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Product equivalence of quasihomogeneous Toeplitz operators on the harmonic Bergman space

by

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Abstract. We present here a quite unexpected result: If the product of two quasihomogeneous Toeplitz operators $T_f T_g$ on the harmonic Bergman space is equal to a Toeplitz operator T_h, then the product $T_g T_f$ is also the Toeplitz operator T_h, and hence T_f commutes with T_g. From this we give necessary and sufficient conditions for the product of two Toeplitz operators, one quasihomogeneous and the other monomial, to be a Toeplitz operator.

1. Introduction. Let dA denote the Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1, and let $L^2(D, dA)$ be the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\, dA(z).$$

The harmonic Bergman space L_h^2 is the closed subspace of $L^2(D, dA)$ consisting of the harmonic functions on D. We will write Q for the orthogonal projection from $L^2(D, dA)$ onto L_h^2. Each point evaluation is easily verified to be a bounded linear functional on L_h^2. Hence, for each $z \in D$, there exists a unique function R_z (called the *harmonic Bergman kernel*) in L_h^2 that has the following reproducing property:

$$f(z) = \langle f, R_z\rangle \quad \text{for every } f \in L_h^2.$$

Given $z \in D$, let $K_z(w) = 1/(1 - w\overline{z})^2$ be the well-known reproducing kernel for the analytic Bergman space L_a^2 consisting of all L^2-analytic functions on D. The well-known Bergman projection P is then the integral operator

$$Pf(z) = \int_D f(w)\overline{K_z(w)}\, dA(w)$$

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for $f \in L^2(D, dA)$. Since $L_h^2 = L_a^2 + \overline{L_a^2}$, it is easily checked that $R_z = K_z + \overline{K_z} - 1$. Thus, Q can be represented by

$$Qf = Pf + \overline{P\overline{f}} - Pf(0).$$

For $u \in L^1(D, dA)$, the *Toeplitz operator* T_u with symbol u is the operator on L_h^2 defined by

$$T_u f = Q(uf) \tag{1.1}$$

for $f \in L_h^2$. This operator is always densely defined on the polynomials and not bounded in general. We are interested in the case where it is bounded in the L_h^2 norm. In this paper we will consider the case where u is a T-function. Recall that u is a *T-function* if the equation (1.1) defines a bounded operator on L_h^2. Also, if u is a T-function, we write T_u for the continuous extension of the operator defined by (1.1) (see [DZ3] or [LSZ]). Generally, the T-functions form a proper subset of $L^1(D, dA)$ which contains all bounded and "nearly bounded" functions.

A function f is said to be *quasihomogeneous* of degree $k \in \mathbb{Z}$ if

$$f(re^{i\theta}) = e^{ik\theta}\varphi(r),$$

where φ is a radial function. In this case the associated Toeplitz operator T_f is also called a *quasihomogeneous Toeplitz operator* of degree k.

In 1964, Brown and Halmos [BH] proved that $T_f T_g = T_h$ on the classical Hardy space H^2 if and only if either (I) g is analytic, or (II) f is conjugate analytic. They also showed that in both cases $h = fg$. In the setting of analytic Bergman space, conditions (I) and (II) are still sufficient, but they are no longer necessary. Ahern and Čučković [AC] showed that a Brown–Halmos type result holds for Toeplitz operators with harmonic symbols on L_a^2. Later in [LSZ], Louhichi, Strouse and Zakariasy gave necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators on L_a^2 to be a Toeplitz operator. Recently, we studied some algebraic properties of quasihomogeneous Toeplitz operators on the analytic Bergman space of the unit ball in [DZ1] and [ZD].

The theory of Toeplitz operators on L_h^2 is quite different from that on L_a^2. We list here some examples.

1. Choe and Lee [CL] showed that two analytic Toeplitz operators on L_h^2 commute only when their symbols and the constant function 1 are linearly dependent.
2. Ding [D] showed that an analytic Toeplitz operator and a co-analytic Toeplitz operator on L_h^2 can commute only when one of their symbols is constant.
3. We showed in [DZ3] that a Toeplitz operator with an analytic or co-analytic monomial symbol commutes with another Toeplitz operator

only when their symbols and the constant function 1 are linearly dependent.

4. We showed in [DZ2] that the product of two Toeplitz operators on L_h^2 with monomial symbols is a Toeplitz operator only when both of them are radial.
5. Louhichi and Zakariasy [LZ] proved that the product of two Toeplitz operators on L_h^2, one quasihomogeneous and the other radial, is a Toeplitz operator only in the trivial case.

In general, it requires more on their symbols for two quasihomogeneous Toeplitz operators to commute on L_h^2 than on L_a^2, as the orthonormal basis for L_h^2 involves a co-analytic monomial. In fact, the above five results confirm this view. However, we were quite surprised to find in [DZ3] that to check the commutativity of two quasihomogeneous Toeplitz operators on L_h^2, the vanishing of the commutator on only "half" of the orthonormal basis is needed.

In this paper we will show some other quite unexpected results. First, we give the following theorem.

THEOREM 1.1. *Let f_1 and f_2 be two quasihomogeneous T-functions on D. If there exists a T-function f such that $T_{f_1}T_{f_2} = T_f$, then $T_{f_2}T_{f_1} = T_f$.*

Obviously, Theorem 1.1 yields the following result which shows the connection between the product and the commutativity of two quasihomogeneous Toeplitz operators on L_h^2.

THEOREM 1.2. *Let f_1 and f_2 be two quasihomogeneous T-functions on D. If there exists a T-function f such that $T_{f_1}T_{f_2} = T_f$, then $T_{f_1}T_{f_2} = T_{f_2}T_{f_1}$.*

We would like to point out that we have not seen any similar results for Toeplitz operators on function spaces. So the research of algebraic properties of Toeplitz operators on L_h^2 is quite meaningful and interesting.

According to Theorem 1.2 we can use the commutativity of two quasihomogeneous Toeplitz operators to discuss when their product is a Toeplitz operator. The next theorem will show that the product of $T_{e^{ik_1\theta}r^m}$ and $T_{e^{ik_2\theta}\varphi(r)}$ is a Toeplitz operator only in the trivial cases.

THEOREM 1.3. *Let $k_1, k_2 \in \mathbb{Z}$ and let m be a nonnegative real number. Then for a nonzero bounded radial function φ on D, $T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}$ is a Toeplitz operator if and only if one of the following conditions holds:*

(1) $k_1 = m = 0$.
(2) $k_1 = k_2 = 0$.
(3) $k_2 = 0$ *and φ is a constant function.*

A special case of Theorem 1.3 with $\varphi = r^n$ was proved by us in [DZ2].

2. Some preliminary results. Before we state our results, we need to introduce the concept of the Mellin transform. For $f \in L^1([0,1], rdr)$, the Mellin transform of f is the function $\hat{f}$ defined by

$$\hat{f}(z) = \int_0^1 f(s)s^{z-1}\,ds.$$

It is known that $\hat{f}$ is well defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$.

In [DZ2] we proved the following results which we shall use often in this paper.

LEMMA 2.1. *Let $k \in \mathbb{Z}$ and let φ be a radial T-function. Then for each $n \in \mathbb{N}$,*

$$T_{e^{ik\theta}\varphi}(z^n) = \begin{cases} 2(n+k+1)\widehat{\varphi}(2n+k+2)z^{n+k} & \text{if } n \geq -k, \\ 2(-n-k+1)\widehat{\varphi}(-k+2)\overline{z}^{-n-k} & \text{if } n < -k; \end{cases}$$

$$T_{e^{ik\theta}\varphi}(\overline{z}^n) = \begin{cases} 2(n-k+1)\widehat{\varphi}(2n-k+2)\overline{z}^{n-k} & \text{if } n \geq k, \\ 2(k-n+1)\widehat{\varphi}(k+2)z^{k-n} & \text{if } n < k. \end{cases}$$

A direct calculation gives the following essential lemma.

LEMMA 2.2. *Let $k_1, k_2 \in \mathbb{Z}$ and let φ_1, φ_2, ψ be radial T-functions.*

(a) *If $k_1 + k_2 \geq 0$, then the following properties hold:*

(I) *For any $n \in \mathbb{N}$, $T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(z^n) = T_{e^{i(k_1+k_2)\theta}\psi}(z^n)$*
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{n+k_1+k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{n+k_1+k_2})$.

(II) *For any $n \in \mathbb{N}$ such that $n \geq k_1 + k_2$,*
$T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)$
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(z^{n-k_1-k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(z^{n-k_1-k_2})$.

(III) *For any $n \in \mathbb{N}$ such that $1 \leq n < k_1 + k_2$,*
$T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)$
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{k_1+k_2-n}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{k_1+k_2-n})$.

(b) *If $k_1 + k_2 < 0$, then the following properties hold:*

(I) *For any $n \in \mathbb{N}$, $T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)$*
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(z^{n-k_1-k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(z^{n-k_1-k_2})$.

(II) *For any $n \in \mathbb{N}$ such that $n \geq -k_1 - k_2$,*
$T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(z^n) = T_{e^{i(k_1+k_2)\theta}\psi}(z^n)$
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{n+k_1+k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{n+k_1+k_2})$.

(III) *For any $n \in \mathbb{N}$ such that $1 \leq n < -k_1 - k_2$,*
$T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(z^n) = T_{e^{i(k_1+k_2)\theta}\psi}(z^n)$
$\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(z^{-k_1-k_2-n}) = T_{e^{i(k_1+k_2)\theta}\psi}(z^{-k_1-k_2-n})$.

Proof. First suppose that $k_1+k_2 \geq 0$. Without loss of generality, we can also assume that $k_1 \geq 0$.

For any $n \in \mathbb{N}$, by Lemma 2.1 we get

$$
\begin{aligned}
&T_{e^{ik_1\theta}\varphi_1} T_{e^{ik_2\theta}\varphi_2}(z^n)\\
&\quad= \begin{cases} 2(n+k_1+k_2+1)\widehat{\varphi_1}(2n+k_1+2k_2+2) \\ \qquad \times 2(n+k_2+1)\widehat{\varphi_2}(2n+k_2+2) z^{n+k_1+k_2} & \text{if } n \geq -k_2, \\ 2(n+k_1+k_2+1)\widehat{\varphi_1}(k_1+2) \\ \qquad \times 2(-n-k_2+1)\widehat{\varphi_2}(-k_2+2) z^{n+k_1+k_2} & \text{if } n < -k_2; \end{cases}\\
&T_{e^{ik_2\theta}\varphi_2} T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{n+k_1+k_2})\\
&\quad= \begin{cases} 2(n+1)\widehat{\varphi_2}(2n+k_2+2) \\ \qquad \times 2(n+k_2+1)\widehat{\varphi_1}(2n+k_1+2k_2+2)\overline{z}^n & \text{if } n \geq -k_2, \\ 2(n+1)\widehat{\varphi_2}(-k_2+2) 2(-n-k_2+1)\widehat{\varphi_1}(k_1+2)\overline{z}^n & \text{if } n < -k_2; \end{cases}\\
&T_{e^{i(k_1+k_2)\theta}\psi}(z^n) = 2(n+k_1+k_2+1)\widehat{\psi}(2n+k_1+k_2+2) z^{n+k_1+k_2};\\
&T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)\\
&\quad= \begin{cases} 2(n-k_1-k_2+1)\widehat{\psi}(2n-k_1-k_2+2)\overline{z}^{n-k_1-k_2} & \text{if } n \geq k_1+k_2, \\ 2(k_1+k_2-n+1)\widehat{\psi}(k_1+k_2+2)\overline{z}^n & \text{if } n < k_1+k_2. \end{cases}
\end{aligned}
$$

Thus, if $n \geq -k_2$,

$$
\begin{aligned}
&T_{e^{ik_1\theta}\varphi_1} T_{e^{ik_2\theta}\varphi_2}(z^n) = T_{e^{i(k_1+k_2)\theta}\psi}(z^n)\\
&\quad\Leftrightarrow 2(n+k_2+1)\widehat{\varphi_1}(2n+k_1+2k_2+2)\widehat{\varphi_2}(2n+k_2+2)\\
&\qquad = \widehat{\psi}(2n+k_1+k_2+2)\\
&\quad\Leftrightarrow T_{e^{ik_2\theta}\varphi_2} T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{n+k_1+k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{n+k_1+k_2}),
\end{aligned} \tag{2.1}
$$

and if $n < -k_2$,

$$
\begin{aligned}
&T_{e^{ik_1\theta}\varphi_1} T_{e^{ik_2\theta}\varphi_2}(z^n) = T_{e^{i(k_1+k_2)\theta}\psi}(z^n)\\
&\quad\Leftrightarrow 2(-n-k_2+1)\widehat{\varphi_1}(k_1+2)\widehat{\varphi_2}(-k_2+2)\\
&\qquad = \widehat{\psi}(2n+k_1+k_2+2)\\
&\quad\Leftrightarrow T_{e^{ik_2\theta}\varphi_2} T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{n+k_1+k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{n+k_1+k_2}).
\end{aligned} \tag{2.2}
$$

So condition (I) holds.

Similarly, for any $n \in \mathbb{N}$ such that $n \geq k_1+k_2$, by Lemma 2.1,

$$
\begin{aligned}
T_{e^{ik_1\theta}\varphi_1} T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) &= 2(n-k_1-k_2+1)\widehat{\varphi_1}(2n-k_1-2k_2+2)\\
&\qquad \times 2(n-k_2+1)\widehat{\varphi_2}(2n-k_2+2)\overline{z}^{n-k_1-k_2};
\end{aligned}
$$

$$T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(z^{n-k_1-k_2}) \\ = 2(n-k_2+1)\widehat{\varphi_1}(2n-k_1-2k_2+2)2(n+1)\widehat{\varphi_2}(2n-k_2+2)z^n,$$

and hence

$$\begin{aligned}(2.3)\qquad T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) &= T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)\\ &\Leftrightarrow 2(n-k_2+1)\widehat{\varphi_1}(2n-k_1-2k_2+2)\widehat{\varphi_2}(2n-k_2+2)\\ &\qquad = \widehat{\psi}(2n-k_1-k_2+2)\\ &\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(z^{n-k_1-k_2}) = T_{e^{i(k_1+k_2)\theta}\psi}(z^{n-k_1-k_2}).\end{aligned}$$

So condition (II) holds.

For any $n \in \mathbb{N}$ such that $1 \le n < k_1+k_2$, by Lemma 2.1,

$$T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) \\ = \begin{cases} 2(k_1+k_2-n+1)\widehat{\varphi_1}(k_1+2k_2-2n+2) & \\ \quad \times 2(k_2-n+1)\widehat{\varphi_2}(k_2+2)z^{k_1+k_2-n} & \text{if } 1 \le n < k_2,\\ 2(k_1+k_2-n+1)\widehat{\varphi_1}(k_1+2) & \\ \quad \times 2(n-k_2+1)\widehat{\varphi_2}(2n-k_2+2)z^{k_1+k_2-n} & \text{if } k_2 \le n < k_1+k_2;\end{cases}$$

$$T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{k_1+k_2-n}) \\ = \begin{cases} 2(k_2-n+1)\widehat{\varphi_1}(k_1+2k_2-2n+2) & \\ \quad \times 2(n+1)\widehat{\varphi_2}(k_2+2)z^{n} & \text{if } 1 \le n < k_2,\\ 2(n-k_2+1)\widehat{\varphi_1}(k_1+2) & \\ \quad \times 2(n+1)\widehat{\varphi_2}(2n-k_2+2)z^{n} & \text{if } k_2 \le n < k_1+k_2.\end{cases}$$

Thus, if $1 \le n < k_2$,

$$\begin{aligned}(2.4)\qquad T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) &= T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)\\ &\Leftrightarrow 2(k_2-n+1)\widehat{\varphi_1}(k_1+2k_2-2n+2)\widehat{\varphi_2}(k_2+2) = \widehat{\psi}(k_1+k_2+2)\\ &\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{k_1+k_2-n}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{k_1+k_2-n}),\end{aligned}$$

and if $k_2 \le n < k_1+k_2$,

$$\begin{aligned}(2.5)\qquad T_{e^{ik_1\theta}\varphi_1}T_{e^{ik_2\theta}\varphi_2}(\overline{z}^n) &= T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^n)\\ &\Leftrightarrow 2(n-k_2+1)\widehat{\varphi_1}(k_1+2)\widehat{\varphi_2}(2n-k_2+2) = \widehat{\psi}(k_1+k_2+2)\\ &\Leftrightarrow T_{e^{ik_2\theta}\varphi_2}T_{e^{ik_1\theta}\varphi_1}(\overline{z}^{k_1+k_2-n}) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^{k_1+k_2-n}).\end{aligned}$$

It follows that condition (III) holds.

The proof of (b) is similar. This completes the proof. ∎

Remark 2.3. Let f_1, f_2 and f be three quasihomogeneous T-functions of degree $k_1, k_2, k_1+k_2 \in \mathbb{Z}$ respectively. Then Lemma 2.2 implies that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} &T_{f_1}T_{f_2}(r^{|k|}e^{ik\theta}) = T_f(r^{|k|}e^{ik\theta}) \\ &\quad \Leftrightarrow T_{f_2}T_{f_1}(r^{|-k-k_1-k_2|}e^{i(-k-k_1-k_2)\theta}) = T_f(r^{|-k-k_1-k_2|}e^{i(-k-k_1-k_2)\theta}). \end{aligned}$$

In [DZ3] we characterized the commutativity of two quasihomogeneous Toeplitz operators $T_{e^{ik_1\theta}r^m}$ and $T_{e^{ik_2\theta}\varphi(r)}$ in the case of $|k_1| \le |k_2|$. The following lemma will study another case which will be used later.

LEMMA 2.4. *Let $k_1, k_2 \in \mathbb{Z}$ such that $k_1 > 0$, $k_2 < 0$ and $|k_1| > |k_2|$, and let m be a nonnegative real number. Then for a bounded radial function φ on D,*

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi} = T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}$$

if and only if $\varphi = 0$.

Proof. Suppose $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$. Then the equality

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}(z^n) = T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}(z^n)$$

for each $n \in \mathbb{Z}$ such that $n \ge -k_2$ together with Lemma 2.1 gives

$$\widehat{\varphi}(2n+2k_1+k_2+2) = \widehat{\varphi}(2n+k_2+2)\frac{(2n+2k_2+2)(2n+k_1+m+2)}{(2n+2k_1+2)(2n+k_1+2k_2+m+2)},$$

which is the same as equation (2.4) of [CR]. However, here m is a real number. In fact, by the same proof we can also get

$$\widehat{\varphi}(z) = C\frac{\Gamma\big(\frac{z+k_2}{2k_1}\big)\Gamma\big(\frac{z+m+k_1-k_2}{2k_1}\big)}{\Gamma\big(\frac{z+2k_1-k_2}{2k_1}\big)\Gamma\big(\frac{z+m+k_1+k_2}{2k_1}\big)}$$

for some constant C. In what follows, we will show $C = 0$, and hence $\varphi = 0$.

So assume $C \ne 0$. Noting that $k_2 < 0$, by Theorem 4 of [CR] one of

$$\frac{2k_2-2k_1}{2k_1}, \quad -\frac{m+k_1}{2k_1}, \quad \frac{m-k_1}{2k_1}, \quad -\frac{2k_2}{2k_1}$$

must be an integer. Since $|k_1| > |k_2| > 0$ and $m \ge 0$, it is easy to see that

$$m = (2n+1)k_1$$

for some $n \in \mathbb{N}$. Hence

$$\widehat{\varphi}(z) = C\frac{\Gamma\big(\frac{z+k_2}{2k_1}\big)\Gamma\big(\frac{z-k_2}{2k_1}+n+1\big)}{\Gamma\big(\frac{z-k_2}{2k_1}+1\big)\Gamma\big(\frac{z+k_2}{2k_1}+n+1\big)}.$$

Denote $F(i) = (-k_2 + i)\widehat{\varphi}(-k_2 + 2i)$ for positive integers i. Then

$$\begin{aligned} F(i) &= C(-k_2+i)\frac{\Gamma\big(\frac{i}{k_1}\big)\Gamma\big(\frac{-k_2+i}{k_1}+n+1\big)}{\Gamma\big(\frac{-k_2+i}{k_1}+1\big)\Gamma\big(\frac{i}{k_1}+n+1\big)} \\ &= C(-k_2+i)\frac{\big(\frac{-k_2+i}{k_1}+n\big)\big(\frac{-k_2+i}{k_1}+n-1\big)\cdots\big(\frac{-k_2+i}{k_1}+1\big)}{\big(\frac{i}{k_1}+n\big)\big(\frac{i}{k_1}+n-1\big)\cdots\big(\frac{i}{k_1}+1\big)\frac{i}{k_1}} \\ &= Ck_1\bigg(\frac{-k_2}{i}+1\bigg)\bigg(1+\frac{-k_2}{\frac{i}{k_1}+n}\bigg)\bigg(1+\frac{-k_2}{\frac{i}{k_1}+n-1}\bigg)\cdots\bigg(1+\frac{-k_2}{\frac{i}{k_1}+1}\bigg), \end{aligned}$$

which is strictly monotonic. However, by Lemma 2.1,

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}(z^0) = T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}(z^0)$$

gives

$$(-k_2+1)\widehat{\varphi}(-k_2+2) = (k_1+1)\widehat{\varphi}(2k_1+k_2+2),$$

and consequently

$$F(1) = F(k_1+k_2+1),$$

which is a contradiction since $k_1 + k_2 > 0$.

The converse implication is clear. This completes the proof. ∎

3. Proofs of the theorems. In this section we will prove our main theorems and give some corollaries.

Proof of Theorem 1.1. Suppose the quasihomogeneous degrees of f_1 and f_2 are k_1 and k_2 respectively. If there exists a T-function f such that $T_{f_1}T_{f_2} = T_f$, then by Theorem 1.2 of [DZ2], f must be a quasihomogeneous function of degree $k_1 + k_2$. Now for any $k \in \mathbb{Z}$, the equality

$$T_{f_1}T_{f_2}(r^{|k|}e^{ik\theta}) = T_f(r^{|k|}e^{ik\theta})$$

together with Remark 2.3 gives

$$T_{f_2}T_{f_1}(r^{|-k-k_1-k_2|}e^{i(-k-k_1-k_2)\theta}) = T_f(r^{|-k-k_1-k_2|}e^{i(-k-k_1-k_2)\theta}).$$

Since

$$\{\sqrt{n+1}\,z^n\}_{n=0}^{\infty} \cup \{\sqrt{n+1}\,\overline{z}^n\}_{n=1}^{\infty}$$

is an orthonormal basis for the harmonic Bergman space, the desired result is obvious. ∎

Theorem 1.2 is a direct consequence of Theorem 1.1, but it is very useful. By Theorem 1.2 it seems a natural idea to use the commutativity of two quasihomogeneous Toeplitz operators to discuss when their product is a Toeplitz operator. Moreover, in [DZ3] we have characterized the commuting Toeplitz operators with quasihomogeneous symbols on L_h^2. So in what follows, we will use Theorem 1.2 and the results of [DZ3] to get some corollaries.

Here we need a known fact about the Mellin convolution. If f and g are in $L^1([0,1], rdr)$, then their *Mellin convolution* is defined by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{r}{t}\right) g(t)\, \frac{dt}{t}, \qquad 0 \le r < 1.$$

The following corollary gives a complete description of the product of two quasihomogeneous Toeplitz operators with the same or opposite degrees.

COROLLARY 3.1. *Let φ_1 and φ_2 be two radial T-functions on D. Then the following properties hold:*

(a) *For any integer $p \neq 0$, $T_{e^{ip\theta}\varphi_1} T_{e^{ip\theta}\varphi_2} = T_{e^{i2p\theta}\psi}$ if and only if $\varphi_1 = C\varphi_2$ for some constant C, and ψ such that*

$$\mathbb{I} *_M \psi = (r^p \varphi_1) *_M (r^{-p}\varphi_2)$$

and

$$2(p-n+1)\widehat{\varphi_1}(3p-2n+2)\widehat{\varphi_2}(p+2) = \widehat{\psi}(2p+2), \qquad \forall 1 \le n \le p.$$

(b) *For any integer $p \neq 0$, $T_{e^{ip\theta}\varphi_1} T_{e^{-ip\theta}\varphi_2} = T_\psi$ if and only if $|p| = 1$ and*

$$\varphi_1 *_M \varphi_2 = C(r *_M r^{-1})$$

for some constant C. In this case $\psi = C$.

(c) *For $p = 0$, $T_{e^{ip\theta}\varphi_1} T_{e^{ip\theta}\varphi_2} = T_\psi$ if and only if ψ is a solution of the equation*

$$\mathbb{I} *_M \psi = \varphi_1 *_M \varphi_2.$$

Proof. First assume $p \neq 0$. Without loss of generality, we can assume that $p > 0$, for otherwise we could take the adjoints.

It follows from (2.1), (2.3), (2.4) and (2.5) that $T_{e^{ip\theta}\varphi_1} T_{e^{ip\theta}\varphi_2} = T_{e^{i2p\theta}\psi}$ if and only if

$$2(n+p+1)\widehat{\varphi_1}(2n+3p+2)\widehat{\varphi_2}(2n+p+2) = \widehat{\psi}(2n+2p+2), \qquad \forall n \in \mathbb{N}; \tag{3.1}$$

$$2(n-p+1)\widehat{\varphi_1}(2n-3p+2)\widehat{\varphi_2}(2n-p+2) = \widehat{\psi}(2n-2p+2), \qquad \forall n \in \mathbb{N},\ n \ge 2p; \tag{3.2}$$

$$2(p-n+1)\widehat{\varphi_1}(3p-2n+2)\widehat{\varphi_2}(p+2) = \widehat{\psi}(2p+2), \qquad \forall n \in \mathbb{N},\ 1 \le n < p; \tag{3.3}$$

$$2(n-p+1)\widehat{\varphi_1}(p+2)\widehat{\varphi_2}(2n-p+2) = \widehat{\psi}(2p+2), \qquad \forall n \in \mathbb{N},\ p \le n < 2p. \tag{3.4}$$

First we suppose $T_{e^{ip\theta}\varphi_1} T_{e^{ip\theta}\varphi_2} = T_{e^{i2p\theta}\psi}$; then Theorem 1.2 shows that $T_{e^{ip\theta}\varphi_1}$ commutes with $T_{e^{ip\theta}\varphi_2}$, and according to Proposition 3.4 of [DZ3] we get

$$\varphi_1 = C\varphi_2.$$

Now, we replace n by $n+2p$; then (3.2) turns into

$$2(n+p+1)\widehat{\varphi_1}(2n+p+2)\widehat{\varphi_2}(2n+3p+2)=\widehat{\psi}(2n+2p+2),\qquad \forall n\in\mathbb{N},$$

which is the same as (3.1) since $\varphi_1=C\varphi_2$. Similarly, replacing n by $2p-n$ we see that (3.4) is the same as (3.3). Moreover, using the same reasoning as in the proof of Proposition 4.3 of [LSZ], one can show that (3.1) holds if and only if

$$\mathbb{I}*_M\psi=(r^p\varphi_1)*_M(r^{-p}\varphi_2).$$

The converse implication of (a) is clear, and hence condition (a) holds.

Similarly, (2.1)–(2.3) show that $T_{e^{ip\theta}\varphi_1}T_{e^{-ip\theta}\varphi_2}=T_\psi$ if and only if

$$(3.5)\qquad 2(n-p+1)\widehat{\varphi_1}(2n-p+2)\widehat{\varphi_2}(2n-p+2)=\widehat{\psi}(2n+2),\qquad \forall n\in\mathbb{N},\ n\ge p;$$

$$(3.6)\qquad 2(p-n+1)\widehat{\varphi_1}(p+2)\widehat{\varphi_2}(p+2)=\widehat{\psi}(2n+2),\qquad \forall n\in\mathbb{N},\ 0\le n<p;$$

$$(3.7)\qquad 2(n+p+1)\widehat{\varphi_1}(2n+p+2)\widehat{\varphi_2}(2n+p+2)=\widehat{\psi}(2n+2),\qquad \forall n\in\mathbb{N}.$$

Now we suppose $T_{e^{ip\theta}\varphi_1}T_{e^{-ip\theta}\varphi_2}=T_\psi$; then according to Theorem 1.2 and Proposition 3.4 of [DZ3] we get $p=1$ and

$$\varphi_1*_M\varphi_2=C(r*_M r^{-1}).$$

So (3.5) becomes

$$\widehat{\psi}(2n+2)=\frac{C}{2n+2},$$

which implies $\psi=C$.

Conversely, if $p=1$, $\psi=C$ and $\varphi_1*_M\varphi_2=C(r*_M r^{-1})$, then by a direct calculation, one can get (3.5)–(3.7), and hence condition (b) holds.

If $p=0$, then by Lemma 2.2, one can easily show that $T_{e^{ip\theta}\varphi_1}T_{e^{ip\theta}\varphi_2}=T_\psi$ if and only if

$$2(n+1)\widehat{\varphi_1}(2n+2)\widehat{\varphi_2}(2n+2)=\widehat{\psi}(2n+2),\qquad \forall n\in\mathbb{N},$$

which implies that

$$\mathbb{I}*_M\psi=\varphi_1*_M\varphi_2. \ \blacksquare$$

The following two corollaries concern the product of two quasihomogeneous Toeplitz operators $T_{e^{ik_1\theta}r^m}$ and $T_{e^{ik_2\theta}\varphi(r)}$.

Corollary 3.2. *Let $k_1,k_2\in\mathbb{Z}$ be such that $k_1k_2\le 0$ and $|k_1|\le|k_2|$, and let m be a real number greater than or equal to -1. Then for a nonzero radial T-function φ on D, there exists a radial T-function ψ such that*

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}=T_{e^{i(k_1+k_2)\theta}\psi}$$

if and only if one of the following conditions holds:

(1) $k_1=m=0$.
(2) $k_1=k_2=0$.
(3) $k_1k_2=-1$ *and* $\varphi=C\big(\frac{m+1}{2}r^{-1}-\frac{m-1}{2}r\big)$.

Proof. If condition (1) or (2) holds, then the desired result is obvious. Assume condition (3) holds. A direct calculation shows that

$$(r^m) *_M \left(\frac{m+1}{2}r^{-1} - \frac{m-1}{2}r\right) = \frac{1}{2}\left(\frac{1}{r} - r\right) = r *_M r^{-1},$$

so Corollary 3.1 implies that $T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi} = T_{e^{i(k_1+k_2)\theta}C}$.

Conversely, assume that $T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi} = T_{e^{i(k_1+k_2)\theta}\psi}$. Then by Theorem 1.2, $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$. Thus, Theorem 3.8 of [DZ3] shows that one of conditions (1)–(3) holds. ∎

COROLLARY 3.3. *Let $k_1, k_2 \in \mathbb{Z}$ be such that $k_1k_2 > 0$, and let m be a real number greater than or equal to -1. Then for a radial T-function φ on D, there exists a radial T-function ψ such that*

$$T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi} = T_{e^{i(k_1+k_2)\theta}\psi}$$

if and only if $\varphi = 0$.

Proof. First we assume that $T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi} = T_{e^{i(k_1+k_2)\theta}\psi}$. Combining Theorem 1.1 with the use of adjoint operators, we can further assume $k_1, k_2 > 0$. Thus by (2.1) and (2.4),

$$T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi}(z^0) = T_{e^{i(k_1+k_2)\theta}\psi}(z^0)$$

gives

$$\frac{2(k_2+1)}{m+k_1+2k_2+2}\widehat{\varphi}(k_2+2) = \widehat{\psi}(k_1+k_2+2),$$

and

$$T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi}(\overline{z}^1) = T_{e^{i(k_1+k_2)\theta}\psi}(\overline{z}^1)$$

gives

$$\frac{2k_2}{m+k_1+2k_2}\widehat{\varphi}(k_2+2) = \widehat{\psi}(k_1+k_2+2).$$

Hence

$$\frac{2(k_2+1)}{m+k_1+2k_2+2}\widehat{\varphi}(k_2+2) = \frac{2k_2}{m+k_1+2k_2}\widehat{\varphi}(k_2+2). \tag{3.8}$$

If φ is nonzero, noting that $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$, by taking φ_1 to be r^m in Lemma 3.7 of [DZ3] we obtain

$$\widehat{\varphi}(k_2+2) \neq 0.$$

Then it follows from (3.8) that

$$m = -k_1.$$

Using the fact that $m \geq -1$ and $k_1 > 0$, we get

$$k_1 = -m = 1.$$

Obviously, $1 = k_1 \le k_2$ since $k_2 > 0$; then Theorem 3.8 of [DZ3] shows that

$$k_2 = k_1 = 1 \quad \text{and} \quad \varphi = Cr^m = Cr^{-1}.$$

Now Corollary 3.1 shows that, if $T_{e^{i\theta}r^{-1}} T_{e^{i\theta}r^{-1}}$ were a Toeplitz operator, its symbol would be $e^{i2\theta}r^{-2}$. However, $e^{i2\theta}r^{-2}$ is not a T-function, so $\varphi = 0$.

The converse implication is clear. ∎

Proof of Theorem 1.3. First we suppose $T_{e^{ik_1\theta}r^m} T_{e^{ik_2\theta}\varphi}$ is a Toeplitz operator. We need to discuss several cases.

Case 1. Suppose $k_1 k_2 > 0$. Then Corollary 3.3 implies $\varphi = 0$, which is a contradiction.

Case 2. Suppose $k_1 k_2 \le 0$ and $|k_1| \le |k_2|$. Noticing that φ is bounded, by Corollary 3.2 we infer that one of conditions (1) or (2) holds.

Case 3. Suppose $k_1 k_2 < 0$ and $|k_1| > |k_2|$. Combining Theorem 1.1 with the use of adjoint operators, we can further assume $k_1 > 0$ and $k_2 < 0$. However, noticing that φ is bounded and $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$, Lemma 2.4 implies $\varphi = 0$, which is a contradiction.

Case 4. Suppose $k_1 k_2 = 0$ and $|k_1| > |k_2|$. This implies that $k_2 = 0$ and $k_1 \ne 0$; then Theorem 2 of [LZ] shows that φ must be a constant function, and so condition (3) holds.

The converse implication is clear. ∎

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