 Tianjin, 300071, China
 E-mail: leilee@nankai.edu.cn

Received October 9, 2012
Revised version October 2, 2013

(7646)