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# On the composition of Frostman Blaschke products

by

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**Abstract.** We construct an infinite uniform Frostman Blaschke product $B$ such that $B \circ B$ is also a uniform Frostman Blaschke product. We also show that the set of uniform Frostman Blaschke products is open in the set of inner functions with the uniform norm.

**1. Introduction.** In 1922, Ritt [19] studied *prime* or *indecomposable* finite Blaschke products, that is, Blaschke products that cannot be written as a composition of two nontrivial Blaschke products. Extending this idea to infinite Blaschke products, or functions of the form

$$B(z) = \lambda \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z},$$

where $0/0$ is taken to be 1 and $|\lambda| = 1$, has proved much more difficult. For example, writing $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ and denoting the Blaschke product by $B$, the composition $\varphi_a \circ B$ may be a singular inner function, while Frostman's theorem [5] shows that the composition $\varphi_a \circ B$ is, with the possible exception of $a$, $|a| < 1$, in a set of logarithmic capacity zero, a Blaschke product. Thus, certain natural questions arose: For example, when is a Blaschke product $B$ indestructible in the sense that $\varphi_a \circ B$ is a Blaschke product for all values of $a$ in the open unit disk $\mathbb{D}$? Work in this direction can be found in McLaughlin [15] and Morse [16].

We are concerned with the following: Which inner functions, outside the class of finite Blaschke products, are prime? As an example of how sensitive this question is we mention that Stephenson, in extending work of Ball on composition (see [1] and [21]), noted that the atomic singular inner function $S(z) = \exp\left(\frac{z+1}{z-1}\right)$ is not prime (just consider $z^n \circ S^{1/n}$), but the function $zS(z)$ is prime! This led the second author, Laroco, Mortini, and Rupp to

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study prime inner functions and pose several questions, [8]. We answer one of them here. Before proceeding, we note that the constant $\lambda$ appearing in the definition of Blaschke product will usually not affect computations or theorems and, in these cases, we will choose $\lambda = 1$.

To state the problem, we introduce the notion of Frostman and uniform Frostman Blaschke products and give a more precise definition of *prime*. Let $\mathbb{T}$ denote the unit circle. Recall that a Blaschke product with zero sequence $\{z_n\}$ (listed according to multiplicity) is a *Frostman Blaschke product* if

$$\sum_{n=1}^{\infty} \frac{1-|z_n|}{|z_n - z|} < \infty$$

for every $z \in \mathbb{T}$, and a *uniform Frostman Blaschke product* if

$$\sup_{z\in\mathbb{T}} \sum_{n=1}^{\infty} \frac{1-|z_n|}{|z_n - z|} < \infty.$$

These are named after Frostman, because he showed [5] that a Blaschke product with zeros $\{z_n\}$ has the property that it and all of its subproducts have radial limits of modulus 1 at $z$ if and only if $\sum_{n=1}^{\infty} \frac{1-|z_n|}{|z_n - z|} < \infty$.

An inner function $I$ is said to be *prime* if whenever $I = U \circ V$ then either $U$ or $V$ is a Möbius transformation.

In [8], the authors studied a small but important class of Blaschke products, the so-called *thin Blaschke products* or the Blaschke products for which the zero sequence $\{z_n\}$ satisfies

$$\lim_{n\to\infty} (1 - |z_n|^2)|B'(z_n)| = 1.$$

When the zeros are distinct, the zero sequences of thin Blaschke products are interpolating sequences for the space of bounded analytic functions in VMO and, consequently, it is a well-studied class. One of the results in [8] shows that no function in the class of thin Blaschke products can be written as the composition of two infinite Blaschke products, though some can be written as compositions of infinite Blaschke products and finite Blaschke products with degree greater than 1. Thus not all are prime but all are *semiprime*, that is, whenever a thin Blaschke product is a composition $U \circ V$ with $U$ and $V$ inner, then either $U$ or $V$ is a finite Blaschke product. In [3], the authors showed that finite products of thin Blaschke products, which are also semiprime, can be approximated uniformly by prime Blaschke products. We denote the class of finite products of thin Blaschke products by $\mathcal{FPTB}$.

The class of uniform Frostman Blaschke products (written $\mathcal{UFB}$) is another small but important class of Blaschke products: Hruščev and Vinogradov [10] showed that these are the inner functions that are multipliers of the space of Cauchy transforms (or Cauchy type integrals). In 1994 the authors of [8] posed a natural question: In many ways, functions in $\mathcal{FPTB}$ and $\mathcal{UFB}$ seem to act similarly; is their behavior under composition similar too? More precisely, can a uniform Frostman Blaschke product be the composition of two infinite Blaschke products?

In [17], Nicolau and Mortini studied yet another class, $\mathcal{M}$, of inner functions satisfying: If $I \in \mathcal{M}$, then $\varphi_a \circ I$ is a finite product of interpolating Blaschke products for all $a \in \mathbb{D} \setminus \{0\}$. Thin Blaschke products are in this class, but not every function in $\mathcal{M}$ is a Blaschke product. Mortini asked whether $\mathcal{UFB} \subset \mathcal{M}$. Later, Matheson and Ross [14] used the multiplier characterization in [10] to show that the composition $\varphi_a \circ B$ of a Möbius transformation $\varphi_a$ with a uniform Frostman Blaschke product $B$ is again a uniform Frostman Blaschke product. Thus, we know that

$$\mathcal{FPTB} \subset \mathcal{M} \quad \text{and} \quad \mathcal{UFB} \subset \mathcal{M},$$

and we know that $\mathcal{M}$ is not closed under composition, so different techniques are needed to answer the question of whether a Blaschke product in $\mathcal{UFB}$ can be a composition of two infinite Blaschke products, [8].

In [3], the authors showed (among other things) that a Frostman Blaschke product can be the composition of two infinite Blaschke products and that, given a uniform Frostman Blaschke product, $B$, there is a conformal automorphism $\varphi_a$ such that $\varphi_a^2 B$ cannot be written as the composition of two infinite Blaschke products. In this paper, we answer the original question that was posed in [8] by giving an example of a uniform Frostman Blaschke product that can be written as a composition of two infinite Blaschke products. Another way of thinking about this is the following: A finite product of multipliers of the space of Cauchy transforms is obviously a multiplier. When can an infinite product also be a multiplier? What is nice about this particular example is that the uniform Frostman Blaschke product we construct is a composition of a Blaschke product with itself. It is also connected with a question about the range of composition operators: Can a uniform Frostman Blaschke product be in the range of a composition operator? If the symbol is an automorphism, the answer is obviously yes. But what if the symbol is not a finite Blaschke product?

Our proof requires computations that allow us to present another interesting fact: $\mathcal{UFB}$ is open in the set of inner functions in $H^\infty$, the space of bounded analytic functions with the uniform norm.

**2. Preliminaries and notation.** As above, we say that a Blaschke product $B$ with zeros $\{z_n\}$ (listed according to multiplicity) is a *uniform*

*Frostman Blaschke product* if

$$\sup_{z\in\mathbb{T}}\sum_{n=1}^{\infty}\frac{1-|z_n|}{|z_n-z|}<\infty.$$

We denote the class of uniform Frostman Blaschke products by $\mathcal{UFB}$. Recall that a Blaschke product $B$ is interpolating if its zero sequence $\{z_n\}$ is an interpolating sequence for $H^\infty$; that is, whenever $\{w_n\}$ is a bounded sequence of complex numbers, there exists a function $f\in H^\infty$ such that $f(z_n)=w_n$ for all $n$. Every function in $\mathcal{UFB}$ is a finite product of interpolating Blaschke products [4, p. 140]. Matheson studied the spectrum of such Blaschke products in [13], and Matheson and Ross [14] showed that $\mathcal{UFB}$ is a subset of the class of indestructible Blaschke products and that, in fact, more is true:

Lemma 2.1 (Matheson and Ross, [14]). *If $B\in\mathcal{UFB}$ and $\varphi_a(z):=\frac{a-z}{1-\bar{a}z}$ is a disk automorphism with $a\in\mathbb{D}$, then $\varphi_a\circ B\in\mathcal{UFB}$.*

In [11], Kraus and Roth showed that indestructible Blaschke products form a semigroup. The lemma below provides a different proof of their result.

Lemma 2.2. *If $B$ is a Blaschke product and $C$ is an indestructible Blaschke product, then $B\circ C$ is a Blaschke product.*

*Proof.* Note that if we consider $C_1=\varphi_{C(0)}\circ C$, then $C_1$ is a Blaschke product because $C$ is indestructible and, moreover, $C_1(0)=0$. Now, $B\circ\varphi_{C(0)}^{-1}$ is also a Blaschke product and

$$B\circ C=(B\circ\varphi_{C(0)}^{-1})\circ(\varphi_{C(0)}\circ C),$$

so we may assume that $C(0)=0$. Having done so, note that if $B(0)=0$ we may write $B(z)=z^nB_1(z)$ with $B_1(0)\neq 0$ and $B(C(z))=C^n(z)(B_1\circ C)(z)$. Since $C^n$ is a Blaschke product, $B\circ C$ is a Blaschke product if and only if $B_1\circ C$ is too, so we may assume that $(B\circ C)(0)\neq 0$.

Write

$$B(z)=\prod_{n=1}^{\infty}\frac{|w_n|}{w_n}\,\frac{w_n-z}{1-\overline{w_n}z}\quad\text{so that}\quad(B\circ C)(z)=\prod_{n=1}^{\infty}\frac{|w_n|}{w_n}\,\frac{w_n-C(z)}{1-\overline{w_n}C(z)}.$$

Now each factor in the product above is a Blaschke product, because we assume $C$ is indestructible. Let $\{z_{n,j}\}_n$ denote the zeros of the $j$th factor, for $j=1,2,\dots$. Since $B\circ C\not\equiv 0$, the sequence $\{z_{n,j}\}_{n,j}$ is a Blaschke sequence and we may form the corresponding Blaschke product $D$. We claim that $B\circ C=D$.

First, it is clear that the zeros of $D$ are zeros of $B\circ C$, so $D$ is a subfactor of $B\circ C$. Now, $|(B\circ C)(0)|=\prod_{n=1}^{\infty}|w_n|$. But

$$\left|\frac{w_j-C(z)}{1-\overline{w_j}C(z)}\right|=\prod_{k=1}^{\infty}\left|\frac{z_{k,j}-z}{1-\overline{z_{k,j}}z}\right|.$$

Therefore $|w_j| = \prod_{k=1}^{\infty} |z_{k,j}|$. Consequently,

$$|(B \circ C)(0)| = \prod_{j=1}^{\infty} |w_j| = \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} |z_{k,j}| = |D(0)|,$$

since the product converges absolutely. Since $D$ is a subfactor of $B \circ C$ and $|D(0)| = |(B \circ C)(0)| \neq 0$, we see that $B \circ C = \lambda D$ for some $\lambda \in \mathbb{T}$. ∎

Since Matheson and Ross showed that every Blaschke product in $\mathcal{UFB}$ is indestructible, we have the following corollary.

Corollary 2.3. *If $C \in \mathcal{UFB}$ and $B$ is a Blaschke product, then $B \circ C$ is a Blaschke product.*

**3. An example.** We now construct an infinite, uniform Frostman Blaschke product $B$ such that $B \circ B$ is also a uniform Frostman Blaschke product, answering in the affirmative a question posed in [8]. The zeros $\{a_n\}_{n=1}^{\infty}$ of $B$ are chosen such that $a_n := r_n e^{i\theta_n}$, where $1 - 8^{-(n+1)} \leq r_n < r_{n+1} < 1$ and $\theta_n = \pi(1 - 2^{-(n+1)})$. Therefore, $\{a_n\}_{n=1}^{\infty}$ converges to $-1$ tangentially through the upper half-disk $\mathbb{D}^+ := \{z \in \mathbb{D} : \operatorname{Im}(z) > 0\}$. For any positive integer $n$, let $J_n$ be the arc of $\mathbb{T}$ given by $J_n = \{e^{i\theta} : |\theta - \theta_n| \leq \pi 2^{-(n+3)}\}$. Notice that $J_n$ is centered at $e^{i\theta_n}$ and that $J_n \cap J_m = \emptyset$ if $m \neq n$. By our choice of $a_n$ (in particular, $r_n$) there is a positive constant $c$ such that

$$\frac{1 - |a_n|}{|a_n - z|} \leq \frac{c 8^{-(n+1)}}{2^{-(n+3)}} = c 4^{-n} \quad \text{whenever } z \in \mathbb{T} \setminus J_n.$$

From this it follows that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|a_n - z|} \leq 1 + c/3$$

for all $z$ in $\mathbb{T}$, and hence $B$ is a uniform Frostman Blaschke product. So, by Corollary 2.3, $B \circ B$ is a Blaschke product.

Remark 3.1. Furthermore, if $0 < \varepsilon < 1/8$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{D}$ such that, for all $n$,

$$|a_n| \leq |z_n| \quad \text{and} \quad |z_n - a_n| < \varepsilon |1 + a_n|,$$

then $z_n$ lies over the arc $J_n$ and, for $z$ in $\mathbb{T} \setminus J_n$,

$$|z - z_n| \geq |z - a_n| - |a_n - z_n| > |z - a_n| - \varepsilon |1 + a_n| \geq |z - a_n|/2.$$

From this it follows that

$$\sum_{n=1}^{\infty} \frac{1 - |z_n|}{|z_n - z|} \leq 2 + 2c/3$$

for all $z$ in $\mathbb{T}$, and hence the Blaschke product with distinct zeros $\{z_n\}_{n=1}^{\infty}$ is itself uniform Frostman, with a bound on its Frostman sum independent of the particular choice of $\{z_n\}_{n=1}^{\infty}$.

Notation 3.2. Let $b_n(z) = \frac{|a_n|}{a_n}\frac{a_n - z}{1-\bar{a}_n z}$, the Blaschke factor of $B$ that is built around the zero $a_n$ – and let $B_n = B/b_n$. For positive integers $n$, let $T_n$ be the *tent region* whose base is $J_n$ and has vertex angle of $\pi/2$; see the figure below. The vertex of $T_n$ is expressible as $s_n e^{i\theta_n}$, where $s_n \approx \pi(1 - 2^{-(n+3)})$. Therefore, $a_n \in T_n$ and $a_n$ is much closer to $e^{i\theta_n}$ (in $J_n$) than it is to any point in $\mathbb{D} \cap \partial T_n$.

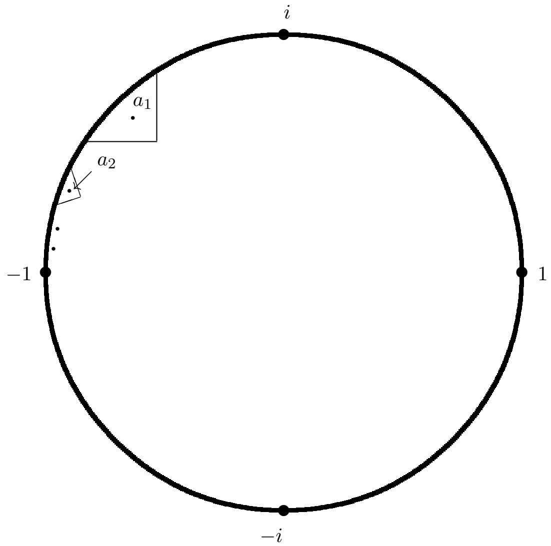


Fig. 1

The next lemma is an immediate consequence of results on pages 38, 145 and 149 of [4], namely, inequality (1.11.6) of Theorem 1.11.5, part (2) of Lemma 6.6.30 and inequality (6.6.42) of Lemma 6.6.41 in this reference. For points $z$ and $w$ in $\mathbb{D}$ we let $\rho(z,w) := \left|\frac{z-w}{1-\bar{z}w}\right|$, the *pseudohyperbolic distance* between $z$ and $w$.

Lemma 3.3. *Let $\Lambda$ be a Blaschke sequence such that*

$$\delta(\Lambda) := \inf_{a\in\Lambda} \prod_{b\in\Lambda\setminus\{a\}} \rho(a,b) > 0$$

*and*

$$\Sigma_\Lambda := \sup_{\zeta\in\mathbb{T}} \sum_{a\in\Lambda} \frac{1-|a|}{|\zeta - a|} < \infty.$$

*Then there is a constant $C$, depending only on $\delta(\Lambda)$ and $\Sigma_\Lambda$, such that*

$$\sum_{a\in\psi(\Lambda)} \frac{1-|a|}{|\zeta-a|} \le C$$

*whenever $\zeta \in \mathbb{T}$ and $\psi$ is a disk automorphism (i.e., a Möbius transformation from the disk onto itself).*

We now resume the construction of $B$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be any sequence in the interval $(0,1)$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. By the definitions of $T_n$ and of the $n$th Blaschke factor $b_n$, if $r_n$ is sufficiently near 1, then $|1-b_n(z)| < \varepsilon_n$ and $|b_n'(z)| < \varepsilon_n$, for all $z$ in $\overline{\mathbb{D}} \setminus \overline{T}_n$. Therefore, we have established:

Claim I. *For any $\varepsilon$ $(0 < \varepsilon < 1/8)$, we can choose $\{r_n\}_{n=1}^{\infty}$ converging sufficiently quickly to $\mathbb{T}$ so that $B$ (whose $n$th zero is $a_n := r_n e^{i\theta_n}$, as defined above), satisfies:*

(1) $|1-B(z)| < \varepsilon$ *whenever* $z \in \overline{\mathbb{D}} \setminus \bigcup_{n=1}^{\infty} \overline{T}_n$,
(2) $|1-B_n(z)| < \varepsilon$ *whenever* $z \in \overline{T}_n$, *and*
(3) $|B_n'(z)| < \varepsilon$ *whenever* $z \in \overline{T}_n$.

Choosing $\{r_n\}_{n=1}^{\infty}$ so that (1) holds, we find that the zeros of $B \circ B$ lie in $\bigcup_{n=1}^{\infty} T_n$. For any positive integer $n$, let $c_n$ be the unimodular constant given by $c_n := B_n(e^{i\theta_n})$. Notice that $z \mapsto c_n b_n(z)$ is a disk automorphism. Moreover, by (2) and our choice of $c_n$, $B_n \approx c_n$ on $\overline{T}_n$, and $|c_n - B_n(z)| \to 0$ as $z$ in $\overline{T}_n$ tends to $e^{i\theta_n}$. So, for $z$ in $\overline{T}_n$, $B(c_n b_n(z))$ is a very good approximation of $B(B(z))$.

For positive integers $n$, let $S_n = \{z \in \mathbb{D} : \mathrm{Re}(b_n(z)) < 0\}$. The closure of $\mathbb{D} \cap \partial S_n$ is an arc of a circle that contains $a_n$ and intersects $\mathbb{T}$ in two points, with contact angle $\pi/2$ at each. From this we find that $S_n \subseteq T_n$ for all positive integers $n$. Since $B_n \approx 1$ on $T_n$, it follows that any zero of $B \circ B$ in $T_n$ is in fact in $S_n$. And, since symmetry of points with respect to a circle or line is preserved under mapping by a Möbius transformation, we find that the aforementioned arc is part of the circle of radius $\frac{1}{2}(1/|a_n| - |a_n|)$ that is centered at $\frac{1}{2}(1/\overline{a}_n + a_n)$. This circle contains the point $a_n$, as does the circle with center $1/\overline{a}_n$ and radius $1/|a_n| - |a_n|$. From this we find that the region $S_n$ is contained within this second circle and hence

$$|1-\overline{a}_n z| < 1 - |a_n|^2, \tag{4}$$

whenever $z \in S_n$.

Claim II. *For any positive integer $n$, the function $B$ (which equals $B_n b_n$) is univalent and continuous on $\overline{S}_n$.*

*Proof.* That $B$ is continuous on $\overline{S}_n$ is easily observed, and no issue at all. For the univalence assertion, we digress momentarily and observe that, by (4), if $z \in \overline{S}_n$, then

$$|b_n'(z)| = \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2} \geq \frac{1}{1-|a_n|^2}.$$

So, by (2) and (3), if $z \in \overline{S}_n$, then

$$(5) \qquad |B'(z)| \geq |b_n'(z)|\,|B_n(z)| - |b_n(z)|\,|B_n'(z)| \geq \frac{1}{2(1-|a_n|^2)}.$$

And, again by (4), if $z$ and $w$ are distinct points in $\overline{S}_n$, then

$$|b_n(z) - b_n(w)| = \frac{(1-|a_n|^2)|z-w|}{|1-\overline{a}_n z|\,|1-\overline{a}_n w|} \geq \frac{|z-w|}{1-|a_n|^2}.$$

Therefore, by (2) and (3), for distinct points $z$ and $w$ in $\overline{S}_n$,

$$\begin{aligned}(6)\quad & |B(z)-B(w)| \\ &= |B_n(z)b_n(z) - B_n(w)b_n(z) + B_n(w)b_n(z) - B_n(w)b_n(w)| \\ &\geq |B_n(w)|\,|b_n(z)-b_n(w)| - |B_n(z)-B_n(w)|\,|b_n(z)| \\ &\geq |B_n(w)|\,|z-w|/(1-|a_n|^2) - \varepsilon|z-w|\,|b_n(z)| \\ &> |z-w| \neq 0.\end{aligned}$$

So, we find that $B$ is also univalent on $\overline{S}_n$, which completes the proof of Claim II.

By our work above, $B$ is a homeomorphism on $\overline{S}_n$ and consequently $\Gamma_n := B(\partial S_n)$ is a Jordan curve. Now $b_n(\mathbb{T} \cap \partial S_n) = \{\zeta \in \mathbb{T} : \operatorname{Re}(\zeta) \leq 0\}$, $b_n(\mathbb{D} \cap \partial S_n) = \{iy : -1 < y < 1\}$, $B_n$ is unimodular on $\mathbb{T} \cap \partial S_n$ and $|1 - B_n(z)| \leq \varepsilon$ for all $z$ in $\overline{S}_n$. From this it follows that if $\varepsilon < 1/8$ (which we henceforth assume), then all of the zeros of $B$, namely $\{a_k\}_{k=1}^{\infty}$, are on the inside of the Jordan curve $\Gamma_n$; and thus, $\{a_k\}_{k=1}^{\infty} \subseteq B(S_n)$. Therefore, there is a sequence of points $\{\alpha_{n,k}\}_{k=1}^{\infty}$ in $S_n$ such that $B(\alpha_{n,k}) = a_k$ for all positive integers $k$. Now $\{\alpha_{n,k}\}_{k=1}^{\infty}$ are the zeros of $B \circ B$ in $S_n$ (and $T_n$). Since $\lim_{k\to\infty} a_k = -1$ and $B$ is a homeomorphism on $\overline{S}_n$, we find that $\{\alpha_{n,k}\}_{k=1}^{\infty}$ converges to some value $\alpha_n$ in $\mathbb{T} \cap \overline{S}_n$ as $k \to \infty$, where $B(\alpha_n) = -1$.

Let $\sigma_n = B_n(\alpha_n)$, let $\beta_n$ be the (functional) inverse of $b_n$ and define $\omega_{n,k} = b_n(\alpha_{n,k})$. Then, among other things, $\beta_n(0) = a_n$, $\alpha_{n,k} = \beta_n(\omega_{n,k})$ and

$$a_k = B_n(\alpha_{n,k})b_n(\alpha_{n,k}) = B_n(\beta_n(\omega_{n,k})) \cdot \omega_{n,k}.$$

Moreover, by (3) and (6),

$$\begin{aligned}(7)\quad |\overline{\sigma}_n a_k - \omega_{n,k}| &= |a_k - \sigma_n \cdot \omega_{n,k}| = |B_n(\beta_n(\omega_{n,k})) \cdot \omega_{n,k} - \sigma_n \cdot \omega_{n,k}| \\ &= |\omega_{n,k}|\,|\sigma_n - B_n(\alpha_{n,k})| = |\omega_{n,k}|\,|B_n(\alpha_n) - B_n(\alpha_{n,k})| \\ &< |B_n(\alpha_n) - B_n(\alpha_{n,k})| < \varepsilon|\alpha_n - \alpha_{n,k}| \\ &< \varepsilon|B(\alpha_n) - B(\alpha_{n,k})| = \varepsilon|1 + a_k|.\end{aligned}$$

Since $a_k \to -1$ as $k \to \infty$, we have $\omega_{n,k} \to -\overline{\sigma}_n$ as $k \to \infty$. Furthermore, as $a_k = B_n(\beta_n(\omega_{n,k})) \cdot \omega_{n,k}$ and $|B_n(\beta_n(\omega_{n,k}))| < 1$, we find that $|\omega_{n,k}| > |a_k|$ for all $k$. Now, define $\varphi_n$ on $\mathbb{T}$ by

$$\varphi_n(z) := \sum_{k=1}^{\infty} \frac{1 - |\alpha_{n,k}|}{|z - \alpha_{n,k}|},$$

the *Frostman sum* over the sequence $\{\alpha_{n,k}\}_{k=1}^{\infty}$. Part of our string of inequalities in (7) shows that, for all positive integers $k$,

$$|a_k - \sigma_n \cdot \omega_{n,k}| < \varepsilon|1 + a_k|.$$

And, as already observed, $|\sigma_n \cdot \omega_{n,k}| = |\omega_{n,k}| > |a_k|$ for all $k$. So, by Remark 3.1, the Blaschke product having distinct zeros $\{\sigma_n \cdot \omega_{n,k}\}_{k=1}^{\infty}$ is uniform Frostman, with Frostman sum bounded on $\mathbb{T}$, independently of $n$. Moreover, we have:

LEMMA 3.4. *Let $\Lambda_n$ denote the Blaschke sequence $\{\sigma_n \cdot \omega_{n,k}\}_{k=1}^{\infty}$. If $\{a_k\}_{k=1}^{\infty}$ converges to $\mathbb{T}$ sufficiently fast, then* $\inf_n \delta(\Lambda_n) > 0$.

*Proof.* We already know that $|\sigma_n \cdot \omega_{n,k}| = |\omega_{n,k}| > |a_k|$ and also that $|a_k - \sigma_n \cdot \omega_{n,k}| < \varepsilon|1 + a_k|$ for all $k$, independently of $n$. So, by Remark 3.1, we additionally see that $\sigma_n \cdot \omega_{n,k}$ lies over the arc $J_k$, independent of $n$. Our analysis here is based on these things and the following well-known fact. Let $E$ be a nonempty subset of $\mathbb{T}$ and for $\eta > 0$, let $E_\eta = \{z \in \mathbb{D} : |z - \zeta| \geq \eta \text{ for all } \zeta \text{ in } E\}$. Then $\inf_{z \in E_\eta} \rho(w, z)$ can be made arbitrarily near 1 by choosing $w$ in $\mathbb{D}$ sufficiently near $E$. Now, recall that there is a positive gap between any pair of consecutive arcs $J_k$ and $J_{k+1}$, and the sequence of arcs $\{J_k\}_{k=1}^{\infty}$ tends to $-1$ through $\{\zeta \in \mathbb{T} : \operatorname{Im}(\zeta) > 0\}$ as $k \to \infty$. So, for any nondecreasing sequence $\{\rho_k\}_{k=1}^{\infty}$ in the interval $[0, 1)$, we can work inductively with respect to $k$ and radially move $a_k$ $(k = 1, 2, \ldots)$ sufficiently near $\mathbb{T}$ to obtain

$$\rho(\sigma_n \cdot \omega_{n,K}, \sigma_n \cdot \omega_{n,k}) \geq \rho_K$$

whenever $K \geq 2$ and $1 \leq k \leq K - 1$, independently of $n$. If $\{\rho_k\}_{k=1}^{\infty}$ converges to 1 sufficiently fast, then $\inf_K \rho_K^{K-1} \cdot \prod_{k=K+1}^{\infty} \rho_k > 0$ and hence $\inf_n \delta(\Lambda_n) > 0$. ∎

We have now established that, provided $\{a_k\}_{k=1}^{\infty}$ converges to $\mathbb{T}$ sufficiently fast, $\Lambda_n := \{\sigma_n \cdot \omega_{n,k}\}_{k=1}^{\infty}$ is a uniform Frostman Blaschke sequence with a bound on its Frostman sum independent of $n$ and that $\inf_n \delta(\Lambda_n) > 0$. Since $\sigma_n \cdot \omega_{n,k} = \sigma_n b_n(\alpha_{n,k})$, where $\sigma_n b_n$ is a disk automorphism, we can apply Lemma 3.3 and find there is a positive constant $M$ (independent of $n$) that bounds $\varphi_n$ on $\mathbb{T}$. There is one last ingredient that makes things fit together here.

Claim III. *If $\{a_n\}_{n=1}^\infty$ converges to $\mathbb{T}$ sufficiently fast, then the Frostman sum $\varphi_n$ is bounded by $1/2^n$ on $\mathbb{T} \setminus J_n$.*

*Proof.* Since $B$ maps $\mathbb{D}$ into itself, the Schwarz–Pick Lemma (cf. [7, p. 2]) tells us that

$$|B'(z)|(1-|z|^2) \le 1-|B(z)|^2$$

for all $z$ in $\mathbb{D}$. By (5), $|B'(z)| \ge \frac{1}{2(1-|a_n|^2)}$ for all $z$ in $S_n$. Since $\{\alpha_{n,k}\}_{k=1}^\infty \subseteq S_n$ and $B(\alpha_{n,k}) = a_k$, we thus have

$$1-|\alpha_{n,k}|^2 \le 2(1-|a_n|^2)(1-|a_k|^2)$$

for all positive integers $k$ and $n$. So, we can make $\sum_{k=1}^\infty (1-|\alpha_{n,k}|^2)$ as small as we like by choosing $a_n$ sufficiently near $\mathbb{T}$. Moreover, if $a_n$ is moved radially to $\mathbb{T}$, then $S_n$ shrinks uniformly to $e^{i\theta_n}$, which is a positive distance from $\mathbb{T} \setminus J_n$. Therefore, if $a_n$ is chosen sufficiently near $\mathbb{T}$, then

$$\varphi_n(z) = \sum_{k=1}^\infty \frac{1-|\alpha_{n,k}|}{|z-\alpha_{n,k}|} \le 1/2^n$$

for all $z$ in $\mathbb{T} \setminus J_n$. ∎

Coupling Claim III with our earlier work, we find that if $\{a_n\}_{n=1}^\infty$ converges to $\mathbb{T}$ sufficiently fast, then

$$\varphi(z) := \sum_{n=1}^\infty \varphi_n(z) \le M+1,$$

for all $z$ in $\mathbb{T}$. That is, the Frostman sum for $B \circ B$ is bounded on $\mathbb{T}$; hence, $B \circ B$ is a uniform Frostman Blaschke product.

**4. Approximation.** In this section we show that the class $\mathcal{UFB}$ is open in the set of inner functions in the space of bounded analytic functions, $H^\infty$, on the open unit disk $\mathbb{D}$. We begin by mentioning some related results on approximation and Blaschke products, starting with a lemma that is of interest in its own right and that holds for Frostman Blaschke products, too. For points $z, w \in \mathbb{D}$ let $\rho(z,w) := \left|\frac{z-w}{1-\bar{z}w}\right|$ denote the pseudohyperbolic distance between $z$ and $w$.

Lemma 4.1. *Let $B$ and $B^*$ be two Blaschke products with zero sequences $\{z_n\}_n$ and $\{z_n^*\}_n$, respectively, satisfying (for some appropriate ordering of the zeros): there exists $s$ with $0<s<1$ such that $\sup_n \rho(z_n, z_n^*) \le s$. Then, for $z$ in $\mathbb{T}$,*

$$\frac{1-s}{\sqrt{2}\,(1+s)} \sum_n \frac{1-|z_n|}{|z-z_n|} \le \sum_n \frac{1-|z_n^*|}{|z-z_n^*|} \le \frac{\sqrt{2}\,(1+s)}{1-s} \sum_n \frac{1-|z_n|}{|z-z_n|}.$$

*Proof.* If $0 < a < 1$ and $\varphi_a(x) = s$, then $x = \varphi_a(s)$ and $1 - \frac{a-s}{1-as} = \frac{(1-a)(1+s)}{1-as} \le (1-a)\frac{1+s}{1-s}$. If $\varphi_a(x) = -s$, then $x = \frac{s+a}{1+as}$ and $1 - \frac{s+a}{1+as} = \frac{(1-s)(1-a)}{1+as} \ge (1-a)\frac{1-s}{1+s}$. Therefore, if $a \in \mathbb{D}$ and $\rho(a,b) \le s$, then

$$(1-|a|)\frac{1-s}{1+s} \le 1-|b| \le (1-|a|)\frac{1+s}{1-s}. \tag{8}$$

For $\alpha$ in $\mathbb{D}$, let $P_\alpha(z) := \frac{1-|\alpha|^2}{|z-\alpha|^2}$ denote the Poisson kernel on $\mathbb{T}$ for evaluation at $\alpha$. For any point $z$ in $\mathbb{T}$,

$$\alpha \mapsto P_\alpha(z)$$

is positive and harmonic in $\mathbb{D}$. Therefore, by Harnack's inequality and the conformal invariance of the pseudohyperbolic metric,

$$\frac{1-|b|^2}{|z-b|^2} \le \frac{1+s}{1-s}\,\frac{1-|a|^2}{|z-a|^2}$$

whenever $\rho(a,b) \le s$ and $z \in \mathbb{T}$. And so, by the second inequality in (8),

$$(1+|b|)\left(\frac{1-|b|}{|z-b|}\right)^2 \le \left(\frac{1+s}{1-s}\right)^2 (1+|a|)\left(\frac{1-|a|}{|z-a|}\right)^2.$$

Consequently,

$$\frac{1-|b|}{|z-b|} \le \frac{\sqrt{2}\,(1+s)}{1-s}\,\frac{1-|a|}{|z-a|}$$

whenever $z \in \mathbb{T}$ and $\rho(a,b) \le s$. ■

Every uniform Frostman Blaschke product is a finite product of interpolating Blaschke products [4, p. 140]. It is also known [12] that every Blaschke product that is a finite product of interpolating Blaschke products can be approximated by an interpolating Blaschke product. Therefore, every uniform Frostman Blaschke product can be approximated by an interpolating Blaschke product. If $C \in \mathcal{UFB}$ and $B$ is an interpolating Blaschke product that approximates $C$, is $B$ in $\mathcal{UFB}$? The answer is affirmative if the approximation is sufficiently close to $C$, which is the content of Theorem 4.4 below.

Our original proof of this theorem used maximal ideal space techniques. The referee provided us with a more elementary proof; one that depends only on elementary techniques and knowledge of the behavior of finite products of interpolating Blaschke products, which have been well studied (see [18], [22] or [9]). Such products have been characterized by, for example, Vasyunin [18, p. 217] who showed that a Blaschke product $B$ is a finite product of interpolating Blaschke products if and only if there is a constant $c > 0$ such that for all $z \in \mathbb{D}$,

$$|B(z)| \ge c \inf\{\rho(z,w) : B(w) = 0\}.$$

We also have the following well known lemma due to K. Hoffman.

LEMMA 4.2 ([7, Hoffman's Lemma 1.4, p. 395]). *Let $B$ be an interpolating Blaschke product with zeros $\{z_n\}$ satisfying*

$$(1-|z_n|^2)|B'(z_n)| \geq \delta > 0.$$

*Then there exist $\lambda := \lambda(\delta)$ and $r := r(\delta)$ both between $0$ and $1$ such that*

$$\lim_{\delta\to 1} \lambda(\delta) = 1 \quad \text{and} \quad \lim_{\delta\to 1} r(\delta) = 1$$

*and the set $\{z : |B(z)| < r\}$ is contained in disjoint domains $V_n$ with $z_n \in V_n \subset \{z : \rho(z, z_n) < \lambda\}$. Further, $B$ maps each $V_n$ univalently onto $\{w : |w| < r\}$. If $|w| < r$, then $\varphi_w \circ B$ is an interpolating Blaschke product with one zero in each $V_n$.*

Our next theorem relies on a similar property of such products, one with an elementary proof that we provide for completeness.

LEMMA 4.3. *For $1 \leq k \leq n$, let $\Lambda_k$ be an interpolating sequence in $\mathbb{D}$, let $\Lambda = \bigcup_{k=1}^n \Lambda_k$ and let $B$ be the Blaschke product whose (simple) zeros are the points in $\Lambda$. Then, for any $\varepsilon > 0$, there is an interpolating subsequence $\Omega$ of $\Lambda$ and a positive function $\delta$ defined on $\Omega$ that is bounded above by $\varepsilon$ and has at most $2^n - 1$ values on $\Omega$, such that:*

(i) *there is a constant $\sigma$, $0 < \sigma < 1$, such that $\rho(z, w) > \sigma$ whenever $z$ and $w$ are in distinct pseudohyperbolic disks of the form $D_{\delta(a)}(a) := \{z \in \mathbb{D} : \rho(z, a) < \delta(a)\}$, where $a \in \Omega$,*
(ii) *$D_{\delta(a)}(a)$ contains at most $n$ points of $\Lambda$ for any $a$ in $\Omega$, and*
(iii) *there is a positive lower bound for*

$$\Big\{\rho(a, z) : z \in \mathbb{D} \setminus \bigcup_{a\in\Omega} D_{\delta(a)}(a) \text{ and } a \in \Lambda\Big\}.$$

*Hence, there is a positive constant $\eta$ such that $|B(z)| > \eta$ whenever $z \in \mathbb{D} \setminus \bigcup_{a\in\Omega} D_{\delta(a)}(a)$.*

*Proof.* If $n = 1$ there is (essentially) nothing to show. We proceed via induction on $n$, assuming that the conclusion holds for some positive integer $n$. Let $\Lambda_k$ be an interpolating sequence in $\mathbb{D}$ for $1 \leq k \leq n+1$. Then there exists a constant $r$, $0 < r < 1$, such that $D_r(a) \cap D_r(a') = \emptyset$, whenever $a$ and $a'$ are distinct points in $\Lambda_k$ for any $k$ in the range $1 \leq k \leq n+1$. By our induction hypothesis, for any $\varepsilon$, $0 < \varepsilon < r/8$, there is an interpolating subsequence $\Omega$ of $\Lambda := \bigcup_{k=1}^n \Lambda_k$ and a positive function $\delta$ defined on $\Omega$ that is bounded above by $\varepsilon/3$ and has at most $2^n - 1$ values on $\Omega$, such that (i)–(iii) above hold. Let $t = \min\{\delta(a) : a \in \Omega\}$ and let $s = \min\{t, \sigma/8\}$. We partition $\Lambda_{n+1}$ into $V_{n+1} = \{a \in \Lambda_{n+1} :$ there exists $a'$ in $\Omega$ such that $D_s(a) \cap D_{\delta(a')}(a') \neq \emptyset\}$

and $W_{n+1} = \Lambda_{n+1} \setminus V_{n+1}$. Let $\Omega^* = \Omega \cup W_{n+1}$ and define $\delta^*$ on $\Omega^*$ by

$$\delta^*(a) = \begin{cases} s/2 & \text{if } a \in W_{n+1}, \\ \delta(a) & \text{if } a \in \Omega \text{ and } D_{\delta(a)}(a) \cap D_s(a') = \emptyset \text{ for all } a' \in V_{n+1}, \\ \delta(a) + 2s & \text{if } a \in \Omega \text{ and } \exists a' \in V_{n+1} \text{ with } D_{\delta(a)}(a) \cap D_s(a') \neq \emptyset. \end{cases}$$

Notice that $\Omega^*$ is an interpolating subequence of $\Lambda^* := \Lambda \cup \Lambda_{n+1}$, and, by our choice of $s$, $\delta^*$ is bounded above by $\varepsilon$ on $\Omega^*$ and satisfies conditions (i)–(iii) with $\Lambda^*$ in place of $\Lambda$, $n+1$ in place of $n$ and smaller specified positive constants. ■

We thank the referee for suggesting the previous lemma to us as well as the following simplification of our original proof of Theorem 4.4 below.

THEOREM 4.4. *The set of uniform Frostman Blaschke products is open in the set of inner functions in $H^\infty$ with the uniform norm.*

*Proof.* Let $B \in \mathcal{UFB}$. Since $B$ is a finite product of interpolating Blaschke products we apply Lemma 4.3 to conclude that there is a subsequence $\{z_n\}$ of the zeros of $B$ that is interpolating and constants $\delta > 0$ and $\eta > 0$ such that $|B(z)| > \eta$ if $\inf_n \rho(z, z_n) > \delta$. Moreover, we may assume that the pseudohyperbolic disks $D_n := D_\delta(z_n)$ centered at $z_n$ and of radius $\delta$ are pairwise disjoint.

Assume that $B^\star$ is an inner function with $\|B - B^\star\|_\infty < \eta$. We claim that $B^\star \in \mathcal{UFB}$.

Since $B \in \mathcal{UFB}$, we know that $B$ has radial limits of modulus one at each point of the unit circle. Therefore $B^\star$ cannot have radial limit zero at any point and we conclude that $B^\star$ is a Blaschke product.

Further, $B^\star$ can only have zeros in $\bigcup_n D_n$. By Rouché's theorem, the number of zeros of $B^\star$ in each $D_n$ is equal to the number of zeros of $B$ in $D_n$ and is therefore uniformly bounded.

Now, the result follows from Lemma 4.1, or one may note that if $w, w^\star$ are in $D_\eta$, then $1-|w|$ and $1-|w^\star|$ are comparable. Also, for any $\lambda \in \mathbb{T}$, we know that $|\lambda - w|$ and $|\lambda - w^\star|$ are comparable. Since $B$ is uniform Frostman, so is $B^\star$. ■

**5. Open questions.** We end this paper with two questions. The first one is about certain closed subalgebras of the algebra $L^\infty$ of essentially bounded measurable functions on $\mathbb{T}$. We begin with some simple remarks: Write $A = H^\infty[\overline{B}]$ for the closed subalgebra of $L^\infty$ generated by $\overline{B}$, the complex conjugate of the Blaschke product $B$, and $H^\infty$. Note that saying that a Blaschke product $C$ is invertible in $A$ is equivalent to saying that the conjugate, $\overline{C}$, is in $A$. Hence, since $\overline{B} \in A$, if we write $B = B_1 B_2$ then $\overline{B_1} = B_2 \overline{B} \in A$. So all subproducts of $B$ are invertible in the algebra $A$. With these preliminaries behind us, we turn to the question we wish to pose.

Consider $A_0 = H^\infty[\overline{B}]$, where $B$ is a thin Blaschke product. It follows from Hedenmalm's work that every Blaschke product that is invertible in $A_0$ is a finite product of thin products. Now consider the algebra $A_1 = H^\infty[\overline{B}]$ where $B$ is a Blaschke product that has finite angular derivative at each point of $\mathbb{T}$. The same is true here [6]: Every Blaschke product that is invertible in $A_1$ has finite angular derivative at every point of $\mathbb{T}$, a fact that is even true locally. Now, since a Blaschke product with zeros $\{z_n\}$ has finite angular derivative at $\lambda \in \mathbb{T}$ if and only if $\sum_n \frac{1-|z_n|}{|\lambda-\overline{z_n}|^2} < \infty$ [5], it is natural to pose the following question: Letting $A_2 = H^\infty[\overline{B}]$ where $B \in \mathcal{UFB}$, is every Blaschke product invertible in $A_2$ in the class $\mathcal{UFB}$? An example due to W. Rudin [20] plus a theorem of Cargo [2] show that there is a Blaschke product such that $\sum_n \frac{1-|z_n|}{|z-\overline{z_n}|} = \infty$ at almost every point $z \in \mathbb{T}$. Since Blaschke products have radial limits of modulus one at almost every point of $\mathbb{T}$, this example plus Frostman's theorem [5, Theorem 1] imply that a Blaschke product can have radial limit of modulus one at a point, while some subproduct does not. Therefore, letting $B_0$ denote this Blaschke product, we see that there is a Blaschke product invertible in $H^\infty[\overline{B_0}]$ that does not have a radial limit at some point where $B_0$ does have a radial limit.

The second question we pose is the following: Prime finite Blaschke products are uniformly dense in the set of finite Blaschke products (see also [9] and [3]). All examples so far suggest that the same is true for infinite Blaschke products. Thus, we conjecture that the class of prime Blaschke products is uniformly dense in the set of all Blaschke products.

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