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On the infimum convolution inequality

by

R. Latała and J. O. Wojtaszczyk (Warszawa)

Abstract. We study the *infimum convolution* inequalities. Such an inequality was first introduced by B. Maurey to give the optimal concentration of measure behaviour for the product exponential measure. We show how IC inequalities are tied to concentration and study the optimal cost functions for an arbitrary probability measure μ. In particular, we prove an optimal IC inequality for product log-concave measures and for uniform measures on the ℓ_p^n balls. Such an optimal inequality implies, for a given measure, the central limit theorem of Klartag and the tail estimates of Paouris.

1. Introduction and notation. In the seminal paper [20], B. Maurey introduced the so called property (τ) for a probability measure μ with a cost function φ (see Definition 2.1 below) and established a very elegant and simple proof of Talagrand's two-level concentration for the product exponential distribution ν^n using (τ) for this distribution and an appropriate cost function w.

It is natural to ask what other pairs (μ, φ) have property (τ). As any μ satisfies (τ) with $\varphi \equiv 0$, one will rather ask how big a cost function one can take. In this paper we study the probability measures μ that have property (τ) with respect to the largest (up to a multiplicative factor) possible convex cost function $\Lambda_\mu^\star$. This bound comes from checking property (τ) for linear functions. We say a measure satisfies the *infimum convolution inequality* (IC for short) if the pair $(\mu, \Lambda_\mu^\star)$ satisfies (τ).

It turns out that such an optimal infimum convolution inequality has very strong consequences. It gives the best possible concentration behaviour, governed by the so-called L_p-centroid bodies (Corollary 3.11). This, in turn, implies in particular a weak-strong moment comparison (Proposition 3.15), the central limit theorem of Klartag [14] and the tail estimates of Paouris [23]

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(Proposition 3.18). We believe that IC holds for any log-concave probability measure, which is the main motivation for this paper.

Maurey's inequality for the exponential measure is of this optimal type. We transport this to any log-concave measure on the real line, and as the inequality tensorizes, any product log-concave measure satisfies IC (Corollary 2.19). However, the main challenge is to provide nonproduct examples of measures satisfying IC. We show how such an optimal result can be obtained from concentration inequalites, and follow on to prove IC for the uniform measure on any ℓ_p^n ball for $p \geq 1$ (Theorem 5.27).

With the techniques developed we also prove a few other results. We give a proof of the Gaussian-type isoperimetry for uniform measures on ℓ_p^n balls, where $p \geq 2$ (Theorem 5.29), and provide a new concentration inequality for the exponential measure for sets lying far away from the origin (Theorem 4.6).

Organization of the paper. This section, apart from the above introduction, defines the notation used throughout the paper. The second section is devoted to studying the general properties of the inequality IC. In Subsection 2.1 we recall the definition of property (τ) and its ties to concentration from [20]. In Subsection 2.2 we study the opposite implication: what additional assumptions one needs to infer (τ) from concentration inequalities. In Subsection 2.3 we show that $\Lambda_\mu^\star$ is indeed the largest possible cost function and define the inequality IC. In Subsection 2.4 we show that product log-concave measures satisfy IC.

In the third section we give more attention to the concentration inequalities tied to IC. In Subsection 3.1 we show the connection to $\mathcal{Z}_p$ bodies. In Subsection 3.2 we continue in this vein with the additional assumption that our measure is α-regular. In Subsection 3.3 we show how IC implies a comparison of weak and strong moments and the results of [14] and [23].

In the fourth section we give a modification of the two-level concentration for the exponential measure, in which for sets lying far away from the origin only an enlargement by tB_1^n is used. This will be used in the fifth section, which focuses on the uniform measure on the B_p^n ball. In Subsection 5.1 we define and study two rather standard transports of measure used further on. In Subsection 5.2 we use these transports along with the concentration from Section 4 and a Cheeger inequality from [24] to give a proof of IC for $p \leq 2$. In Subsection 5.3 we prove IC for $p \geq 2$ and the Gaussian-type isoperimetric inequality for $p \geq 2$.

We conclude with a few possible extensions of the results of the paper in the sixth section.

Notation. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we put $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|x\|_\infty = \max_i |x_i|$, we also use $|x|$ for $\|x\|_2$. We write B_p^n for the unit ball in l_p^n, i.e. $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$.

We let ν denote the symmetric exponential distribution on $\mathbb{R}$, i.e. the probability measure with density $\frac{1}{2}\exp(-|x|)$. For $p \ge 1$, ν_p is the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1}\exp(-|x|^p)$, where $\gamma_p = \Gamma(1+1/p)$, in particular $\nu_1 = \nu$. For a probability measure μ we write μ^n for the product measure $\mu^{\otimes n}$, thus ν_p^n has the density $(2\gamma_p)^{-n}\exp(-\|x\|_p^p)$.

$\mathcal{B}(\mathbb{R}^n)$ will denote the family of Borel sets on $\mathbb{R}^n$. The Lebesgue measure of $A \in \mathcal{B}(\mathbb{R}^n)$ is denoted by $|A|$ or $\Lambda_n(A)$. We choose numbers $r_{p,n}$ in such a way that $|r_{p,n}B_p^n| = 1$ and denote by $\mu_{p,n}$ the uniform distribution on B_p^n. The median of a function f with respect to a probability measure μ will be denoted by $\mathrm{Med}_\mu f$.

The letters c, C denote absolute numerical constants, which may change from line to line; $c(p), C(p)$ stand for constants dependent on p (or, formally, a family of absolute constants indexed by p); these may also change from line to line. For any sets of positive real numbers a_i and b_i, $i \in I$, by $a_i \sim b_i$ we indicate that there exist absolute numerical constants $c, C > 0$ such that $ca_i < b_i < Ca_i$ for any $i \in I$. Similarly, for collections of sets A_i and B_i the notation $A_i \sim B_i$ means $cA_i \subset B_i \subset CA_i$ for any $i \in I$, where again $c, C > 0$ are absolute numerical constants. By writing $\sim_p$ we mean that the constants above may depend on p.

2. Infimum convolution inequality

2.1. *Property* (τ). The following property was introduced by B. Maurey [20]:

DEFINITION 2.1. Let μ be a probability measure on $\mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \to [0,\infty]$ be a measurable function. We say that the pair (μ,φ) has *property* (τ) if for any bounded measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} e^{f\square\varphi}\,d\mu \int_{\mathbb{R}^n} e^{-f}\,d\mu \le 1, \tag{1}$$

where for two functions f and g on $\mathbb{R}^n$,

$$f \square g(x) := \inf\{f(x-y) + g(y) : y \in \mathbb{R}^n\}$$

denotes the *infimum convolution* of f and g.

The following two easy observations are almost immediate (cf. [20]):

PROPOSITION 2.2 (Tensorization). *If pairs* (μ_i,φ_i), $i = 1,\dots,k$, *have property* (τ) *and* $\varphi(x_1,\dots,x_k) = \varphi_1(x_1) + \dots + \varphi_k(x_k)$, *then the couple* $(\bigotimes_{i=1}^k \mu_i, \varphi)$ *also has property* (τ).

Proposition 2.3 (Transport of measure). *Suppose that* (μ, φ) *has property* (τ) *and* $T\colon \mathbb{R}^n \to \mathbb{R}^m$ *is such that*

$$\psi(Tx - Ty) \le \varphi(x-y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Then the pair $(\mu \circ T^{-1}, \psi)$ *has property* (τ).

Maurey noticed that property (τ) implies $\mu(A + B_\varphi(t)) \ge 1 - \mu(A)^{-1}e^{-t}$, where

$$B_\varphi(t) := \{x \in \mathbb{R}^n \colon \varphi(x) \le t\}.$$

We will need a slight modification of this estimate.

Proposition 2.4. *Property* (τ) *for* (φ, μ) *implies that for any Borel set* A *and* $t \ge 0$,

$$\mu(A + B_\varphi(t)) \ge \frac{e^t \mu(A)}{(e^t - 1)\mu(A) + 1}. \tag{2}$$

In particular, *for all* $t > 0$,

$$\mu(A) > 0 \ \Rightarrow\ \mu(A + B_\varphi(t)) > \min\{e^{t/2}\mu(A), 1/2\}, \tag{3}$$

$$\mu(A) \ge 1/2 \ \Rightarrow\ 1 - \mu(A + B_\varphi(t)) < e^{-t/2}(1 - \mu(A)), \tag{4}$$

$$\mu(A) = \nu(-\infty, x] \ \Rightarrow\ \mu(A + B_\varphi(t)) \ge \nu(-\infty, x + t/2]. \tag{5}$$

Proof. Set $f(x) = t\mathbf{1}_{\mathbb{R}^n \setminus A}$. Then $f(x)$ is nonnegative on $\mathbb{R}^n$, so $f \,\square\, \varphi$ is nonnegative (recall that by definition we consider only nonnegative cost functions). For $x \notin A + B_\varphi(t)$ we have $f \,\square\, \varphi(x) = \inf_y (f(y) + \varphi(x-y)) \ge t$, since either $y \notin A$, and then $f(y) = t$, or $y \in A$, and then $\varphi(x-y) \ge t$ as $x \notin A + B_\varphi(t)$.

Thus from property (τ) for f we have

$$\begin{aligned} 1 &\ge \int e^{f \square \varphi(x)}\, d\mu(x) \int e^{-f(x)}\, d\mu(x) \\ &\ge [\mu(A + B_\varphi(t)) + e^t(1 - \mu(A + B_\varphi(t)))][\mu(A) + e^{-t}(1 - \mu(A))], \end{aligned}$$

from which, extracting the condition upon $\mu(A + B_\varphi(t))$ by direct calculation, we get (2).

Let $f_t(p) := e^t p/((e^t - 1)p + 1)$. Then f_t is increasing in p and for $p \le e^{-t/2}/2$,

$$(e^t - 1)p + 1 \le e^{t/2} + 1 - \tfrac{1}{2}(e^{t/2} + e^{-t/2}) < e^{t/2},$$

hence $f_t(p) > \min(e^{t/2}p, 1/2)$ and (3) follows. Moreover for $p \ge 1/2$,

$$1 - f_t(p) = \frac{1-p}{(e^t - 1)p + 1} \le \frac{1-p}{(e^t + 1)/2} < e^{-t/2}(1-p)$$

and we get (4).

Let $F(x) = \nu(-\infty, x]$ and $g_t(p) = F(F^{-1}(p) + t)$. Previous calculations show that for $t, p > 0$, $f_t(p) \ge g_{t/2}(p)$ if $F^{-1}(p) + t/2 \le 0$ or $F^{-1}(p) \ge 0$.

Since $g_{t+s} = g_t \circ g_s$ and $f_{t+s} = f_t \circ f_s$, we see that $f_t(p) \geq g_{t/2}(p)$ for all $t, p > 0$, hence (2) implies (5). ■

The main theorem of [20] states that ν satisfies (τ) with a suitably chosen cost function.

THEOREM 2.5. *Let $w(x) = \frac{1}{36}x^2$ for $|x| \leq 4$ and $w(x) = \frac{2}{9}(|x| - 2)$ otherwise. Then the pair $(\nu^n, \sum_{i=1}^n w(x_i))$ has property (τ).*

Theorem 2.5 together with Proposition 2.4 immediately gives the following two-level concentration:

$$\nu^n(A) = \nu(-\infty, x] \ \Rightarrow\ \forall_{t\geq 0}\ \nu^n(A + 6\sqrt{2t}\, B_2^n + 18tB_1^n) \geq \nu(-\infty, x+t], \tag{6}$$

first established (with different universal, rather large constants) by Talagrand [26].

2.2. *From concentration to property (τ).* Proposition 2.4 shows that property (τ) implies concentration. We will show a few results in the opposite direction: how to recover (τ) from concentration.

COROLLARY 2.6. *Suppose that the cost function φ is radius-wise nondecreasing, μ is a Borel probability measure on $\mathbb{R}^n$, and $\beta > 0$ is such that for any $t > 0$ and $A \in \mathcal{B}(\mathbb{R}^n)$,*

$$\mu(A) = \nu(-\infty, x] \ \Rightarrow\ \mu(A + \beta B_\varphi(t)) \geq \nu(-\infty, x + \max\{t, \sqrt{t}\}]. \tag{7}$$

Then the pair $\big(\mu, \frac{1}{36}\varphi(\frac{\cdot}{\beta})\big)$ has property (τ). In particular if φ is convex, symmetric and $\varphi(0) = 0$ then (7) implies property (τ) for $\big(\mu, \varphi(\frac{\cdot}{36\beta})\big)$.

Proof. Fix $f\colon \mathbb{R}^n \to \mathbb{R}$. For any measurable function h on $\mathbb{R}^k$ and $t \in \mathbb{R}$ put

$$A(h, t) := \{x \in \mathbb{R}^k \colon h(x) < t\}.$$

Let g be a nondecreasing right-continuous function on $\mathbb{R}$ such that $\mu(A(f,t)) = \nu(A(g,t))$. Then the distribution of g with respect to ν is the same as the distribution of f with respect to μ and thus

$$\int_{\mathbb{R}^n} e^{-f(x)}\, d\mu(x) = \int_{\mathbb{R}} e^{-g(x)}\, d\nu(x).$$

To finish the proof of the first assertion, by Theorem 2.5 it is enough to show that

$$\int_{\mathbb{R}^n} e^{f \Box \frac{1}{36}\varphi(\frac{\cdot}{\beta})}\, d\mu \leq \int_{\mathbb{R}} e^{g \Box w}\, d\nu,$$

where w is as in Theorem 2.5. We will establish a stronger property:

$$\forall_u \quad \mu\bigg(A\bigg(f \Box \frac{1}{36}\varphi\Big(\frac{\cdot}{\beta}\Big), u\bigg)\bigg) \geq \nu(A(g \Box w, u)).$$

Since the set $A(g \,\square\, w, u)$ is a halfline, it is enough to prove that

$$g(x_1) + w(x_2) < u \;\Rightarrow\; \mu\biggl(A\biggl(f \,\square\, \frac{1}{36}\,\varphi\Bigl(\frac{\cdot}{\beta}\Bigr), u\biggr)\biggr) \ge \nu(-\infty, x_1 + x_2]. \tag{8}$$

Fix x_1 and x_2 with $g(x_1) + w(x_2) < u$ and take $s_1 > g(x_1)$ and $s_2 = w(x_2)$ with $s_1 + s_2 < u$. Put $A := A(f, s_1)$. Then $\mu(A) = \nu(A(g, s_1)) \ge \nu(-\infty, x_1]$. By the definition of w it easily follows that $x_2 \le \max\{6\sqrt{s_2}, 9s_2\}$, hence by (7), $\mu(A + \beta B_\varphi(36s_2)) \ge \nu(-\infty, x_1 + x_2]$. Since

$$A + \beta B_\varphi(36s_2) = A(f, s_1) + B_{\varphi(\frac{\cdot}{\beta})/36}(s_2) \subset A\biggl(f \,\square\, \frac{1}{36}\,\varphi\Bigl(\frac{\cdot}{\beta}\Bigr), s_1 + s_2\biggr),$$

we obtain the property (8).

The last part of the statement immediately follows since any symmetric convex function φ is radius-wise nondecreasing and if additionally $\varphi(0) = 0$, then $\varphi(x/36) \le \varphi(x)/36$ for any x. ∎

The next proposition shows that inequalities (3) and (4) are strongly related.

Proposition 2.7. *The following two conditions are equivalent for any Borel set K and $\gamma > 1$:*

$$\forall_{A \in \mathcal{B}(\mathbb{R}^n)} \quad \mu(A) > 0 \;\Rightarrow\; \mu(A + K) > \min\{\gamma\mu(A), 1/2\}, \tag{9}$$

$$\forall_{\widetilde{A} \in \mathcal{B}(\mathbb{R}^n)} \quad \mu(\widetilde{A}) \ge 1/2 \;\Rightarrow\; 1 - \mu(\widetilde{A} - K) < \frac{1}{\gamma}\,(1 - \mu(\widetilde{A})). \tag{10}$$

Proof. (9)$\Rightarrow$(10). Suppose $\mu(\widetilde{A}) \ge 1/2$ and $1 - \mu(\widetilde{A} - K) \ge \gamma^{-1}(1 - \mu(\widetilde{A}))$. Let $A := \mathbb{R}^n \setminus (\widetilde{A} - K)$. Then $(A + K) \cap \widetilde{A} = \emptyset$, so $\mu(A + K) \le 1/2$ and

$$\mu(A + K) \le 1 - \mu(\widetilde{A}) \le \gamma(1 - \mu(\widetilde{A} - K)) = \gamma\mu(A),$$

and this contradicts (9).

(10)$\Rightarrow$(9). Fix $A \subset \mathbb{R}^n$ with $\mu(A) > 0$ and $\mu(A + K) \le \min\{\gamma\mu(A), 1/2\}$. Let $\widetilde{A} := \mathbb{R}^n \setminus (A + K)$. Then $\mu(\widetilde{A}) \ge 1/2$. Moreover, $(\widetilde{A} - K) \cap A = \emptyset$, thus

$$1 - \mu(\widetilde{A} - K) \ge \mu(A) \ge \frac{1}{\gamma}\,\mu(A + K) = \frac{1}{\gamma}\,(1 - \mu(\widetilde{A})),$$

contradicting (10). ∎

Corollary 2.8. *Suppose that $t > 0$ and K is a symmetric convex set in $\mathbb{R}^n$ such that*

$$\forall_{A \in \mathcal{B}(\mathbb{R}^n)} \quad \mu(A) > 0 \;\Rightarrow\; \mu(A + K) > \min\{e^t\mu(A), 1/2\}.$$

Then for any Borel set A,

$$\mu(A) = \nu(-\infty, x] \;\Rightarrow\; \mu(A + 2K) > \nu(-\infty, x + t].$$

Proof. Fix A with $\mu(A) = \nu(-\infty, x]$. Notice that $A + 2K = A + K + K \supset A + K$. If $x + t \le 0$, then $\mu(A + K) > e^t \mu(A) = \nu(-\infty, x + t]$. If $x \ge 0$, Proposition 2.7 gives

$$\mu(A + K) > 1 - e^{-t}(1 - \mu(A)) = \nu(-\infty, x + t].$$

Finally, if $x \le 0 \le x + t$, we get $\mu(A + K) \ge 1/2 = \nu(-\infty, 0]$, hence by the previous case,

$$\mu(A + 2K) = \mu((A + K) + K) > \nu(-\infty, t] \ge \nu(-\infty, x + t]. \ \blacksquare$$

Corollary 2.8 shows that if the cost function φ is symmetric and convex, condition (7) (with $\beta = 2\gamma$) for $t \ge 1$ is implied by the following:

$$(11) \qquad \forall_{A \in \mathcal{B}(\mathbb{R}^n)} \quad \mu(A) > 0 \ \Rightarrow \ \mu(A + \gamma B_\varphi(t)) > \min\{e^t \mu(A), 1/2\}.$$

To treat the case $t \le 1$ we will need Cheeger's version of the Poincaré inequality.

We say that a probability measure μ on $\mathbb{R}^n$ satisfies *Cheeger's inequality* with constant κ if for any Borel set A,

$$(12) \qquad \mu^+(A) := \liminf_{t \to 0+} \frac{\mu(A + tB_2^n) - \mu(A)}{t} \ge \kappa \min\{\mu(A), 1 - \mu(A)\}.$$

It is not hard to check (cf. [7, Theorem 2.1]) that Cheeger's inequality implies

$$\mu(A) = \nu(-\infty, x] \ \Rightarrow \ \mu(A + tB_2^n) \ge \nu(-\infty, x + \kappa t].$$

Finally, we may summarize this section with the following statement.

PROPOSITION 2.9. *Suppose that the cost function φ is convex, symmetric with $\varphi(0) = 0$ and $1 \wedge \varphi(x) \le (\alpha|x|)^2$ for all x. If the measure μ satisfies Cheeger's inequality* (12) *and the condition* (11) *is satisfied for all $t \ge 1$ then $(\mu, \varphi(\cdot/C))$ has property (τ) with the constant $C = 36 \max\{2\gamma, \alpha/\kappa\}$.*

Proof. Notice that $\alpha B_\varphi(t) \supset \sqrt{t}\, B_2^n$ for all $t < 1$, hence Cheeger's inequality implies that condition (7) holds for $t < 1$ with $\beta = \alpha/\kappa$. Therefore (7) holds for all $t \ge 0$ with $\beta = \max\{2\gamma, \alpha/\kappa\}$ and the assertion follows by Corollary 2.6. $\blacksquare$

2.3. *Optimal cost functions.* A natural question arises: what other pairs (μ, φ) have property (τ)? First we have to choose the right cost function. To do this let us recall the following definitions.

DEFINITION 2.10. Let $f : \mathbb{R}^n \to (-\infty, \infty]$. The *Legendre transform* of f, denoted $\mathcal{L}f$, is defined by $\mathcal{L}f(x) := \sup_{y \in \mathbb{R}^n}\{\langle x, y\rangle - f(y)\}$.

The Legendre transform of any function is a convex function. If f is convex and lower semicontinuous, then $\mathcal{L}\mathcal{L}f = f$, and otherwise $\mathcal{L}\mathcal{L}f \le f$. In general, if $f \ge g$, then $\mathcal{L}f \le \mathcal{L}g$. The Legendre transform satisfies $\mathcal{L}(Cf)(x) = C\mathcal{L}f(x/C)$, and if $g(x) = f(x/C)$, then $\mathcal{L}g(x) = \mathcal{L}f(Cx)$. For these and other properties of $\mathcal{L}$, see [19]. The Legendre transform has been previously used in the context of convex geometry (see for instance [2] and [15]).

Definition 2.11. Let μ be a probability measure on $\mathbb{R}^n$. We define

$$M_\mu(v) := \int_{\mathbb{R}^n} e^{\langle v,x\rangle} d\mu(x), \qquad \Lambda_\mu(v) := \log M_\mu(v)$$

and

$$\Lambda_\mu^\star(v) := \mathcal{L}\Lambda_\mu(v) = \sup_{u\in\mathbb{R}^n} \Big\{\langle v,u\rangle - \ln \int_{\mathbb{R}^n} e^{\langle u,x\rangle}\, d\mu(x)\Big\}.$$

The function $\Lambda_\mu^\star$ plays a crucial role in the theory of large deviations (cf. [10]).

It is a common phenomenon in many places of the theory that the "worst" (in some sense) functions are linear functionals. Thus it is worth checking what happens when we take f in the definition of property (τ) to be a linear functional. This approach is at the heart of the following results.

Remark 2.12. *Let μ be a symmetric probability measure on $\mathbb{R}^n$ and let φ be a convex cost function such that (μ,φ) has property (τ). Then*

$$\varphi(v) \le 2\Lambda_\mu^\star(v/2) \le \Lambda_\mu^\star(v).$$

Proof. Set $f(x) = \langle x,v\rangle$. Then

$$f \,\Box\, \varphi(x) = \inf_y\{f(x-y)+\varphi(y)\} = \inf_y\{\langle x-y,v\rangle + \varphi(y)\} = \langle x,v\rangle - \mathcal{L}\varphi(v).$$

Property (τ) yields

$$1 \ge \int e^{f\Box\varphi}\, d\mu \int e^{-f}\, d\mu = e^{-\mathcal{L}\varphi(v)}\int e^{\langle x,v\rangle}\, d\mu \int e^{-\langle x,v\rangle}\, d\mu = e^{-\mathcal{L}\varphi(v)} M_\mu^2(v),$$

where the last equality uses the fact that μ is symmetric. Thus by taking the logarithm we get $\mathcal{L}\varphi(v) \ge 2\Lambda_\mu(v)$, and by applying the Legendre transform we obtain $\varphi(v) = \mathcal{L}\mathcal{L}\varphi(v) \le 2\Lambda_\mu^\star(v/2)$. The inequality $2\Lambda_\mu^\star(v/2) \le \Lambda_\mu^\star(v)$ follows by the convexity of $\Lambda_\mu^\star$. ∎

The above remark motivates the following definition.

Definition 2.13. We say that a symmetric probability measure μ satisfies the *infimum convolution inequality with constant* β (IC(β) for short) if the pair $(\mu, \Lambda_\mu^\star(\cdot/\beta))$ has property (τ).

Tensorization properties of (τ) and additive properties of $\Lambda_\mu^\star$ imply the tensorization of the IC inequality:

Proposition 2.14. *If μ_i are symmetric probability measures on $\mathbb{R}^{n_i}$, $1 \le i \le k$, satisfying* IC(β_i), *then $\mu = \bigotimes_{i=1}^k \mu_i$ satisfies* IC(β) *with $\beta = \max_i \beta_i$.*

Proof. By independence, we have $\Lambda_\mu(x_1,\dots,x_k) = \sum_{i=1}^k \Lambda_{\mu_i}(x_i)$ and $\Lambda_\mu^\star(x_1,\dots,x_k) = \sum_{i=1}^k \Lambda_{\mu_i}^\star(x_i)$. Since IC($\beta$) implies IC($\beta'$) with any $\beta' \ge \beta$, the result immediately follows by Proposition 2.2. ∎

In the next proposition we give an equivalent form of property IC.

PROPOSITION 2.15. *For $v = (v_0, v_1, \ldots, v_n)$ in $\mathbb{R}^{n+1}$ let $\widetilde{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$. A probability measure μ on $\mathbb{R}^n$ satisfies* IC(β) *if and only if for any nonempty $V \subset \mathbb{R}^{n+1}$ and any bounded measurable function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} e^{f \Box \psi_V} \, d\mu \int_{\mathbb{R}^n} e^{-f} \, d\mu \le \sup_{v \in V} \Big(e^{v_0} \int_{\mathbb{R}^n} e^{\beta \langle x, \widetilde{v} \rangle} \, d\mu(x) \Big), \tag{13}$$

where

$$\psi_V(x) := \sup_{v \in V} \{ v_0 + \langle x, \widetilde{v} \rangle \}.$$

Proof. If we put $V = \{(v_0, \widetilde{v}) \colon v_0 = -\Lambda_\mu(\beta \widetilde{v})\}$, then the right-hand side of (13) is equal to 1 and $\psi_V(x) = \Lambda_\mu^\star(x/\beta)$, so if μ satisfies (13) for this V, it satisfies IC(β).

Conversely, suppose μ satisfies IC(β). Take an arbitrary nonempty set V. If the right-hand side supremum is infinite, the inequality is obvious, so we may assume it is equal to some $s < \infty$. This means that for any $(v_0, \widetilde{v}) \in V$ we have $v_0 + \Lambda_\mu(\beta \widetilde{v}) \le \log s$, that is, $v_0 \le \log s - \Lambda_\mu(\beta \widetilde{v})$. Thus

$$\begin{aligned}\psi_V(x) = \sup_{v \in V} \{ v_0 + \langle x, \widetilde{v} \rangle \} &\le \log s + \sup_{v \in V} \{ \langle x, \widetilde{v} \rangle - \Lambda_\mu(\beta \widetilde{v}) \} \\ &\le \log s + \sup_{\widetilde{v} \in \mathbb{R}^n} \{ \langle x, \widetilde{v} \rangle - \Lambda_\mu(\beta \widetilde{v}) \} = \log s + \Lambda_\mu^\star(x/\beta),\end{aligned}$$

which in turn means from IC(β) that the left-hand side is no larger than s. ∎

The previous proposition easily implies that property IC is invariant under linear transformations.

PROPOSITION 2.16. *Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map and suppose that a probability measure μ on $\mathbb{R}^n$ satisfies* IC(β). *Then the probability measure $\mu \circ L^{-1}$ satisfies* IC(β).

Proof. For any set $V \subset \mathbb{R} \times \mathbb{R}^k$ and any function $f \colon \mathbb{R}^k \to \mathbb{R}$ put $\bar{f}(x) := f(L(x))$ and $\bar{V} := \{(v_0, L^\star(\widetilde{v})) \colon (v_0, \widetilde{v}) \in V\}$, where $L^\star$ is the Hermitian conjugate of L. Then direct calculation shows $\psi_V(L(x)) = \psi_{\bar{V}}(x)$ and $f \Box \psi_V(L(x)) \le \bar{f} \Box \psi_{\bar{V}}(x)$, thus

$$\int_{\mathbb{R}^k} e^{f \Box \psi_V} \, d(\mu \circ L^{-1}) \le \int_{\mathbb{R}^n} e^{\bar{f} \Box \psi_{\bar{V}}} \, d\mu$$

and

$$\int_{\mathbb{R}^k} e^{-f} \, d(\mu \circ L^{-1}) = \int_{\mathbb{R}^n} e^{-\bar{f}} \, d\mu$$

and finally

$$\sup_{v \in V} \Big\{ e^{v_0} \int_{\mathbb{R}^k} e^{\beta \langle x, \widetilde{v} \rangle} \, d(\mu \circ L^{-1}) \Big\} = \sup_{v \in \bar{V}} \Big\{ e^{v_0} \int_{\mathbb{R}^n} e^{\beta \langle x, \widetilde{v} \rangle} \, d\mu \Big\},$$

which substituted into (13) gives the conclusion. ∎

Proposition 2.17. *For any* $x \in \mathbb{R}$,

$$\tfrac{1}{5}\min(x^2,|x|) \le \Lambda_\nu^\star(x) \le \min(x^2,|x|),$$

in particular the measure ν *satisfies* IC(9).

Proof. Direct calculation shows that $\Lambda_\nu(x) = -\ln(1-x^2)$ for $|x|<1$ and

$$\Lambda_\nu^\star(x) = \sqrt{1+x^2} - 1 - \ln\biggl(\frac{\sqrt{1+x^2}+1}{2}\biggr).$$

Since $a/2 \le a - \ln(1+a/2) \le a$ for $a \ge 0$, we get $\frac{1}{2}(\sqrt{1+x^2}-1) \le \Lambda_\nu^\star(x) \le \sqrt{1+x^2}-1$. Finally,

$$\min(x,|x|^2) \ge \sqrt{1+x^2}-1 = \frac{x^2}{\sqrt{1+x^2}+1} \ge \frac{1}{\sqrt{2}+1}\min(|x|,x^2).$$

The last statement follows from Theorem 2.5, since $\min((x/9)^2,|x|/9) \le w(x)$. ■

2.4. *Logarithmically concave product measures.* A measure μ on $\mathbb{R}^n$ is *logarithmically concave* (log-concave for short) if for all nonempty compact sets A, B and $t \in [0,1]$,

$$\mu(tA+(1-t)B) \ge \mu(A)^t\mu(B)^{1-t}.$$

By Borell's theorem [8] a measure μ on $\mathbb{R}^n$ with a full-dimensional support is logarithmically concave if and only if it is absolutely continuous with respect to the Lebesgue measure and has a logarithmically concave density, i.e. $d\mu(x) = e^{h(x)}dx$ for some concave function $h\colon \mathbb{R}^n \to [-\infty,\infty)$.

Note that if μ is a symmetric probability measure on $\mathbb{R}^n$, then both Λ_μ and $\Lambda_\mu^\star$ are convex and symmetric, and $\Lambda_\mu(0) = \Lambda_\mu^\star(0) = 0$.

Recall also that a probability measure μ on $\mathbb{R}^n$ is called *isotropic* if

$$\int \langle u,x\rangle\, d\mu(x) = 0 \quad \text{and} \quad \int \langle u,x\rangle^2\, d\mu(x) = |u|^2 \quad \text{for all } u \in \mathbb{R}^n.$$

It is easy to check that for any measure μ with a full-dimensional support there exists a linear map L such that $\mu \circ L^{-1}$ is isotropic.

The next theorem (with a different universal, but rather large constant) may be deduced from the results of Gozlan [11]. We give the following, relatively short proof for the sake of completeness.

Theorem 2.18. *Any symmetric log-concave measure on* $\mathbb{R}$ *satisfies* IC(48).

Proof. Let μ be a symmetric log-concave probability measure on $\mathbb{R}$. We may assume that μ is isotropic by Proposition 2.16. Denote the density of μ by $g(x)$ and let the tail function be $\mu[x,\infty) = e^{-h(x)}$. By the Hensley inequality [12] we obtain

$$g(0) = g(0)\Big(\int_{\mathbb{R}} x^2 g(x)\,dx\Big)^{1/2} \geq \frac{1}{2\sqrt{3}} \geq \frac{1}{8}.$$

Let $T\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\nu(-\infty, x) = \mu(-\infty, Tx)$. Then $\mu = \nu \circ T^{-1}$, and T is odd and concave on $[0,\infty)$. In particular, $|Tx - Ty| \leq 2|T(x-y)|$ for all $x, y \in \mathbb{R}$.

Notice that $T'(0) = 1/(2g(0)) \leq 4$, thus by concavity of T, $Tx \leq 4x$ for $x \geq 0$. Moreover, for $x \geq 0$, $h(Tx) = x + \ln 2$.

Define

$$\widetilde{h}(x) := \begin{cases} x^2 & \text{for } |x| \leq 2/3, \\ \max\{4/9, h(|x|)\} & \text{for } |x| > 2/3. \end{cases}$$

We claim that $(\mu, \widetilde{h}(\cdot/48))$ has property (τ). Notice that $\widetilde{h}((Tx-Ty)/48) \leq \widetilde{h}(T(|x-y|)/24)$ so by Proposition 2.3 it is enough to check that

$$\widetilde{h}\Big(\frac{Tx}{24}\Big) \leq w(x) \quad \text{for } x \geq 0, \tag{14}$$

where $w(x)$ is as in Theorem 2.5. We have two cases.

(i) $Tx \leq 16$; then

$$\widetilde{h}\Big(\frac{Tx}{24}\Big) = \Big(\frac{Tx}{24}\Big)^2 \leq \min\Big\{\frac{4}{9}, \Big(\frac{x}{6}\Big)^2\Big\} \leq w(x).$$

(ii) $Tx \geq 16$; then $x \geq 4$ and

$$\begin{aligned} \widetilde{h}\Big(\frac{Tx}{24}\Big) &= \max\Big\{\frac{4}{9}, h\Big(\frac{Tx}{24}\Big)\Big\} \leq \max\Big\{\frac{4}{9}, \frac{h(Tx)}{24}\Big\} \\ &= \max\Big\{\frac{4}{9}, \frac{x+\ln 2}{24}\Big\} \leq \frac{x}{9} \leq w(x). \end{aligned}$$

So (14) holds in both cases.

To conclude we need to show that $\Lambda_\mu^\star(x) \leq \widetilde{h}(x)$. For $|x| \leq 2/3$ this follows from the more general Proposition 3.3 below. Notice that for any $t, x \geq 0$, $\Lambda_\mu(t) \geq tx + \ln \mu[x,\infty) = tx - h(x)$, hence

$$\Lambda_\mu^\star(x) = \Lambda_\mu^\star(|x|) = \sup_{t \geq 0}\{t|x| - \Lambda_\mu(t)\} \leq h(|x|) \leq \widetilde{h}(x)$$

for $|x| > 2/3$. ∎

Using Proposition 2.14 we get

COROLLARY 2.19. *Any symmetric, log-concave product probability measure on* $\mathbb{R}^n$ *satisfies* IC(48).

We expect that in fact a more general fact holds.

Conjecture 1. *Any symmetric log-concave probability measure satisfies* IC(C) *with a uniform constant* C.

3. Concentration inequalities. In this section we shall translate the concentration obtained from IC into an alternative form, which in particular will allow us to prove that IC implies several strong results, known by other means to be true for any log-concave measure.

3.1. *L_p-centroid bodies and related sets*

Definition 3.1. Let μ be a probability measure on $\mathbb{R}^n$. For $p \geq 1$ we define

$$\mathcal{M}_p(\mu) := \Big\{v \in \mathbb{R}^n \colon \int |\langle v, x\rangle|^p \, d\mu(x) \leq 1\Big\},$$

$$\mathcal{Z}_p(\mu) := (\mathcal{M}_p(\mu))^\circ = \Big\{x \in \mathbb{R}^n \colon |\langle v, x\rangle|^p \leq \int |\langle v, y\rangle|^p \, d\mu(y) \text{ for all } v \in \mathbb{R}^n\Big\}$$

and for $p > 0$ we put

$$B_p(\mu) := \{v \in \mathbb{R}^n \colon \Lambda_\mu^\star(v) \leq p\}.$$

The sets $\mathcal{Z}_p(\mu_K)$ for $p \geq 1$, when μ_K is the uniform distribution on the convex body K, are called the *L_p-centroid bodies* of K. They were introduced (under a different normalization) in [18]; their properties were also investigated in [23].

Proposition 3.2. *For any symmetric probability measure μ on $\mathbb{R}^n$ and $p \geq 1$,*

$$\mathcal{Z}_p(\mu) \subset 2^{1/p} e B_p(\mu).$$

Proof. Let $v \in \mathcal{Z}_p(\mu)$. We need to show that $\Lambda_\mu^\star(v/(2^{1/p}e)) \leq p$, that is,

$$\frac{\langle u, v\rangle}{2^{1/p}e} - \Lambda_\mu(u) \leq p \quad \text{ for all } u \in \mathbb{R}^n.$$

Fix $u \in \mathbb{R}^n$ with $\int |\langle u, x\rangle|^p \, d\mu(x) = \beta^p$. Then $u/\beta \in \mathcal{M}_p(\mu)$. We will consider two cases.

(i) $\beta \leq 2^{1/p}ep$. Then, since $\Lambda_\mu(u) \geq \int \langle u, x\rangle \, d\mu(x) = 0$,

$$\frac{\langle u, v\rangle}{2^{1/p}e} - \Lambda_\mu(u) \leq \frac{\beta}{2^{1/p}e}\Big\langle \frac{u}{\beta}, v\Big\rangle \leq p \cdot 1.$$

(ii) $\beta > 2^{1/p}ep$. We have

$$\begin{aligned}\int e^{\langle u, x\rangle} \, d\mu(x) &\geq \int |e^{\langle u, x\rangle/p}|^p I_{\{\langle u, x\rangle \geq 0\}} \, d\mu(x) \geq \int \Big|\frac{\langle u, x\rangle}{p}\Big|^p I_{\{\langle u, x\rangle \geq 0\}} \, d\mu(x) \\ &\geq \frac{1}{2}\int \Big|\frac{\langle u, x\rangle}{p}\Big|^p \, d\mu(x),\end{aligned}$$

thus

$$\int e^{2^{1/p}ep\langle u,x\rangle/\beta}\,d\mu(x) \ge \frac{1}{2}\int \left|\frac{2^{1/p}e\langle u,x\rangle}{\beta}\right|^p d\mu(x) = e^p.$$

Hence $\Lambda_\mu(2^{1/p}epu/\beta) \ge p$ and

$$\Lambda_\mu(u) \ge \frac{\beta}{2^{1/p}ep}\,\Lambda_\mu(2^{1/p}epu/\beta) \ge \frac{\beta}{2^{1/p}e}.$$

Therefore

$$\frac{\langle u,v\rangle}{2^{1/p}e} - \Lambda_\mu(u) \le \frac{\beta}{2^{1/p}e}\left\langle \frac{u}{\beta}, v\right\rangle - \frac{\beta}{2^{1/p}e} \le 0. \ ■$$

Proposition 3.3. *If μ is a symmetric, isotropic probability measure on $\mathbb{R}^n$, then* $\min\{1,\Lambda_\mu^\star(u)\} \le |u|^2$ *for all u, in particular*

$$\sqrt{p}\,B_2^n \subset B_p(\mu) \quad \text{for } p\in(0,1).$$

Proof. Using the symmetry and isotropy of μ, we get

$$\int e^{\langle u,x\rangle}\,d\mu(x) = 1+\sum_{k=1}^{\infty}\frac{1}{(2k)!}\int \langle u,x\rangle^{2k}\,d\mu(x) \ge 1+\sum_{k=1}^{\infty}\frac{|u|^{2k}}{(2k)!} = \cosh(|u|).$$

Hence for $|u|<1$,

$$\Lambda_\mu^\star(u) \le \mathcal{L}(\ln\cosh)(|u|) = \tfrac{1}{2}[(1+|u|)\ln(1+|u|)+(1-|u|)\ln(1-|u|)] \le |u|^2,$$

where to get the last inequality we used $\ln(1+x)\le x$ for $x>-1$. ■

3.2. *α-regular measures.* To establish inclusions opposite to those in the previous subsection, we introduce the following property:

Definition 3.4. We say that a measure μ on $\mathbb{R}^n$ is *α-regular* if for any $p\ge q\ge 2$ and $v\in\mathbb{R}^n$,

$$\Big(\int |\langle v,x\rangle|^p\,d\mu(x)\Big)^{1/p} \le \alpha\,\frac{p}{q}\Big(\int |\langle v,x\rangle|^q\,d\mu(x)\Big)^{1/q}.$$

Proposition 3.5. *If μ is α-regular for some $\alpha\ge 1$, then for any $p\ge 2$,*

$$B_p(\mu) \subset 4e\alpha\mathcal{Z}_p(\mu).$$

Proof. First we will show that

$$u\in\mathcal{M}_p(\mu) \ \Rightarrow\ \Lambda_\mu\Big(\frac{pu}{2e\alpha}\Big) \le p. \tag{15}$$

Indeed, if we fix $u\in\mathcal{M}_p(\mu)$ and put $\widetilde{u} := \frac{pu}{2e\alpha}$, then

$$\Big(\int |\langle \widetilde{u},x\rangle|^k\,d\mu(x)\Big)^{1/k} = \frac{p}{2e\alpha}\Big(\int |\langle u,x\rangle|^k\,d\mu(x)\Big)^{1/k} \le \begin{cases} \dfrac{p}{2e\alpha}, & k\le p,\\[2mm] \dfrac{k}{2e}, & k>p.\end{cases}$$

Hence

$$\int e^{\langle \widetilde{u},x\rangle}\,d\mu(x) \le \int e^{|\langle \widetilde{u},x\rangle|}\,d\mu(x) = \sum_{k=0}^{\infty}\frac{1}{k!}\int |\langle \widetilde{u},x\rangle|^k\,d\mu(x)$$

$$\le \sum_{k\le p}\frac{1}{k!}\left|\frac{p}{2e\alpha}\right|^k + \sum_{k>p}\frac{1}{k!}\left|\frac{k}{2e}\right|^k \le e^{\frac{p}{2e\alpha}} + 1 \le e^p$$

and (15) follows.

Now, for any $v \notin 4e\alpha\mathcal{Z}_p(\mu)$ we may find $u \in \mathcal{M}_p(\mu)$ such that $\langle v,u\rangle > 4e\alpha$ to obtain

$$\Lambda_\mu^\star(v) \ge \left\langle v, \frac{pu}{2e\alpha}\right\rangle - \Lambda_\mu\left(\frac{pu}{2e\alpha}\right) > \frac{p}{2e\alpha}4e\alpha - p = p. \ \blacksquare$$

Proposition 3.6. *If μ is symmetric, isotropic and α-regular for some $\alpha \ge 1$, then*

$$\Lambda_\mu^\star(u) \ge \min\left\{\frac{|u|}{2\alpha e}, \frac{|u|^2}{2\alpha^2e^2}\right\},$$

in particular

$$B_p(\mu) \subset \max\{2\alpha e p, \alpha e\sqrt{2p}\}B_2^n \quad \text{for all } p > 0.$$

Proof. By the symmetry, isotropy and regularity of μ, we have

$$\int e^{\langle u,x\rangle}\,d\mu(x) = \sum_{k=0}^{\infty}\frac{1}{(2k)!}\int \langle u,x\rangle^{2k}\,d\mu(x) \le 1 + \frac{|u|^2}{2} + \sum_{k=2}^{\infty}\frac{(\alpha k|u|)^{2k}}{(2k)!}$$

$$\le 1 + \frac{|u|^2}{2} + \sum_{k=2}^{\infty}\left(\frac{\alpha e|u|}{2}\right)^{2k}.$$

Hence if $\alpha e|u| \le 1$,

$$\int e^{\langle u,x\rangle}\,d\mu(x) \le 1 + \frac{|u|^2}{2} + \frac{4}{3}\left(\frac{\alpha e|u|}{2}\right)^4 \le 1 + \frac{\alpha^2e^2|u|^2}{2} + \frac{(\alpha e|u|)^4}{8} \le e^{\alpha^2e^2|u|^2/2},$$

so $\Lambda_\mu(u) \le \alpha^2e^2|u|^2/2$ for $\alpha e|u| \le 1$. Thus $\Lambda_\mu^\star(u) \ge \min\{\frac{|u|}{2\alpha e}, \frac{|u|^2}{2\alpha^2e^2}\}$ for all u. $\blacksquare$

Remark 3.7. For $p \ge q$, we always have $\mathcal{M}_p(\mu) \subset \mathcal{M}_q(\mu)$ and $\mathcal{Z}_q(\mu) \subset \mathcal{Z}_p(\mu)$. If the measure μ is α-regular, then $\mathcal{M}_q(\mu) \subset (\alpha p/q)\mathcal{M}_p(\mu)$ and $\mathcal{Z}_p(\mu) \subset (\alpha p/q)\mathcal{Z}_q(\mu)$ for $p \ge q \ge 2$. Moreover, for any symmetric measure μ, $\Lambda_\mu^\star(0) = 0$, hence by the convexity of $\Lambda_\mu^\star$, $B_q(\mu) \subset B_p(\mu) \subset (p/q)B_q(\mu)$ for all $p \ge q > 0$.

Proposition 3.8. *Symmetric log-concave measures are* 1*-regular.*

Proof. If X is distributed according to a symmetric, log-concave measure μ and $u \in \mathbb{R}^n$, then the random variable $S = \langle u,X\rangle$ has a log-concave symmetric distribution on the real line. We need to show that $(\mathbb{E}|S|^p)^{1/p} \le$

$(p/q)(\mathbb{E}|S|^q)^{1/q}$ for $p \geq q \geq 2$. Barlow, Marshall and Proschan [3] (see also proof of Remark 5 in [16]) showed that

$$(\mathbb{E}|S|^p)^{1/p} \leq \frac{(\Gamma(p+1))^{1/p}}{(\Gamma(q+1))^{1/q}} (\mathbb{E}|S|^q)^{1/q},$$

so it is enough to show that the function $f(x) := \frac{1}{x}(\Gamma(x+1))^{1/x}$ is nonincreasing on $[2,\infty)$. Binet's form of the Stirling formula (cf. [1, Theorem 1.6.3]) gives

$$\Gamma(x+1) = x\Gamma(x) = \sqrt{2\pi}\, x^{x+1/2} e^{-x+\mu(x)},$$

where $\mu(x) = \int_0^\infty \arctan(t/x)(e^{2\pi t}-1)^{-1}\,dt$ is a decreasing function. Thus

$$\ln f(x) = \frac{\mu(x)}{x} + \frac{\ln(2\pi x)}{2x} - 1$$

is indeed nonincreasing on $[2,\infty)$. ∎

Let us introduce the following notion:

Definition 3.9. We say that a measure μ satisfies the *concentration inequality with constant* β (CI(β) for short) if

$$(16)\quad \forall_{p\geq 2}\forall_{A\in\mathcal{B}(\mathbb{R}^n)} \qquad \mu(A) \geq 1/2 \;\Rightarrow\; 1-\mu(A+\beta\mathcal{Z}_p(\mu)) \leq e^{-p}(1-\mu(A)).$$

The next proposition shows that property (16) is in a sense optimal.

Proposition 3.10. *Suppose that μ is an α-regular, symmetric probability measure on $\mathbb{R}^n$, and K is a convex set such that for any halfspace A,*

$$\mu(A) \geq 1/2 \;\Rightarrow\; 1-\mu(A+K) \leq e^{-p}/2.$$

Then $K \supset c(\alpha)\mathcal{Z}_p$ for $p \geq p(\alpha)$, where $c(\alpha)$ and $p(\alpha)$ depend only on α.

Proof. Fix $v \in \mathbb{R}^n$ and set $A = \{x\colon \langle v,x\rangle < 0\}$. Then $A+K = \{x\colon \langle v,x\rangle < a(v)\}$, where $a(v) = \sup_{x\in K}\langle x,v\rangle$. Let X be a random variable with the same distribution as $\langle v,x\rangle$ under μ. Then

$$\mathbb{P}(|X| \geq a(v)) = 2\mathbb{P}(X \geq a(v)) = 2(1-\mu(A+K)) \leq e^{-p}.$$

Regularity of μ implies $\|X\|_p \leq \alpha p\|X\|_q/q$ for any $p \geq q \geq 2$, where $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$. Hence by the Paley–Zygmund inequality (cf. [17, Lemma 0.2.1]) we obtain for $q \geq 2$,

$$\begin{aligned}\mathbb{P}(|X| \geq \|X\|_q/2) &= \mathbb{P}(|X|^q \geq 2^{-q}\mathbb{E}|X|^q) \geq (1-2^{-q})^2\|X\|_q^{2q}/\|X\|_{2q}^{2q}\\ &\geq \tfrac{9}{16}(2\alpha)^{-2q} > (3\alpha)^{-2q}.\end{aligned}$$

Thus if $p \geq p(\alpha) = 4\ln(3\alpha)$ and $c(\alpha) = (4\alpha\ln(3\alpha))^{-1}$,

$$\mathbb{P}(|X| \geq c(\alpha)\|X\|_p) \geq \mathbb{P}(|X| \geq \tfrac{1}{2}\|X\|_{p/(2\ln(3\alpha))}) > (3\alpha)^{-p/\ln(3\alpha)} = e^{-p}.$$

Hence $c(\alpha)\|X\|_p = c(\alpha)(\int|\langle v,x\rangle|^p d\mu(x))^{1/p} \leq a(v)$ and $c(\alpha)\mathcal{Z}_p(\mu) \subset K$. ∎

Another motivation for the definition of CI is the following corollary:

Corollary 3.11. *Let μ be an α-regular symmetric and isotropic probability measure with $\alpha \geq 1$. Then*

(i) *If μ satisfies* $\mathrm{IC}(\beta)$, *then it satisfies* $\mathrm{CI}(8e\alpha\beta)$.
(ii) *If μ satisfies* $\mathrm{CI}(\beta)$ *and Cheeger's inequality* (12) *with constant* $1/\gamma$, *then it satisfies* $\mathrm{IC}(36\max\{6e\beta,\gamma\})$.

Proof. (i) Suppose that μ satisfies $\mathrm{IC}(\beta)$. By Remark 3.7, Proposition 2.4 and the definition of $B_p(\mu)$ we have

$$\mu(A + 2\beta B_p(\mu)) \geq \mu(A + \beta B_{2p}(\mu)) \geq 1 - e^{-p}(1-\mu(A)),$$

so (i) follows immediately from Proposition 3.5.

(ii) On the other hand, if μ satisfies $\mathrm{CI}(\beta)$, then by Remark 3.7 and Proposition 3.2 we have for $\mu(A) \geq 1/2$ and $p \geq 1$,

$$\begin{aligned} e^{-p}(1-\mu(A)) > e^{-2p}(1-\mu(A)) &\geq 1 - \mu(A + \beta\mathcal{Z}_{2p}(\mu)) \\ &\geq 1 - \mu(A + e2^{1/2p}\beta B_{2p}(\mu)) \geq 1 - \mu(A + 3e\beta B_p(\mu)). \end{aligned}$$

By Proposition 2.7 this implies property (11) with $\gamma = 3e\beta$. Additionally $\Lambda_\mu^\star$ is convex, symmetric and $\Lambda_\mu^\star(0) = 0$. Finally, from Proposition 3.3 we have $\min\{1, \Lambda_\mu^\star(u)\} \leq |u|^2$. Thus, Proposition 2.9 yields (ii). ∎

By Proposition 2.7, in Definition 3.9 we could use the equivalent condition $\mu(A + \beta\mathcal{Z}_p(\mu)) \geq \min\{e^p\mu(A), 1/2\}$. The next proposition shows that for log-concave measures these conditions are satisfied for large p and for small sets.

Proposition 3.12. *Let μ be a symmetric log-concave probability measure on $\mathbb{R}^n$ and $c \in (0,1]$. Then*

$$\mu\bigg(A + \frac{40}{c}\,\mathcal{Z}_p(\mu)\bigg) \geq \frac{1}{2}\min\{e^p\mu(A), 1\}$$

for $p \geq cn$ or for $\mu(A) \leq e^{-cn}$.

Proof. Using a standard volumetric estimate for any $r > 0$ we may choose $S \subset \mathcal{M}_r(\mu)$ with $\#S \leq 5^n$ such that $\mathcal{M}_r(\mu) \subset \bigcup_{u\in S}\big(u + \frac{1}{2}\mathcal{M}_r(\mu)\big)$. Then for $t > 0$,

$$x \notin t\mathcal{Z}_r(\mu) \ \Rightarrow\ \max_{u\in S}\langle u, x\rangle \geq t/2$$

and by the Chebyshev inequality,

$$\begin{aligned} \mu(\mathbb{R}^n \setminus t\mathcal{Z}_r(\mu)) &\leq \sum_{u\in S}\mu\bigg\{x\colon \langle u,x\rangle \geq \frac{t}{2}\bigg\} \leq \sum_{u\in S}\bigg(\frac{2}{t}\bigg)^r \int \langle u,x\rangle_+^r\,d\mu \\ &\leq \frac{1}{2}\,5^n\bigg(\frac{2}{t}\bigg)^r. \end{aligned}$$

Let $\mu(A) = e^{-q}$. We will consider two cases.

(i) $p \ge \max\{q, cn\}$. Then by Remark 3.7,

$$\mu(30c^{-1}\mathcal{Z}_p(\mu)) > \mu(30\mathcal{Z}_{\max\{p,n\}}) \ge 1 - \tfrac{1}{2}e^{-\max\{p,n\}} \ge 1 - \mu(A),$$

so $A \cap 30c^{-1}\mathcal{Z}_p(\mu) \neq \emptyset$, hence $0 \in A + 30c^{-1}\mathcal{Z}_p(\mu)$ and

$$\mu(A + 40c^{-1}\mathcal{Z}_p(\mu)) \ge \mu(10c^{-1}\mathcal{Z}_p(\mu)) \ge 1/2.$$

(ii) $q \ge \max\{p, cn\}$. Let $\widetilde{q} := \max\{q, n\}$ and

$$\widetilde{A} := A \cap 30c^{-1}\mathcal{Z}_q(\mu).$$

As in (i), we have $\mu(30c^{-1}\mathcal{Z}_q(\mu)) > 1 - e^{-\widetilde{q}}/2$, thus $\mu(\widetilde{A}) \ge \mu(A)/2$. Moreover,

$$\left(1 - \frac{p}{q}\right)\widetilde{A} \subset A - \frac{p}{q}\,30c^{-1}\mathcal{Z}_q(\mu) \subset A + 30c^{-1}\mathcal{Z}_p(\mu)$$

and

$$\begin{aligned}
\mu(A + 40c^{-1}\mathcal{Z}_p(\mu)) &\ge \mu\biggl(\left(1 - \frac{p}{q}\right)\widetilde{A} + \frac{p}{q}\,10c^{-1}\mathcal{Z}_q(\mu)\biggr)\\
&\ge \mu\biggl(\left(1 - \frac{p}{q}\right)\widetilde{A} + \frac{p}{q}\,10\mathcal{Z}_{\widetilde{q}}(\mu)\biggr) \ge \mu(\widetilde{A})^{1-p/q}\mu(10\mathcal{Z}_{\widetilde{q}})^{p/q}\\
&\ge \left(\frac{1}{2}\,\mu(A)\right)^{1-p/q}\left(\frac{1}{2}\right)^{p/q} \ge \frac{1}{2}\,\mu(A)\mu(A)^{-p/q}\\
&= \frac{1}{2}\,e^{-p}\mu(A). \ \blacksquare
\end{aligned}$$

We conclude this part with a proof that for log-concave probability measures, IC and CI are equivalent and (with the additional assumption of isotropy) imply the Cheeger and Poincaré inequalities. We begin by deriving from CI a concentration of Lipschitz functions for isotropic measures.

Proposition 3.13. *If μ is a log-concave isotropic probability measure on $\mathbb{R}^n$ satisfying* CI(C) *and f is a* 1*-Lipschitz function* (*with respect to the standard Euclidean norm*) *then*

$$\mu(\{x \in \mathbb{R}^n \colon |f(x) - \mathrm{Med}_\mu f(x)| > t\}) \le e^{1-t/C_1}, \tag{17}$$

where $C_1 = 4Ce^2$. Moreover,

$$\mu(\{x \in \mathbb{R}^n \colon |f(x) - \mathbb{E}_\mu f(x)| > t\}) \le e^{1-t/C_2},$$

where $C_2 = 8Ce^3$.

Proof. Let $A_t = \{x \in \mathbb{R}^n \colon f(x) - \mathrm{Med}_\mu f > t\}$ and $A = \{x \colon f(x) \le \mathrm{Med}_\mu f\}$. We have $\mu(A) \ge 1/2$, and thus by CI(C), $1 - \mu(A + CZ_p(\mu)) \le e^{-p}(1 - \mu(A)) \le e^{-p}/2$. Assume $p \ge 1$. Then by Propositions 3.2 and 3.6 we have $\mathcal{Z}_p(\mu) \subset 2eB_p(\mu) \subset 4e^2pB_2^n$. Take $t = 4Ce^2p$ (this entails $t \ge 4Ce^2 = C_1$). Then as f is 1-Lipschitz, $A_t \cap (A + tB_2^n) = \emptyset$, thus $\mu(A_t) \le 1 - \mu(A + tB_2^n) \le 1 - \mu(A + CZ_p) \le e^{-t/C_1}/2$. Similarly one proves that for

$t > C_1$ we have $\mu(\{x\colon f(x) - \mathrm{Med}_\mu f(x) < -t\}) \le e^{-t/C_1}/2$, thus (17) holds for $t \ge C_1$. If $t \le C_1$, then obviously $\mu(A_t) \le 1 \le e^{1-t/C_1}$.

By integration by parts we get $|\mathbb{E}_\mu f - \mathrm{Med}_\mu f| \le \int_0^\infty \mu(\{x\colon |f(x) - \mathrm{Med}_\mu f| \ge t) dt \le eC_1$, thus considering $t \ge 2eC_1$ and $t < 2eC_1$ we get the second assertion. ∎

The property (17) is called *exponential concentration of Lipschitz functions*. Theorem 1.5 of [21] states in particular that under convexity assumptions (satisfied by any log-concave measure) exponential concentration is equivalent to Cheeger's inequality (12) and the Poincaré inequality. Thus we have the following corollary:

Corollary 3.14. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then*:

(i) *If μ satisfies* $\mathrm{IC}(C)$, *then it satisfies* $\mathrm{CI}(C')$ *with $C' \simeq C$.*
(ii) *If μ satisfies* $\mathrm{CI}(C')$, *then it satisfies* $\mathrm{IC}(C)$ *with $C \simeq C'$.*
(iii) *If μ satisfies either* $\mathrm{IC}(C)$ *or* $\mathrm{CI}(C)$ *and is in addition isotropic, then it satisfies Cheeger's inequality with constant $\kappa \simeq C$.*

Proof. Any probability measure can be transported by an affine map onto an isotropic measure, so let L be an affine map such that $\mu \circ L^{-1}$ is isotropic. Also note that $\mathcal{Z}_p(\mu \circ L^{-1}) = L(\mathcal{Z}_p(\mu))$, thus $\mathrm{CI}(C)$ is affine invariant, and by Proposition 2.16, $\mathrm{IC}(C)$ is affine invariant. Thus in (i) and (ii) we may assume μ is isotropic. Also note that by Proposition 3.8, μ is 1-regular.

Thus (i) is a direct consequence of Corollary 3.11. For (iii) we may, by (i), assume μ satisfies $\mathrm{CI}(C)$. Then by Proposition 3.13 we have exponential concentration of Lipschitz functions, and the conclusion follows from Theorem 1.5 of [21]. For (ii) we can use Corollary 3.11 again, as by (iii) we know μ satisfies Cheeger's inequality. ∎

Thus Conjecture 1 (see end of Section 2) is equivalent to the statement that any log-concave measure satisfies $\mathrm{CI}(C)$. Moreover, it would imply the following conjecture of Kannan, Lovász and Simonovits:

Conjecture 2 (Kannan–Lovász–Simonovits [13]). *There exists an absolute constant C such that any symmetric isotropic log-concave probability measure satisfies Cheeger's inequality with constant $1/C$.*

3.3. *Comparison of weak and strong moments.* In this subsection we use standard techniques to derive a concentration of norms from the concentration of measure. We also show several consequences of the CI property.

Proposition 3.15. *Suppose that a probability measure μ on $\mathbb{R}^n$ is α-regular and satisfies* $\mathrm{CI}(\beta)$. *Then for any norm $\|\cdot\|$ on $\mathbb{R}^n$ and $p \ge 2$,*

$$\Big(\int \big|\,\|x\| - \mathrm{Med}_\mu(\|x\|)\big|^p\,d\mu\Big)^{1/p} \le 2\alpha\beta \sup_{\|u\|_*\le 1}\Big(\int |\langle u,x\rangle|^p\,d\mu\Big)^{1/p},$$

where $\|\cdot\|_*$ *denotes the norm dual to* $\|\cdot\|$.

Proof. For $p \ge 2$ we define

$$m_p := \sup_{\|u\|_*\le 1}\Big(\int |\langle u,x\rangle|^p\,d\mu\Big)^{1/p}.$$

Let $M := \mathrm{Med}_\mu(\|x\|)$, $A := \{x\colon \|x\| \le M\}$ and $\widetilde{A} := \{x\colon \|x\| \ge M\}$. Then $\mu(A), \mu(\widetilde{A}) \ge 1/2$, so by CI($\beta$) and Remark 3.7,

$$\forall_{t\ge p}\quad 1 - \mu\Big(A + \beta\,\frac{\alpha t}{p}\,\mathcal{Z}_p(\mu)\Big) \le \tfrac12 e^{-t},\ 1 - \mu\Big(\widetilde{A} + \beta\,\frac{\alpha t}{p}\,\mathcal{Z}_p(\mu)\Big) \le \tfrac12 e^{-t}.$$

Let $y \in \mathcal{Z}_p$. Then there exists $u \in \mathbb{R}^n$ with $\|u\|_* \le 1$ such that

$$\|y\| = \langle u,y\rangle \le \Big(\int |\langle u,x\rangle|^p\,d\mu(x)\Big)^{1/p} \le m_p,$$

hence $\|x\| \le M + tm_p$ for $x \in A + t\mathcal{Z}_p(\mu)$. Thus for $t \ge p$,

$$\mu\Big\{x\colon \|x\| \ge M + \frac{\alpha\beta t}{p}\,m_p\Big\} \le 1 - \mu\Big(A + \beta\,\frac{\alpha t}{p}\,\mathcal{Z}_p(\mu)\Big) \le \tfrac12 e^{-t}.$$

In a similar way we show $\|x\| \ge M - tm_p$ for $x \in \widetilde{A} + t\mathcal{Z}_p(\mu)$ and $\mu\{x\colon \|x\| \le M - \alpha\beta t m_p/p\} \le e^{-t}/2$, therefore

$$\mu\Big\{x\colon \big|\,\|x\| - M\big| \ge \frac{\alpha\beta t}{p}\,m_p\Big\} \le e^{-t} \quad \text{for } t \ge p.$$

Now integrating by parts gives

$$\begin{aligned}
&\Big(\int \big|\,\|x\| - M\big|^p\,d\mu\Big)^{1/p} \\
&\qquad \le \frac{\alpha\beta m_p}{p}\Big[p + \Big(p\int_p^\infty t^{p-1}\mu\Big\{x\colon \big|\,\|x\| - M\big| \ge \frac{\alpha\beta t}{p}\,m_p\Big\}\,dt\Big)^{1/p}\Big] \\
&\qquad \le \frac{\alpha\beta m_p}{p}\Big[p + \Big(p\int_p^\infty t^{p-1}e^{-t}\,dt\Big)^{1/p}\Big] \\
&\qquad \le \alpha\beta m_p\Big(1 + \frac{\Gamma(p+1)^{1/p}}{p}\Big) \le 2\alpha\beta m_p. \ \blacksquare
\end{aligned}$$

REMARK 3.16. Under the assumptions of Proposition 3.15, by the triangle inequality for $\gamma = 4\alpha\beta$ we get

$$(18)\qquad \forall_{p\ge q\ge 2}\quad \Big(\int\Big|\|x\|-\Big(\int\|x\|^q\,d\mu\Big)^{1/q}\Big|^p d\mu\Big)^{1/p} \le \gamma \sup_{\|u\|^\star\le 1}\Big(\int|\langle u,x\rangle|^p\,d\mu\Big)^{1/p}.$$

This motivates the following definition.

Definition 3.17. We say that a probability measure μ on $\mathbb{R}^n$ has *comparable weak and strong moments* with constant γ (CWSM(γ) for short) if (18) holds for any norm $\|\cdot\|$ on $\mathbb{R}^n$.

Conjecture 3. *Every symmetric log-concave probability measure on $\mathbb{R}^n$ satisfies* CWSM(C).

Proposition 3.18. *Let μ be an isotropic probability measure on $\mathbb{R}^n$ satisfying* CWSM(γ). *Then*

(i) $\int|\,\|x\|_2-\sqrt{n}|^2\,d\mu(x)\le\gamma^2$,

(ii) *if μ is also α-regular, then for all $p>2$,*

$$\Big(\int\|x\|_2^p\,d\mu\Big)^{1/p}\le\sqrt{n}+\frac{\gamma\alpha}{2}\,p.$$

Proof. Notice that $\int\|x\|_2^2\,d\mu=n$ and $\|u\|_2^\star=\|u\|_2$. Hence (i) follows directly from (18) with $p=q=2$. Moreover, (18) with $q=2$ implies

$$\Big(\int\|x\|_2^p\,d\mu\Big)^{1/p}\le\sqrt{n}+\gamma\sup_{\|u\|_2\le 1}\Big(\int|\langle u,x\rangle|^p\,d\mu\Big)^{1/p}\le\sqrt{n}+\frac{\gamma\alpha}{2}\,p$$

by the α-regularity and isotropy of μ. ∎

Remark 3.19. Property (i) plays a crucial role in Klartag's proof of the central limit theorem for convex bodies [14]. Paouris [23] showed that moments of the Euclidean norm for symmetric isotropic log-concave measures are bounded by $C(p+\sqrt{n})$. Thus Conjecture 3 would imply both Klartag's CLT (with the optimal speed of convergence) and Paouris concentration.

We conclude this section with the estimate that shows comparability of weak and strong moments for any probability measure and $p>n/C$.

Proposition 3.20. *For any $p\ge 1$ we have*

$$\Big(\int|\,\|x\|-\mathrm{Med}_\mu(\|x\|)|^p\,d\mu\Big)^{1/p}\le\Big(\int\|x\|^p\,d\mu\Big)^{1/p} \le 2\cdot 5^{n/p}\sup_{\|u\|_*\le 1}\Big(\int|\langle u,x\rangle|^p\,d\mu\Big)^{1/p}.$$

Proof. As in the proof of Proposition 3.12 we can find $u_1,\dots,u_N$ with $\|u_i\|_\star\le 1$, $N\le 5^n$ such that $\|x\|\le 2\max_{i\le N}\langle u_i,x\rangle$ for all x. Then

$$\int\|x\|^p\,d\mu\le 2^p\int\sum_{i\le N}|\langle u_i,x\rangle|^p\,d\mu\le 2^p5^n\sup_{\|u\|_*\le 1}\int|\langle u,x\rangle|^p\,d\mu.$$

Moreover,

$$\int_{\{\|x\|\ge M\}} (\|x\|-M)^p\, d\mu(x) \le \int_{\{\|x\|\ge M\}} (\|x\|^p-M^p)\, d\mu(x) \le \int \|x\|^p\, d\mu(x)-\tfrac{1}{2}M^p$$

and

$$\int_{\{\|x\|<M\}} (M-\|x\|)^p\, d\mu(x) \le M^p\mu\{x\colon \|x\|<M\} \le \tfrac{1}{2}M^p. \ \blacksquare$$

4. Modified Talagrand concentration for exponential measure. In this section we show that for a set lying far from the origin, Talagrand's two-level concentration for the exponential measure may be somewhat improved, namely (for sufficiently large t) it is enough to enlarge the set by tB_1^n instead of $tB_1^n+\sqrt{t}\,B_2^n$.

We will need this result for sets which are far away from the origin in the Euclidean norm. The first lemma, however, will consider sets which are far away from the origin in one coordinate direction. The proof is an application of the Brunn–Minkowski inequality for the Lebesgue measure.

LEMMA 4.1. *If $u \ge t > 0$ then for any $i \in \{1,\dots,n\}$ we have*

$$|(A+tB_1^n)\cap nB_1^n \cap \{x\colon |x_i|\ge u-t\}| \ge e^{t/2}|A\cap nB_1^n\cap\{x\colon |x_i|\ge u\}|.$$

Proof. Obviously we may assume that $i=1$ and $u\le n$. Let $A_1 := A\cap nB_1^n\cap\{x\colon x_1\ge u\}$ and $B := \{x\in B_1^n\colon x_1 \ge \sum_{i\ge 2}|x_i|\}$. From the definition of B and A_1 we have $A_1 - tB\subset nB_1^n$. On the other hand, $B = \{x : |x_1-1/2|+\sum_{i\ge2}|x_i|\le 1/2\}$, so $|B| = 2^{-n}|B_1^n| = (2r_{1,n})^{-n}$. Thus

$$|(A_1+tB_1^n)\cap nB_1^n| \ge |(A_1-tB)\cap nB_1^n| = |A_1-tB|.$$

Now set

$$s := \frac{2|A_1|^{1/n}r_{1,n}}{t+2|A_1|^{1/n}r_{1,n}}.$$

Then we easily check that $|tB/(1-s)| = |A_1/s|$. As $A_1\subset\{x\in nB_1^n\colon x_1\ge t\}$ we get $|A_1|^{1/n}\le (n-t)/r_{1,n}$ and $s\le 2(n-t)/(2n-t)$. Now we can use the Brunn–Minkowski inequality to obtain

$$\begin{aligned}
|A_1-tB| &= \left|s\,\frac{A_1}{s}+(1-s)\,\frac{-t}{1-s}\,B\right| \ge \left|\frac{A_1}{s}\right|^s\left|\frac{-t}{1-s}\,B\right|^{1-s} = \left|\frac{A_1}{s}\right| \\
&= s^{-n}|A_1| \ge \left(\frac{2n-t}{2n-2t}\right)^n|A_1| = \left(\frac{1}{1-\frac{t}{2n-t}}\right)^n|A_1| \\
&\ge e^{tn/(2n-t)}|A_1| \ge e^{t/2}|A_1|.
\end{aligned}$$

Notice that $A_1+tB_1^n\subset\{x\colon x_1\ge u-t\}$, so we obtain

$$|(A+tB_1^n)\cap nB_1^n\cap\{x\colon x_1\ge u-t\}| \ge e^{t/2}|A\cap nB_1^n\cap\{x\colon x_1\ge u\}|;$$

in the same way we show

$$|(A+tB_1^n)\cap nB_1^n\cap\{x\colon x_1\le -u+t\}|\ge e^{t/2}|A\cap nB_1^n\cap\{x\colon x_1\le -u\}|. \ \blacksquare$$

Remark 4.2. A similar result (although with a constant multiplicative factor) can be obtained using the same technique and more calculations for $n^{1/p}B_p^n$ instead of nB_1^n for $p\in[1,2]$.

Lemma 4.3. *If $u\ge t>0$ then for any $i\in\{1,\dots,n\}$ we have*

$$\nu^n((A+tB_1^n)\cap\{x\colon |x_i|\ge u-t\})\ge e^{t/2}\nu^n(A\cap\{x\colon |x_i|\ge u\}).$$

Proof. Take an arbitrary $k\in\mathbb{N}$. Let $P:\mathbb{R}^{n+k}\to\mathbb{R}^n$ be the projection onto first n coordinates. Let ϱ_k be the uniform probability measure on $(n+k)B_1^{n+k}$, and $\widetilde{\nu}_k$ the measure defined by $\widetilde{\nu}_k(A)=\varrho_k(P^{-1}(A))$. Take an arbitrary set $A\subset\mathbb{R}^n$. Notice that for any set $C\subset\mathbb{R}^n$ we have

$$C\cap\{x\colon |x_i|\ge s\}=P(P^{-1}(C)\cap\{x\colon |x_i|\ge s\})$$

and also $P^{-1}(A)+B_1^{n+k}\subset P^{-1}(A+B_1^n)$. From Lemma 4.1 we have

$$\varrho_k((P^{-1}(A)+tB_1^{n+k})\cap\{x\colon |x_i|\ge u-t\})\ge e^{t/2}\varrho_k(P^{-1}(A)\cap\{x\colon |x_i|\ge u\}),$$

and thus

$$\widetilde{\nu}_k((A+tB_1^n)\cap\{x\colon |x_i|\ge u-t\})\ge e^{t/2}\widetilde{\nu}_k(A\cap\{x\colon |x_i|\ge u\}).$$

When $k\to\infty$, we have $\widetilde{\nu}_k(C)\to\nu^n(C)$ for any set $C\in\mathcal{B}(\mathbb{R}^n)$. Thus by going to the limit we get the assertion. $\blacksquare$

To pass from sets with one coordinate large to sets far away from the origin in the Euclidean norm we will, instead of considering the measure of a set, consider the integral of the square of the Euclidean norm over the set. Splitting our set into subsets on which the square of the norm is roughly constant we will be able to tie the two quantities, while applying integration by parts we are able to estimate the integral.

Proposition 4.4. *For any $t>0$ and any $n\in\mathbb{N}$ we have*

$$\int_{A+tB_1^n}|x|^2\,d\nu^n(x)\ge e^{t/2}\int_A(|x|-t\sqrt{n})_+^2\,d\nu^n(x).$$

Proof. Let $A_t=A+tB_1^n$. By Lemma 4.3 we get, for any $s\ge 0$ and any i,

$$\int_{A_t}I_{\{|x_i|\ge s\}}\,d\nu^n(x)\ge e^{t/2}\int_A I_{\{|x_i|\ge s+t\}}\,d\nu^n(x).$$

Thus

$$\int_{A_t} x_i^2\,d\nu^n(x) = \int_{A_t}\int_0^\infty 2sI_{\{|x_i|\ge s\}}\,ds\,d\nu^n(x) = \int_0^\infty 2s\int_{A_t} I_{\{|x_i|\ge s\}}\,d\nu^n(x)\,ds$$
$$\ge e^{t/2}\int_0^\infty 2s\int_A I_{\{|x_i|\ge s+t\}}\,d\nu^n(x)\,ds$$
$$= e^{t/2}\int_A\int_0^\infty 2sI_{\{|x_i|\ge s+t\}}\,ds\,d\nu^n(x) = e^{t/2}\int_A (|x_i|-t)_+^2\,d\nu^n(x).$$

To get the assertion it is enough to sum over all i and notice that the function $f(y) := (\sqrt{y}-t)_+^2$ is convex on $[0,\infty)$, hence

$$\sum_{i=1}^n (|x_i|-t)_+^2 = \sum_{i=1}^n f(x_i^2) \ge nf\Big(\frac{1}{n}\sum_{i=1}^n x_i^2\Big) = (|x|-t\sqrt{n})_+^2. \ \blacksquare$$

LEMMA 4.5. *Suppose that* $A \subset \{x\in\mathbb{R}^n : |x| \ge 5t\sqrt{n}\}$. *Then*

$$\nu^n(A+tB_1^n) \ge \tfrac{1}{8}e^{t/2}\nu^n(A).$$

Proof. Let

$$A_k := A\cap\{x : 5t\sqrt{n}+2t(k-1) \le |x| < 5t\sqrt{n}+2tk\}, \quad k=1,2,\ldots.$$

Then $A_k + tB_1^n \subset \{x : 5t\sqrt{n}+t(2k-3) \le |x| < 5t\sqrt{n}+t(2k+1)\}$, hence

$$\nu^n(A+tB_1^n) \ge \frac{1}{2}\sum_{k\ge1}\nu(A_k+tB_1^n).$$

From Proposition 4.4 applied for A_k we have

$$(5t\sqrt{n}+t(2k+1))^2\nu^n(A_k+tB_1^n) \ge \int_{A_k+tB_1^n} |x|^2\,d\nu^n(x)$$
$$\ge e^{t/2}\int_{A_k}(|x|-t\sqrt{n})_+^2\,d\nu^n(x) \ge e^{t/2}(4t\sqrt{n}+2t(k-1))^2\nu^n(A_k).$$

Thus

$$\nu^n(A_k+tB_1^n) \ge \left(\frac{4t\sqrt{n}+2t(k-1)}{5t\sqrt{n}+t(2k+1)}\right)^2 e^{t/2}\nu_n(A_k) \ge \tfrac{1}{4}e^{t/2}\nu^n(A_k)$$

and

$$\nu^n(A+tB_1^n) \ge \frac{1}{2}\sum_{k\ge1}\frac{1}{4}\,e^{t/2}\nu^n(A_k) = \frac{1}{8}\,e^{t/2}\nu^n(A). \ \blacksquare$$

The final step is to make the distance from the origin that is required for the argument to work independent of t. We do this by increasing our set A "step by step", by increments of $10B_1^n$ at a time, and checking the effects of each enlargement. At each step either a large part of our set gets pushed close to the origin (where we will be able to deal with it using different methods) or it stays outside, but increases its volume. It may be useful to note that in this part we strongly use the fact that we are considering enlargements by tB_1^n only, and not by the standard $tB_1^n + \sqrt{t}\,B_2^n$, as the second set is not linear in t (thus a composition of two enlargements with coefficient t does not yield an enlargement with coefficient $2t$).

THEOREM 4.6. *For any $A \in \mathcal{B}(\mathbb{R}^n)$ and any $t \geq 10$, either*

$$\nu^n((A + tB_1^n) \cap 50\sqrt{n}\,B_2^n) \geq \tfrac{1}{2}\nu^n(A)$$

or

$$\nu^n(A + tB_1^n) \geq e^{t/10}\nu^n(A). \tag{19}$$

In particular, (19) *holds if $A \cap (50\sqrt{n}\,B_2^n + tB_1^n) = \emptyset$.*

Proof. Let A_k denote $A + 10kB_1^n$ for $k = 0, 1, \ldots$. If for any $0 \leq k \leq t/10$ we have $\nu^n(A_k \cap 50\sqrt{n}\,B_2^n) \geq \nu^n(A)/2$, the assertion is proved. So assume otherwise. Let $A'_k := A_k \setminus 50\sqrt{n}\,B_2^n$. From Lemma 4.5 we have

$$\nu^n(A_{k+1}) \geq \nu^n(A'_k + 10B_1^n) \geq \tfrac{1}{8}e^5\nu^n(A'_k) \geq \tfrac{1}{16}e^5\nu^n(A_k) \geq e^2\nu^n(A_k).$$

By a simple induction we get $\nu^n(A_k) \geq e^{2k}\nu^n(A)$ for any $k \leq t/10$. Thus

$$\nu^n(A + tB_1^n) \geq \nu^n(A_{\lfloor t/10\rfloor}) \geq e^{2\lfloor t/10\rfloor}\nu^n(A) \geq e^{t/10}\nu^n(A). \ ∎$$

5. Uniform measure on B_p^n. In this section we will prove the infimum convolution property IC(C) for B_p^n balls. Recall that ν_p^n is a product measure, while $\mu_{p,n}$ denotes the uniform measure on $r_{p,n}B_p^n$. We have

$$r_{p,n}^{-n} = |B_p^n| = \frac{2^n\Gamma(1+1/p)^n}{\Gamma(1+n/p)} \sim \frac{(2\Gamma(1+1/p))^n(ep)^{n/p}}{n^{n/p}(\sqrt{n/p}+1)},$$

where the last part follows from Stirling's formula. Thus $r_{p,n} \sim n^{1/p}$.

For ν_p^n we have IC(48) by Corollary 2.19. Let us first try to understand what sort of concentration this implies, that is, how the function $\Lambda^\star$ behaves for ν_p^n.

PROPOSITION 5.1. *For any $p \geq 1$ and $t \in \mathbb{R}$ we have*

$$B_t(\nu_p) \sim \{x\colon f_p(|x|) \leq t\} \quad \textit{and} \quad \Lambda^\star_{\nu_p}(t/C) \leq f_p(|t|) \leq \Lambda^\star_{\nu_p}(Ct),$$

where $f_p(t) = t^2$ for $t < 1$ and $f_p(t) = t^p$ for $t \geq 1$.

Proof. We shall use the facts proved in Section 3 to approximate $B_t(\nu_p)$. Note that ν_p is log-concave (as its density is log-concave) and symmetric. It is 1-regular from Proposition 3.8. Also

$$\sigma_p^2 := \int_{\mathbb{R}} x^2\,d\nu_p(x) = \frac{1}{2\gamma_p}\int_{\mathbb{R}} x^2 e^{-|x|^p}\,dx = \frac{\Gamma(1+3/p)}{3\Gamma(1+1/p)} \sim 1$$

for $p \in [1,\infty)$. The measure $\widetilde{\nu}_p$ with density $\sigma_p d\nu_p(\sigma_p x)$ is isotropic, hence Propositions 3.3 and 3.6 yield $B_t(\widetilde{\nu}_p) \sim \sqrt{t}\, B_2^1 = [-\sqrt{t},\sqrt{t}]$ for $t \le 1$. Thus, as $B_t(\nu_p) = \sigma_p B_t(\widetilde{\nu}_p)$, we get $B_t(\nu_p) \sim [-\sqrt{t},\sqrt{t}]$ for $t \le 1$.

For $t \ge 1$ we have

$$\begin{aligned}\mathcal{M}_t(\nu_p) &= \Big\{u \in \mathbb{R}\colon \frac{1}{2\gamma_p}\int_{\mathbb{R}} |u|^t|x|^t e^{-|x|^p}\,dx \le 1\Big\}\\ &= \Big\{u \in \mathbb{R}\colon |u| \le \sqrt[t]{\frac{(t+1)\Gamma(1+1/p)}{\Gamma\big(1+\frac{t+1}{p}\big)}}\Big\} \sim \{u \in \mathbb{R}\colon |u| \le t^{-1/p}\}.\end{aligned}$$

Thus $Z_t(\nu_p) \sim [-t^{1/p}, t^{1/p}]$ for $|t| \ge 1$, so by Propositions 3.2 and 3.5, $B_t(\nu_p) \sim [-t^{1/p}, t^{1/p}]$. Hence, for all $t \ge 0$ we have $\{x\colon f_p(|x|) \le t\} \sim \{x\colon \Lambda^\star_{\nu_p}(x) \le t\}$, so $\Lambda^\star_{\nu_p}(t/C) \le f_p(t) \le \Lambda^\star_{\nu_p}(Ct)$. As $\Lambda^\star_{\nu_p}$ is symmetric, the proof is finished. ∎

COROLLARY 5.2. *For any $t > 0$ and $n \in \mathbb{N}$ we have*

$$B_t(\nu_p^n) \sim \begin{cases} \sqrt{t}\, B_2^n + t^{1/p} B_p^n & \text{for } p \in [1,2],\\ \sqrt{t}\, B_2^n \cap t^{1/p} B_p^n & \text{for } p \ge 2.\end{cases}$$

Proof. By Proposition 5.1,

$$B_t(\nu_p^n) = \Big\{x \in \mathbb{R}^n\colon \sum \Lambda^\star_{\nu_p}(x_i) \le t\Big\} \sim \Big\{x \in \mathbb{R}^n\colon \sum f_p(|x_i|) \le t\Big\}.$$

Simple calculations show that $\{x \in \mathbb{R}^n\colon \sum f_p(|x_i|) \le t\} \sim t^{1/2}B_2^n + t^{1/p}B_p^n$ for $p \in [1,2]$ and $\{x \in \mathbb{R}^n\colon \sum f_p(|x_i|) \le t\} \sim t^{1/2}B_2^n \cap t^{1/p}B_p^n$ for $p \ge 2$. ∎

PROPOSITION 5.3. *For any $t \in [0,n]$, $p \ge 1$ and $n \in \mathbb{N}$ we have $B_t(\mu_{p,n}) \sim B_t(\nu_p^n)$.*

Proof. For $t < 1$ we use Propositions 3.3 and 3.6. Both $\mu_{p,n}$ and ν_p^n are symmetric, log-concave measures, and both can be rescaled as in the proof of Proposition 5.1 to be isotropic, thus $B_t(\mu_{p,n}) \sim \sqrt{t}\, B_2^n \sim B_t(\nu_p^n)$.

Lemma 6 from [4] gives (after rescaling by $r_{p,n}$)

$$\Big(\int |\langle a,x\rangle|^t\, d\mu_{p,n}(x)\Big)^{1/t} \sim \frac{r_{p,n}}{(\max\{n,t\})^{1/p}}\Big(\int |\langle a,x\rangle|^t\, d\nu_p^n(x)\Big)^{1/t} \tag{20}$$

for any $p,t \ge 1$ and $a \in \mathbb{R}^n$. Note that as $r_{p,n} \sim n^{1/p}$, this simply means the equivalence of tth moments of $\mu_{p,n}$ and $\nu_{p,n}$ for $t \in [0,n]$. Thus $\mathcal{M}_t(\mu_{p,n}) \sim \mathcal{M}_t(\nu_{p,n})$ for $t \le n$ and therefore $B_t(\mu_{p,n}) \sim B_t(\nu_{p,n})$. ∎

REMARK 5.4. It is not hard to verify that $B_t(\mu_{p,n}) \sim r_{p,n}B_p^n$ for $t \ge n$.

5.1. *Transports of measure.* We are now going to investigate two transports of measure. They will combine to transport a measure with known concentration properties (ν^n or ν_2^n, that is, the exponential or Gaussian measure) to the uniform measure $\mu_{p,n}$. We will investigate the contractive properties of these transports with respect to various norms. Our motivation is the following:

REMARK 5.5. *Let $U : \mathbb{R}^n \to \mathbb{R}^n$ be a map such that*

$$\|U(x) - U(y)\|_p^p \geq \delta \|x - y\|_q^q \quad \text{for all } x \in \mathbb{R}^n,\ y \in A.$$

Then

$$U(A + t^{1/q} B_q^n) \supset U(\mathbb{R}^n) \cap (U(A) + \delta^{1/p} t^{1/p} B_p^n).$$

Analogously, if

$$\|U(x) - U(y)\|_p^p \leq \delta \|x - y\|_q^q \quad \text{for all } x \in \mathbb{R}^n,\ y \in A,$$

then

$$U(A + t^{1/q} B_q^n) \subset U(A) + \delta^{1/p} t^{1/p} B_p^n.$$

Proof. We prove the first statement; the proof of the second is almost identical. Suppose $U(x) \in U(A) + \delta^{1/p} t^{1/p} B_p^n$. Then there exists $y \in A$ such that $\|U(x) - U(y)\|_p^p \leq \delta t$. From the assumption we have $t \geq \|x - y\|_q^q$, which means $x \in A + t^{1/q} B_q^n$, and $U(x) \in U(A + t^{1/q} B_q^n)$. ∎

The first transport we introduce is the radial transport $T_{p,n}$ which transforms the product measure ν_p^n onto $\mu_{p,n}$, the uniform measure on $r_{p,n} B_p^n$. We will show this transport is Lipschitz with respect to the ℓ_p norm and Lipschitz on a large set with respect to the ℓ_2 norm for $p \leq 2$.

DEFINITION 5.6. For $p \in [1, \infty)$ and $n \in \mathbb{N}$ let $f_{p,n} : [0, \infty) \to [0, \infty)$ be given by the equation

$$\int_0^s e^{-r^p} r^{n-1}\, dr = (2\gamma_p)^n \int_0^{f_{p,n}(s)} r^{n-1}\, dr \tag{21}$$

and $T_{p,n}(x) := x f_{p,n}(\|x\|_p)/\|x\|_p$ for $x \in \mathbb{R}^n$.

Let us first show the following simple estimate.

LEMMA 5.7. *For any $q > 0$ and $0 \leq u \leq q/2$,*

$$q \int_0^u e^{-t} t^{q-1}\, dt \leq e^{-u} u^q \left(1 + 2\,\frac{u}{q}\right).$$

Proof. Let

$$f(u) := e^{-u} u^q \left(1 + 2\,\frac{u}{q}\right) - q \int_0^u e^{-t} t^{q-1}\, dt.$$

Then $f(0) = 0$ and $f'(u) = e^{-u} u^q (1 - 2u/q + 2/q) \geq 0$ for $0 \leq u \leq q/2$. ∎

Now we are ready to state the basic properties of $T_{p,n}$.

PROPOSITION 5.8.

(i) *The map* $T_{p,n}$ *transports the probability measure* ν_p^n *onto the measure* $\mu_{p,n}$.
(ii) *For all* $t>0$ *we have* $e^{-t^p/n}t \le 2\gamma_p f_{p,n}(t) \le t$ *and* $f'_{p,n}(t) \le (2\gamma_p)^{-1} \le 1$.
(iii) *For any* $t>0$, $0 \le f_{p,n}(t)/t - f'_{p,n}(t) \le \min\{1, 2pt^p/n\}$.
(iv) *The function* $t \mapsto f_{p,n}(t)/t$ *is decreasing on* $(0,\infty)$ *and for any* $s,t>0$,

$$|t^{-1}f_{p,n}(t) - s^{-1}f_{p,n}(s)| \le (st)^{-1}|s-t|f_{p,n}(s\wedge t) \le \frac{|s-t|}{\max\{s,t\}}.$$

Obviously properties of $T_{p,n}$ are strongly tied to properties of $f_{p,n}$. Estimate (ii) means that up to $t = n^{1/p}$ the map $T_{p,n}$ is basically a homothety. Bounds (iii) and (iv) will be used when studying the Lipschitz properties of $T_{p,n}$. The fact that $f_{p,n}(t)/t$ is decreasing means that points farther away from the origin are contracted more. Thus we can decompose $T_{p,n}(x)-T_{p,n}(y)$ by first rescaling both points by $f_{p,n}(\|x\|)/\|x\|$, and then estimating the additional error by the inequality in the second part of (iv).

Proof. The definition of $T_{p,n}$ directly implies (i). Differentiation of (21) gives

$$e^{-s^p}s^{n-1} = (2\gamma_p)^n f_{p,n}^{n-1}(s) f'_{p,n}(s). \tag{22}$$

By (21),

$$e^{-t^p}t^n \le n\int_0^t e^{-r^p}r^{n-1}\,dr = (2\gamma_p)^n f_{p,n}^n(t) \le n\int_0^t r^{n-1}\,dr = t^n,$$

which, when the nth root is taken, gives the first part of (ii).

For the second part of (ii) we use (22) and the estimate above to get

$$\begin{aligned} f'_{p,n}(s) &= e^{-s^p}(2\gamma_p)^{-n}\left(\frac{s}{f_{p,n}(s)}\right)^{n-1} \le e^{-s^p}(2\gamma_p)^{-n}(e^{s^p/n}2\gamma_p)^{n-1} \\ &= e^{-s^p/n}(2\gamma_p)^{-1} \le (2\gamma_p)^{-1} \le 1. \end{aligned}$$

To show (iii) first notice that by (22) and (ii),

$$\frac{tf'_{p,n}(t)}{f_{p,n}(t)} = \left(\frac{t}{f_{p,n}(t)}\right)^n e^{-t^p}(2\gamma_p)^{-n} \le (e^{t^p/n}2\gamma_p)^n e^{-t^p}(2\gamma_p)^{-n} = 1,$$

thus $f_{p,n}(t)/t - f'_{p,n}(t) \ge 0$. Moreover by (ii), $f_{p,n}(t)/t - f'_{p,n}(t) \le f_{p,n}(t)/t \le 1$, so we may assume that $2pt^p/n \le 1$. By (21) and Lemma 5.7 we obtain

$$(2\gamma_p)^n f_{p,n}^n(t) = \frac{n}{p}\int_0^{t^p} e^{-u}u^{n/p-1}du \le e^{-t^p}t^n\left(1 + 2\,\frac{pt^p}{n}\right).$$

Thus using again (22) and (ii) we get

$$\frac{f_{p,n}(t)}{t} - f'_{p,n}(t) = \frac{f_{p,n}(t)}{t}\Big(1 - \frac{e^{-t^p}t^n}{(2\gamma_p)^n f^n_{p,n}(t)}\Big) \le 1 - \Big(1 + 2\,\frac{pt^p}{n}\Big)^{-1} \le \frac{2pt^p}{n}.$$

By (iii) we get $(f_{p,n}(t)/t)' \le 0$, which proves the first part of (iv). For the second part suppose that $s > t > 0$. Then

$$0 \le \frac{f_{p,n}(t)}{t} - \frac{f_{p,n}(s)}{s} \le \frac{f_{p,n}(t)}{t} - \frac{f_{p,n}(t)}{s} = \frac{s-t}{st}\, f_{p,n}(t) \le \frac{s-t}{s}. \ \blacksquare$$

The next two propositions apply the idea given in Proposition 5.8. The first of them may also be deduced (with a different constant) from the more general fact proved in [22].

Proposition 5.9. *For any $x, y \in \mathbb{R}^n$ we have $\|T_{p,n}x - T_{p,n}y\|_p \le 2\|x-y\|_p$.*

Proof. Assume $s := \|x\|_p \ge t := \|y\|_p$. We apply Proposition 5.8 to get

$$\begin{aligned}
\|T_{p,n}x - T_{p,n}y\|_p &= \Big(\sum_i |(T_{p,n}x)_i - (T_{p,n}y)_i|^p\Big)^{1/p} \\
&= \Big(\sum_i \Big|\frac{f_{p,n}(t)}{t}\,(x_i - y_i) + \Big(\frac{f_{p,n}(s)}{s} - \frac{f_{p,n}(t)}{t}\Big)x_i\Big|^p\Big)^{1/p} \\
&\le \Big(\sum_i \Big(|x_i - y_i| + \frac{|s-t|}{s}\,|x_i|\Big)^p\Big)^{1/p} \\
&\le \Big(\sum_i |x_i - y_i|^p\Big)^{1/p} + \frac{|s-t|}{s}\Big(\sum_i |x_i|^p\Big)^{1/p} \\
&= \|x-y\|_p + \frac{|\,\|x\|_p - \|y\|_p|}{\|x\|_p}\,\|x\|_p \le 2\|x-y\|_p. \ \blacksquare
\end{aligned}$$

Proposition 5.10. *Let $u \ge 0$, $p \in [1,2]$ and $x \in \mathbb{R}^n$ be such that $\|x\|_2 n^{-1/2} \le u\|x\|_p n^{-1/p}$. Then*

$$\|T_{p,n}x - T_{p,n}y\|_2 \le (1+u)\|x-y\|_2 \quad \text{for all } y \in \mathbb{R}^n.$$

Proof. Let $s = \|x\|_p$ and $t = \|y\|_p$. We use Proposition 5.8 as in the proof of Proposition 5.9, and the Hölder inequality, to obtain

$$\begin{aligned}
\|T_{p,n}x - T_{p,n}y\|_2 &\le \Big(\sum_i \Big(|x_i - y_i| + \frac{|s-t|}{s}\,|x_i|\Big)^2\Big)^{1/2} \\
&\le \|x-y\|_2 + \frac{|s-t|}{s}\,\|x\|_2 \le \|x-y\|_2 + \frac{\|x-y\|_p}{\|x\|_p}\,\|x\|_2 \\
&\le \|x-y\|_2 + \frac{\|x\|_2}{\|x\|_p}\,n^{1/p-1/2}\|x-y\|_2 \le (1+u)\|x-y\|_2. \ \blacksquare
\end{aligned}$$

The second transport we will use is a simple product transport which transports the measure ν_p^n onto ν_q^n. We shall be particularly interested in the cases $p = 1$ and $p = 2$, but most of the results can be stated in the more general setting.

DEFINITION 5.11. For $1 \le p, q < \infty$ we define the map $w_{p,q} : \mathbb{R} \to \mathbb{R}$ by

$$\frac{1}{\gamma_p} \int_x^\infty e^{-t^p}\,dt = \frac{1}{\gamma_q} \int_{w_{p,q}(x)}^\infty e^{-t^q}\,dt. \tag{23}$$

We write v_p for $w_{p,1}$. We also define $W_{p,q}^n : \mathbb{R}^n \to \mathbb{R}^n$ by

$$W_{p,q}^n(x_1, \ldots, x_n) = (w_{p,q}(x_1), \ldots, w_{p,q}(x_n)).$$

Note that $w_{p,q}^{-1} = w_{q,p}$ and $(W_{p,q}^n)^{-1} = W_{q,p}^n$. Differentiating equality (23) we get

$$w'_{p,q}(x) = \frac{\gamma_q}{\gamma_p}\, e^{-x^p + w_{p,q}^q(x)}. \tag{24}$$

As $W_{p,q}$ is a product transport, we will spend most of our time estimating the properties of the one-dimensional version $w_{p,q}$. We will prove that $w_{p,q}$ behaves very much like $x^{p/q}$ for large x, and is more or less linear for small $|x|$. We begin with the bound for $q = 1$.

LEMMA 5.12. *For $p \ge 1$ we have*

(i) $v_p(x) \ge x^p + \ln(p\gamma_p x^{p-1})$ *and* $v'_p(x) \ge p x^{p-1}$ *for* $x \ge 0$,
(ii) $v_p(x) \le e + x^p + \ln(p\gamma_p x^{p-1})$ *and* $v'_p(x) \le e^e p x^{p-1}$ *for* $x \ge 1$,
(iii) $|v_p(x) - v_p(y)| \ge 2^{1-p}|x - y|^p$.

Proof. Note that $\gamma_1 = 1$. For $x \ge 0$, we have

$$e^{-v_p(x)} = \frac{1}{\gamma_p} \int_x^\infty e^{-t^p}\,dt \le \frac{1}{p\gamma_p x^{p-1}} \int_x^\infty p t^{p-1} e^{-t^p}\,dt = \frac{e^{-x^p}}{p\gamma_p x^{p-1}} \tag{25}$$

and for $x \ge 1$, since $(1 + r/p)^p \le e^r \le 1 + er$ for $r \in [0, 1]$, we get

$$e^{-v_p(x)}\,dt \ge \frac{1}{\gamma_p} \int_x^{x + x^{1-p}/p} e^{-t^p}\,dt \ge \frac{1}{p\gamma_p x^{p-1}}\, e^{-(x + x^{1-p}/p)^p} \ge e^{-e}\, \frac{e^{-x^p}}{p\gamma_p x^{p-1}}.$$

Notice that by (24), $v'_p(x) = e^{-x^p + v_p(x)}/\gamma_p$, hence we may estimate v'_p using the just derived bounds on v_p.

The lower bound on v'_p yields $|v_p(x) - v_p(y)| \ge |x - y|^p$ for $x, y \ge 0$. The same estimate holds for $x, y \le 0$, since v_p is odd. Finally, for $x \ge 0 \ge y$ we have

$$|v_p(x) - v_p(y)| = |v_p(x)| + |v_p(y)| \ge |x|^p + |y|^p \ge 2^{1-p}|x - y|^p. \blacksquare$$

The previous lemma shows that for $x \geq 1$, v_p and x^p have comparable derivatives. One could hope that $w_{p,q}$ and $x^{p/q}$ behave in the same fashion. Unfortunately things are not so bright for the case of $q = 2$: while it is true that $w_{p,2}$ is larger than $x^{p/2}$, things start getting messy around $x = 1$ when one considers the derivative. The following estimates are not optimal, but strong enough for our purposes.

Lemma 5.13.

(i) *For* $p \geq q \geq 1$, $|w_{p,q}(x)| \geq |x|^{p/q}$ *and* $w'_{p,q}(x) \geq \gamma_q/\gamma_p \geq 1/2$.
(ii) *For* $p \geq 2$, $w'_{p,2}(x) \geq \frac{1}{8}\sqrt{p}\,|x|^{p/2-1}$.

Proof. Since the function $w_{p,q}$ is odd, we may and will assume that $x \geq 0$.
(i) By the monotonicity of $u^{p/q-1}$ on $[0,\infty)$, we have

$$\frac{1}{\gamma_p}\int_x^\infty e^{-t^p}\,dt = \frac{1}{\gamma_q}\int_{w_{p,q}(x)}^\infty e^{-t^q}\,dt = \frac{\int_{w_{p,q}(x)}^\infty e^{-t^q}\,dt}{\int_0^\infty e^{-t^q}\,dt}$$
$$= \frac{\int_{w_{p,q}(x)^{q/p}}^\infty u^{p/q-1}e^{-u^p}\,du}{\int_0^\infty u^{p/q-1}e^{-u^p}\,du}$$
$$\geq \frac{\int_{w_{p,q}(x)^{q/p}}^\infty e^{-u^p}\,du}{\int_0^\infty e^{-u^p}\,du} = \frac{1}{\gamma_p}\int_{w_{p,q}(x)^{q/p}}^\infty e^{-u^p}\,du,$$

thus $w_{p,q}(x)^{q/p} \geq x$ and $w_{p,q}(x) \geq x^{p/q}$. Formula (24) gives $w'_{p,q}(x) \geq \gamma_q/\gamma_p \geq 1/2$.

(ii) We begin with the following Gaussian tail estimate for $z > 0$:

$$\int_z^\infty e^{-t^2}\,dt \geq \frac{1}{2\sqrt{z^2+1}}\,e^{-z^2}. \tag{26}$$

We have equality when $z \to \infty$, and direct calculation shows the derivative of the left-hand side is no larger than the derivative of the right-hand side.

Let $\kappa := 4\sqrt{\pi}$. We will now show that for all $x > 0$ and $p \geq 2$,

$$w_{p,2}(x) \geq u_p(x) := \max\Big\{\sqrt{\pi}\,x/2, \sqrt{(x^p + \ln(\sqrt{p}\,x^{p/2-1}/\kappa))_+}\Big\}. \tag{27}$$

Suppose on the contrary that $w_{p,2}(x) < u_p(x)$ for some $p \geq 2$ and $x > 0$. Note that by (i) we have $w'_{p,2} \geq \gamma_2/\gamma_p \geq \gamma_2 = \sqrt{\pi}/2$. Thus $u_p(x)$ is equal to the second part of the maximum. This in particular implies that $x \geq 2/3$, since for $x < 2/3$ we have

$$x^p + \ln(\sqrt{p}\,x^{p/2-1}/\kappa) \leq \frac{4}{9} + \left(\frac{p}{2}-1\right)\ln\frac{2}{3} + \frac{\sqrt{p}}{\kappa} - 1 \leq 0.$$

Therefore $u_p(x) \geq \sqrt{\pi}\, x/2 \geq 1/\sqrt{3}$. Now by (25), (23) and (26),

$$\sqrt{\pi}\,\frac{1}{px^{p-1}}\,e^{-x^p} \geq \frac{\gamma_2}{\gamma_p}\,\frac{1}{px^{p-1}}\,e^{-x^p} \geq \frac{\gamma_2}{\gamma_p}\int_x^\infty e^{-t^p}\,dt = \int_{w_{p,2}(x)}^\infty e^{-t^2}\,dt$$

$$> \int_{u_p(x)}^\infty e^{-t^2}\,dt \geq \frac{1}{2\sqrt{u_p^2(x)+1}}\,e^{-u_p^2(x)} \geq \frac{1}{4u_p(x)}\,e^{-u_p^2(x)}$$

$$= \frac{1}{4u_p(x)}\,e^{-(x^p+\ln(\sqrt{p}\,x^{p/2-1}/\kappa))} = \frac{\sqrt{\pi}}{\sqrt{p}\,u_p(x)}\,x^{1-p/2}e^{-x^p}.$$

After simplifying this gives $u_p(x) > \sqrt{p}\,x^{p/2}$. Hence

$$px^p < u_p^2(x) = x^p + \frac{1}{2}\ln(px^p) + \ln\frac{1}{\kappa x} \leq \frac{p}{2}\,x^p + \frac{1}{2}\,px^p = px^p,$$

which is impossible. This contradiction shows that (27) holds.

Thus we have $w_{p,2}(x) \geq u_p(x)$ and by (24) we obtain

$$w'_{p,2}(x) \geq \frac{\gamma_2}{\gamma_p}\,e^{-x^p+u_p^2(x)} \geq \frac{\sqrt{\pi}}{2}\,\frac{1}{\kappa}\,\sqrt{p}\,x^{p/2-1} = \frac{1}{8}\sqrt{p}\,x^{p/2-1}. \ \blacksquare$$

Remark 5.14. By taking

$$u_p(x) = \max\Big\{\sqrt{\pi}\,x/2,\ \sqrt{(x^p+\ln(px^{p/2-1}/(\kappa\ln p)))_+}\Big\}$$

for sufficiently large κ and estimating carefully one may arrive at the bound $w'_{p,2}(x) \geq C^{-1}px^{p/2-1}/\ln p$. One cannot, however, obtain a bound of the order of $px^{p/2-1}$.

Proposition 5.15. *For $p \geq q \geq 1$ we have*

(i) $\nu_p^n(W_{q,p}^n(A)) = \nu_q^n(A)$ *for* $A \in \mathcal{B}(\mathbb{R}^n)$,
(ii) $|w_{q,p}(x) - w_{q,p}(y)| \leq 2|x-y|$ *for* $x, y \in \mathbb{R}$,
(iii) *for* $x, y \in \mathbb{R}^n$ *and* $r \geq 1$,
$$\|W_{q,p}^n(x) - W_{q,p}^n(y)\|_r \leq 2\|x-y\|_r,$$
(iv) *for* $x, y \in \mathbb{R}$,
$$|w_{1,p}(x) - w_{1,p}(y)| \leq 2\min(|x-y|, |x-y|^{1/p}) \leq 2|x-y|^{1/q},$$
(v) $\|W_{1,p}^n(x) - W_{1,p}^n(y)\|_q^q \leq 2^q\|x-y\|_1$ *for* $x, y \in \mathbb{R}^n$.

Proof. Property (i) follows from the definition of $w_{q,p}$ and $W_{q,p}^n$. Since $w_{q,p} = w_{p,q}^{-1}$ we get (ii) by Lemma 5.13(i). Property (iii) is a direct consequence of (ii).

By Lemma 5.12(iii),

$$|w_{1,p}(x) - w_{1,p}(y)| = |v_p^{-1}(x) - v_p^{-1}(y)| \leq 2^{1-1/p}|x-y|^{1/p}.$$

The above inequality together with (ii) gives (iv), and (iv) yields (v). $\blacksquare$

Let us summarize the facts proved so far. We have IC for the measure ν_p^n for any p, thus if the radial transport $T_{p,n}$ were Lipschitz with respect to the second and pth norms, we could transport IC to $\mu_{p,n}$. However, while $T_{p,n}$ is Lipschitz with respect to the pth norm, it is Lipschitz only for points not very far from the origin in the second norm (Proposition 5.10 proves this for $p \le 2$, a similar problem occurs when $p \ge 2$). Thus we will have to deal with the point farther away from the origin separately.

For $p \le 2$ we shall use the results from Section 4. We can, fortunately, transport them easily to ν_p^n, as the product transport $W_{1,p}^n$ is Lipschitz with respect to any norm (in particular the second norm), and also contracts the first norm to the pth norm.

For larger p it turns out it will suffice to combine the transports we already have. While $T_{p,n}$ is not Lipschitz in the second norm far away from zero, it turns out that $W_{1,p}^n$ contracts the points far away from zero strongly enough to compensate for this, and the composition is Lipschitz. To check this we will bound the norm of the derivative matrix, using the estimates for the derivatives of the transports given above.

To this end we define the following transport from the exponential measure ν^n to $\mu_{p,n}$ for $p \ge 2$:

Definition 5.16. For $n \in \mathbb{N}$ and $2 \le p < \infty$ we define the map $S_{p,n} \colon \mathbb{R}^n \to \mathbb{R}^n$ by $S_{p,n}(x) := T_{p,n}(W_{1,p}^n(x))$.

This transport satisfies the following bound:

Proposition 5.17. *We have* $\|S_{p,n}(x) - S_{p,n}(y)\|_2 \le 4\|x-y\|_2$ *for all* $x, y \in \mathbb{R}^n$ *and* $p \ge 2$.

Proof. It is enough to show that $\|DS_{p,n}(x)\| \le 4$, where $DS_{p,n}$ is the derivative matrix, and the norm is the operator norm from ℓ_2^n into ℓ_2^n.

Let $s = \|W_{1,p}^n(x)\|_p$. By direct calculation we get

$$\frac{(\partial S_{p,n})_j}{\partial x_i}(x) = \frac{\delta_{ij} f_{p,n}(s) w_{1,p}'(x_i)}{s} + \alpha(s) w_{1,p}(x_j)\beta(x_i), \tag{28}$$

where

$$\alpha(s) := s^{-p-1}(s f_{p,n}'(s) - f_{p,n}(s)), \qquad \beta(t) := |w_{1,p}(t)|^{p-1}\operatorname{sgn}(w_{1,p}(t)) w_{1,p}'(t).$$

Thus we can bound

$$\|DS_{p,n}(x)\| \le \frac{f_{p,n}(s)}{s} \max_i |w_{1,p}'(x_i)| + |\alpha(s)|\, \|W_{1,p}^n(x)\|_2 \Big(\sum_{i=1}^n \beta^2(x_i)\Big)^{1/2}.$$

Since $w_{1,p} = w_{p,1}^{-1}$, Proposition 5.13(i) implies $|w_{1,p}'(x_j)| \le 2$, while by Proposition 5.8 we have $f_{p,n}(s)/s \le 1$. Thus the first summand can be bounded by 2.

For the second summand note that by Proposition 5.8(iii),

$$|\alpha(s)| = s^{-p}|f'_{p,n}(s) - f_{p,n}(s)/s| \le s^{-p}\min\{1, 2ps^p/n\}. \tag{29}$$

Moreover, $\|W^n_{1,p}(x)\|_2 \le n^{1/2-1/p}s$ by the Hölder inequality and

$$|\beta(t)| = |w_{1,p}(t)|^{p-1}|w'_{1,p}(t)| = \frac{|w_{1,p}(t)|^{p-1}}{v'_p(w_{1,p}(t))} \le \frac{1}{p}$$

by Lemma 5.12. Thus

$$\begin{aligned}\|DS_{p,n}(x)\| &\le 2 + s^{-p}\min\{1, 2ps^p/n\}n^{1/2-1/p}sn^{1/2}/p \\ &\le 2 + 2sn^{-1/p}\min\{ns^{-p}, 1\} \le 4. \ \blacksquare\end{aligned}$$

Recall our aim is to transport the enlargement by $tB_1^n + \sqrt{t}\,B_2^n$ to the enlargement by $t^{1/p}B_p^n \cap \sqrt{t}\,B_2^n$. This means that any vector in either the tB_1^n ball or $\sqrt{t}\,B_2^n$ ball should be mapped by $S_{p,n}$ both into $\sqrt{t}\,B_2^n$ and $t^{1/p}B_p^n$. We know that B_2^n map to B_2^n from the above proposition. Both B_2^n and B_1^n map to B_p^n when transported by W_p^n, and $T_{p,n}$ is Lipschitz with respect to the pth norm, thus it remains to check what happens to vectors from tB_1^n with respect to the second norm. Here direct derivation would be more involved, thus we will change one coordinate at a time and track the changes in the second norm:

Proposition 5.18. *For any $y, z \in \mathbb{R}^n$ and $p \ge 2$ we have*

$$\|S_{p,n}(y) - S_{p,n}(z)\|_2 \le \|W^n_{1,p}(y) - W^n_{1,p}(z)\|_2 + 2n^{-1/2}\|y - z\|_1.$$

Proof. Let $u_i(t) = (y_1, y_2, \ldots, y_{i-1}, t, z_{i+1}, z_{i+2}, \ldots, z_n)$ for $i = 1, \ldots, n$. Note that $u_i(y_i) = u_{i+1}(z_{i+1})$, $u_1(z_1) = z$ and $u_n(y_n) = y$, hence

$$S_{p,n}(z) - S_{p,n}(y) = \sum_{i=1}^{n}(S_{p,n}(u_i(z_i)) - S_{p,n}(u_i(y_i))).$$

Let $s_i(t) := \|w_{1,p}(u_i(t))\|_p$. By vector-valued integration and (28) we get

$$S_{p,n}(u_i(z_i)) - S_{p,n}(u_i(y_i)) = \int_{y_i}^{z_i} \frac{\partial S_{p,n}}{\partial x_i}(u_i(t))\,dt = a_i + b_i,$$

where

$$a_i := \int_{y_i}^{z_i} \frac{f_{p,n}(s_i(t))}{s_i(t)}\, w'_{1,p}(t)e_i\,dt, \qquad b_i := \int_{y_i}^{z_i} \alpha(s_i(t))\beta(t)W^n_{1,p}(u_i(t))\,dt.$$

As in the proof of Proposition 5.17 we show that

$$\|\alpha(s_i(t))\beta(t)W^n_{1,p}(u_i(t))\|_2 \le 2n^{-1/2}s_i(t)n^{-1/p}\min\{ns_i(t)^{-p}, 1\} \le 2n^{-1/2},$$

thus

$$\Big\|\sum_{i=1}^{n} b_i\Big\|_2 \le \sum_{i=1}^{n}\|b_i\|_2 \le 2n^{-1/2}\sum_{i=1}^{n}|y_i - z_i| = 2n^{-1/2}\|y - z\|_1.$$

To deal with the sum of a_i's we notice that, since $f_{p,n}(s)/s \le 1$ and $w'_{1,p}(x) \ge 0$,

$$\Big|\Big\langle \sum_j a_j, e_i\Big\rangle\Big| = |\langle a_i, e_i\rangle| = \Big|\int_{y_i}^{z_i} \frac{f_{p,n}(s_i(t))}{s_i(t)}\, w'_{1,p}(t)\,dt\Big| \le \Big|\int_{y_i}^{z_i} w'_{1,p}(t)\,dt\Big| = |w_{1,p}(z_i) - w_{1,p}(y_i)|.$$

Thus

$$\Big\|\sum_i a_i\Big\|_2 \le \Big\|\sum_i (w_{1,p}(z_i) - w_{1,p}(y_i))e_i\Big\|_2 = \|W^n_{1,p}(z) - W^n_{1,p}(y)\|_2. \ \blacksquare$$

Having these facts, we can put them together in the following corollary:

Corollary 5.19. *If $x - y \in tB^n_1 + t^{1/2}B^n_2$ for some $t > 0$, then for all $p \ge 2$, $S_{p,n}(x) - S_{p,n}(y) \in 8(t^{1/2}B^n_2 \cap t^{1/p}B^n_p)$.*

Proof. Fix x, y with $x - y \in tB^n_1 + t^{1/2}B^n_2$. By Proposition 5.15(iv),

$$\begin{aligned}\|W^n_{1,p}(x) - W^n_{1,p}(y)\|^p_p &= \sum_i |w_{1,p}(x_i) - w_{1,p}(y_i)|^p \\ &\le 2^p \sum_i \min(|x_i - y_i|^p, |x_i - y_i|) \\ &\le 2^p \sum_i \min(|x_i - y_i|^2, |x_i - y_i|) \le 2^{p+2}t.\end{aligned}$$

Thus by Proposition 5.9,

$$\|S_{p,n}(x) - S_{p,n}(y)\|_p \le 2\|W^n_{1,p}(x) - W^n_{1,p}(y)\|_p \le 8t^{1/p}.$$

By Hölder's inequality, $\|S_{p,n}(x) - S_{p,n}(y)\|_2 \le n^{1/2-1/p}\|S_{p,n}(x) - S_{p,n}(y)\|_p \le 8t^{1/2}$ for $t \ge n$.

Assume now that $t \le n$. Let z be such that $x - z \in t^{1/2}B^n_2$ and $z - y \in tB^n_1$. Then $S_{p,n}(x) - S_{p,n}(z) \in 4t^{1/2}B^n_2$ by Proposition 5.17, and $\|W^n_{1,p}(z) - W^n_{1,p}(y)\|_2 \le 2\sqrt{t}$ by Proposition 5.15(v). Thus by Proposition 5.18,

$$\|S_{p,n}(y) - S_{p,n}(z)\|_2 \le 2t^{1/2} + 2n^{-1/2}t \le 4t^{1/2}.$$

Hence $S_{p,n}(x) - S_{p,n}(y) \in 8t^{1/2}B^n_2$. $\blacksquare$

The last function we define transports the Gaussian measure ν^n_2 to $\mu_{p,n}$ for $p \ge 2$.

Definition 5.20. For $n \in \mathbb{N}$ and $2 \le p < \infty$ we define $\widetilde{S}_{p,n} \colon \mathbb{R}^n \to \mathbb{R}^n$ by $\widetilde{S}_{p,n}(x) := T_{p,n}(W^n_{2,p}(x))$.

We argue in much the same way as in the proof of Proposition 5.17, estimating the norm of the derivative matrix:

PROPOSITION 5.21. *We have* $\|\widetilde{S}_{p,n}(x)-\widetilde{S}_{p,n}(y)\|_2 \le 14\|x-y\|_2$ *for all* $x,y\in\mathbb{R}^n$ *and* $p\ge 2$.

Proof. We need to show that $\|D\widetilde{S}_{p,n}(x)\|\le 14$. Direct calculation gives

$$\frac{(\partial \widetilde{S}_{p,n})_j}{\partial x_i}(x)=\frac{\delta_{ij}f_{p,n}(\widetilde{s})w'_{2,p}(x_i)}{\widetilde{s}}+\alpha(\widetilde{s})w_{2,p}(x_j)\widetilde{\beta}(x_i), \tag{30}$$

where $\widetilde{s}=\|W^n_{2,p}(x)\|_p$ and

$$\alpha(s):=s^{-p-1}(sf'_{p,n}(s)-f_{p,n}(s)),\qquad \widetilde{\beta}(t):=|w_{2,p}(t)|^{p-1}\mathrm{sgn}(w_{2,p}(t))w'_{2,p}(t).$$

Thus we can bound

$$\|D\widetilde{S}_{p,n}(x)\|\le \frac{f_{p,n}(\widetilde{s})}{\widetilde{s}}\max_i|w'_{2,p}(x_i)| + |\alpha(\widetilde{s})|\,\|W^n_{2,p}(x)\|_2\Big(\sum_{i=1}^n\widetilde{\beta}^2(x_i)\Big)^{1/2}. \tag{31}$$

The first summand is bounded by 2 as in the proof of Proposition 5.17. Since $w_{2,p}=w^{-1}_{p,2}$ we get by Lemma 5.13(ii),

$$|\widetilde{\beta}(x)|=|w_{2,p}(x)|^{p-1}|w'_{2,p}(x)|=\frac{|w_{2,p}(x)|^{p-1}}{w'_{p,2}(w_{2,p}(x))}\le\frac{8}{\sqrt{p}}\,|w_{2,p}(x)|^{p/2},$$

hence

$$\Big(\sum_{i=1}^n\widetilde{\beta}^2(x_i)\Big)^{1/2}\le\frac{8}{\sqrt{p}}\,\widetilde{s}^{p/2}.$$

Using (29) and $\|W^n_{2,p}(x)\|_2\le n^{1/2-1/p}\widetilde{s}$ we bound the second summand in (31) by

$$\begin{aligned}\widetilde{s}^{-p}\min\Big\{1,\frac{2p\widetilde{s}^p}{n}\Big\}n^{1/2-1/p}\widetilde{s}\,\frac{8}{\sqrt{p}}\,\widetilde{s}^{p/2}&=8p^{-1/p}\min\{u^{-1/2},2u^{1/2}\}u^{1/p}\\&\le 8\sqrt{2}\le 12,\end{aligned}$$

where $u:=p\widetilde{s}^p/n$. ∎

5.2. *Applying* ν_1 *results*: $p\le 2$. In this subsection we need to put carefully together Theorem 4.6, which for a set far away from the origin allows us to either increase its mass or push it closer to the origin by adding a tB^n_1 ball, with the transport $T_{p,n}$, which is Lipschitz close to the origin, and thus will allow us to transport concentration inequalities from ν^n_p to $\mu_{p,n}$ for sets close to the origin.

We start with the version of Theorem 4.6 for ν_p, which is a direct transportation of the ν_1 case.

LEMMA 5.22. *For any* $A\in\mathcal{B}(\mathbb{R}^n)$, $p\in[1,2]$ *and* $t\ge 1$,

- $\nu^n_p(A+20t^{1/p}B^n_p)\ge e^t\nu^n_p(A)$ *or*
- $\nu^n_p((A+20t^{1/p}B^n_p)\cap 100\sqrt{n}\,B^n_2)\ge \frac{1}{2}\nu^n_p(A)$.

Proof. We will use the transport $W^n_{1,p}$ from ν^n to ν^n_p. Proposition 5.15(v) gives $\|W^n_{1,p}(x) - W^n_{1,p}(y)\|^p_p \le 2^p\|x-y\|_1$. By Remark 5.5 this means that $A+2(10t)^{1/p}B^n_p \supset W^n_{1,p}(W^n_{p,1}(A)+10tB^n_1)$. Fix $t \ge 1$ and apply Theorem 4.6 to $W^n_{p,1}(A)$ and $10t$. If the second case of Theorem 4.6 occurs, we have

$$\begin{aligned}\nu^n_p(A+20t^{1/p}B^n_p) &\ge \nu^n_p(W^n_{1,p}(W^n_{p,1}(A)+10tB^n_1)) = \nu^n(W^n_{p,1}(A)+10tB^n_1)\\ &\ge e^t\nu^n(W^n_{p,1}(A)) = e^t\nu^n_p(A).\end{aligned}$$

If the first case occurs, then due to Proposition 5.15(iii) we have $\|W^n_{1,p}(x)\|_2 \le 2\|x\|_2$, so $2\alpha B^n_2 \supset W^n_{1,p}(\alpha B^n_2)$ for any $\alpha > 0$. Thus

$$\begin{aligned}\nu^n_p((A+20t^{1/p}B^n_p)\cap 100\sqrt{n}\,B^n_2) &\ge \nu^n_p(W^n_{1,p}(W^n_{p,1}(A)+10tB^n_1)\cap 100\sqrt{n}\,B^n_2)\\ &= \nu^n_p(W^n_{1,p}((W^n_{p,1}(A)+10tB^n_1)\cap W^n_{p,1}(100\sqrt{n}\,B^n_2)))\\ &\ge \nu^n_p(W^n_{1,p}((W^n_{p,1}(A)+10tB^n_1)\cap 50\sqrt{n}\,B^n_2))\\ &= \nu^n((W^n_{p,1}(A)+10tB^n_1)\cap 50\sqrt{n}\,B^n_2)\\ &\ge \tfrac{1}{2}\nu^n(W^n_{p,1}(A)) = \tfrac{1}{2}\nu^n_p(A). \ \blacksquare\end{aligned}$$

Now recall what IC (or rather, CI) implies for ν^n_p.

LEMMA 5.23. *There exists a constant C such that for any $p \in [1,2]$, $t > 0$ and $n \in \mathbb{N}$ we have*

$$\nu^n_p(A + C(t^{1/p}B^n_p + t^{1/2}B^n_2)) \ge \min\{1/2, e^t\nu^n_p(A)\}.$$

Proof. Corollary 5.2 gives $B_s(\nu^n_p) \subset C(s^{1/p}B^n_p + s^{1/2}B^n_2)$ for $s > 0$. By Corollary 2.19, ν^n_p satisfies IC(48), which, due to Proposition 2.4, implies $\nu^n_p(A + 48B_{2t}(\nu^n_p)) \ge \min\{1/2, e^t\nu^n_p(A)\}$ for any Borel set A. Thus we have

$$\nu^n_p(A + 96C(t^{1/p}B^n_p + t^{1/2}B^n_2)) \ge \min\{1/2, e^t\nu^n_p(A)\}. \ \blacksquare$$

For technical reasons we will need to discard the set of points where the pth norm is small to use Proposition 5.10. The following proposition uses a simple argument to ensure that this set is small (of the order of c^{-n}).

PROPOSITION 5.24. *For any $\alpha > 1$ there exists a constant $c(\alpha)$ such that for any $n \in \mathbb{N}$ and $p \ge 1$ we have*

$$\nu^n_p(\{x\colon \|x\|_p < c(\alpha)n^{1/p}\}) < \alpha^{-n}.$$

Proof. We have

$$\begin{aligned}\nu^n_p(\{x\colon \|x\|_p < c(\alpha)n^{1/p}\}) &= \frac{n}{\Gamma(1+n/p)}\int_0^{c(\alpha)n^{1/p}} e^{-r^p}r^{n-1}\,dr\\ &\le \frac{n}{\Gamma(1+n/p)}\int_0^{c(\alpha)n^{1/p}} r^{n-1} = \frac{c(\alpha)^n n^{n/p}}{\Gamma(1+n/p)} \le (Cc(\alpha))^n,\end{aligned}$$

where in the last step we use the Stirling approximation and C as always denotes a universal constant. Thus it is enough to take $c(\alpha) < (C\alpha)^{-1}$. ∎

THEOREM 5.25. *There exists a universal constant C such that $\mu_{p,n}$ satisfies* CI(C) *and* IC(C) *for any $p \in [1,2]$ and $n \in \mathbb{N}$.*

Proof. By Propositions 2.7, 3.12, 3.5 and 5.3 it is enough to show

$$\mu_{p,n}(A + C(t^{1/p}B_p^n + t^{1/2}B_2^n)) \geq \min\{1/2, e^t\mu_{p,n}(A)\} \tag{32}$$

for $1 \leq t \leq n$ and $\mu_{p,n}(A) \geq e^{-n}$.

Recall that $T_{p,n}$ denotes the map transporting ν_p^n to $\mu_{p,n}$. Apply Lemma 5.22 to $T_{p,n}^{-1}(A)$ and t. If the first case of Lemma 5.2 occurs, we have

$$\nu_p^n(T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \geq e^t\nu_p^n(T_{p,n}^{-1}(A)) = e^t\mu_{p,n}(A).$$

Proposition 5.9 gives $\|T_{p,n}x - T_{p,n}y\|_p \leq 2\|x-y\|_p$, thus by Remark 5.5,

$$\begin{aligned}\mu_{p,n}(A + 40t^{1/p}B_p^n) &= \nu_p^n(T_{p,n}^{-1}(A + 40t^{1/p}B_p^n)) \\ &\geq \nu_p^n(T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \geq e^t\mu_{p,n}(A)\end{aligned}$$

and we obtain (32) in this case.

Hence we may assume that the second case of Lemma 5.22 holds, that is,

$$\nu_p^n(A') \geq \tfrac{1}{2}\nu_p^n(T_{p,n}^{-1}(A)) = \tfrac{1}{2}\mu_{p,n}(A),$$

where

$$A' := (T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \cap 100\sqrt{n}\,B_2^n.$$

In particular, $\nu_p^n(A') \geq e^{-n}/2$. Let

$$A'' := A' \cap \{x\colon \|x\|_p \geq \widetilde{c}n^{1/p}\},$$

where $\widetilde{c} = c(4e)$ is a constant given by Proposition 5.24 for $\alpha = 4e$. Then

$$\nu_p^n(A'') \geq \nu_p^n(A') - (4e)^{-n} \geq \tfrac{1}{2}\nu_p^n(A') \geq \tfrac{1}{4}\mu_{p,n}(A).$$

We apply Lemma 5.23 for A'' and $4t$ to get

$$\begin{aligned}\mu_{p,n}(T_{p,n}(A'' + 4C(t^{1/p}B_p^n + t^{1/2}B_2^n))) &\geq \nu_p^n(A'' + C((4t)^{1/p}B_p^n + (4t)^{1/2}B_2^n)) \\ \geq \min\{1/2, e^{4t}\nu_p^n(A'')\} &\geq \min\{1/2, e^{4t}\mu_{p,n}(A)/4\} \geq \min\{1/2, e^t\mu_{p,n}(A)\}.\end{aligned}$$

Proposition 5.9 and Remark 5.5 imply

$$T_{p,n}(A'' + 4Ct^{1/2}B_2^n + 4Ct^{1/p}B_p^n) \subset T_{p,n}(A'' + 4Ct^{1/2}B_2^n) + 8Ct^{1/p}B_p^n.$$

Moreover, for $x \in A''$ we have $\|x\|_2 \leq 100\sqrt{n}$ and $\|x\|_p \geq \widetilde{c}n^{1/p}$. Thus $n^{-1/2}\|x\|_2 \leq 100\widetilde{c}^{-1}n^{-1/p}\|x\|_p$, so we can use Proposition 5.10 along with Remark 5.5 to get

$$T_{p,n}(A'' + 4Ct^{1/2}B_2^n) \subset T_{p,n}(A'') + \widetilde{C}t^{1/2}B_2^n.$$

Proposition 5.9, Remark 5.5 and the definitions of A' and A'' yield

$$T_{p,n}(A'') \subset T_{p,n}(A') \subset T_{p,n}(T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \subset A + 40t^{1/p}B_p^n.$$

Putting the four estimates together, we can write

$$\begin{aligned}\mu_{p,n}(A+(40+8C)t^{1/p}B_p^n+\widetilde{C}t^{1/2}B_2^n)&\\ \geq \mu_{p,n}(T_{p,n}(A'')+\widetilde{C}t^{1/2}B_2^n+8Ct^{1/p}B_p^n)&\\ \geq \mu_{p,n}(T_{p,n}(A''+4Ct^{1/2}B_2^n)+8Ct^{1/p}B_p^n)&\\ \geq \mu_{p,n}(T_{p,n}(A''+4C(t^{1/p}B_p^n+t^{1/2}B_2^n)))&\geq \min\{1/2,e^t\mu_{p,n}(A)\},\end{aligned}$$

which gives (32) in the second case and ends the proof of CI. IC follows directly from Corollary 3.14. ∎

5.3. *The easy case*: $p \geq 2$. This case will follow easily from the exponential case and the facts from Subsection 5.1.

THEOREM 5.26. *There exists a universal constant C such that for any $A \subset \mathbb{R}^n$, any $t, n \geq 1$ and $p \geq 2$ we have*

$$\mu_{p,n}(A+C(t^{1/p}B_p^n \cap t^{1/2}B_2^n)) \geq \min\{1/2, e^t\mu_p(A)\}.$$

Proof. In this case we will again use the transport $S_{p,n}$. Assume $A \subset r_{p,n}B_p^n$ and let $\widetilde{A} := S_{p,n}^{-1}(A)$. By Talagrand's inequality (6) we have $\nu^n(\widetilde{A} + CtB_1^n + \sqrt{Ct}\,B_2^n) \geq \min\{e^t\nu^n(\widetilde{A}), 1/2\}$. However, by Corollary 5.19,

$$S_{p,n}(\widetilde{A}+CtB_1^n+\sqrt{Ct}\,B_2^n) \subset S_{p,n}(\widetilde{A}) + 8C(\sqrt{t}\,B_2^n \cap t^{1/p}B_p^n).$$

Thus, as $S_{p,n}(\widetilde{A}) = A$ and $S_{p,n}$ transports the measure ν^n to $\mu_{p,n}$, we get the assertion. ∎

By Propositions 2.7, 3.12, 3.5 and 5.3 and Corollary 3.14, Theorem 5.26 along with Theorem 5.25 yields the following.

THEOREM 5.27. *There exists an absolute constant C such that for any $n \in \mathbb{N}$ and any $p \in [1, \infty)$ the measure $\mu_{p,n}$ satisfies* CI(C) *and* IC(C).

By Corollary 3.14 we get Cheeger's concentration inequality for $\mu_{p,n}$. However, arguing this way we loose control of the constant. We can obtain a more precise result by using—as previously—the transport from the exponential measure ν^n.

PROPOSITION 5.28. *For any $p \geq 2$ and $n \geq 1$ the measure $\mu_{p,n}$ satisfies Cheeger's inequality* (12) *with constant* $1/20$.

Proof. By [6] Cheeger's inequality holds for ν^n with constant $\kappa = 1/(2\sqrt{6})$, thus by Proposition 5.17, $\mu_{p,n}$ satisfies (12) with constant $\kappa/4 \geq 1/20$. ∎

We can also show a stronger result, namely a Gaussian-type isoperimetric inequality for $\mu_{p,n}$ with $p \geq 2$. The isoperimetric estimates for $p \leq 2$ were found by Sodin [24].

THEOREM 5.29. *Let $\Phi(x) = (2\pi)^{-1/2}\int_x^\infty \exp(-y^2/2)dy$ be the Gaussian distribution function, $A \in \mathcal{B}(\mathbb{R}^n)$ and $p \ge 2$. Then*

$$\mu_{p,n}(A) = \Phi(x) \ \Rightarrow\ \mu_{p,n}(A + 20tB_2^n) \ge \Phi(x+t) \text{ for all } t > 0.$$

In particular, there exists a universal constant C such that

$$\mu_{p,n}^+(A) \ge \frac{1}{C}\min\left\{\mu_{p,n}(A)\sqrt{\ln\frac{1}{\mu_{p,n}(A)}}, (1-\mu_{p,n}(A))\sqrt{\ln\frac{1}{1-\mu_{p,n}(A)}}\right\}.$$

Proof. By Proposition 5.21, $\widetilde{S}_{p,n}(\sqrt{2}\,\cdot)$ is $14\sqrt{2}$-Lipschitz and transports the canonical Gaussian measure on $\mathbb{R}^n$ onto $\mu_{p,n}$. Hence the first part of the theorem follows by the Gaussian isoperimetric inequality of Borell [9] and Sudakov–Tsirel'son [25]. The last estimate follows immediately from a standard estimate of the Gaussian isoperimetric function. ∎

6. Concluding remarks

1. With the notion of the IC property one may associate IC-domination of symmetric probability measures $\mu, \widetilde{\mu}$ on $\mathbb{R}^n$: we say that μ *is* IC-*dominated by $\widetilde{\mu}$ with constant β* if $(\mu, \Lambda^\star_{\widetilde{\mu}}(\cdot/\beta))$ has property (τ). IC-domination has the tensorization property: if μ_i are IC(β)-dominated by $\widetilde{\mu}_i$, $1 \le i \le n$, then $\bigotimes \mu_i$ is IC(β)-dominated by $\bigotimes \widetilde{\mu}_i$. An easy modification of the proof of Corollary 3.11 shows that if μ is IC(β)-dominated by an α-regular measure $\widetilde{\mu}$, then

$$\forall_{p\ge 2}\forall_{A\in\mathcal{B}(\mathbb{R}^n)} \quad \mu(A) \ge 1/2 \ \Rightarrow\ 1 - \mu(A + c(\alpha)\beta\mathcal{Z}_p(\widetilde{\mu})) \le e^{-p}(1-\mu(A)).$$

Following the proof of Proposition 3.15 we also get, for all $p \ge 2$,

$$\Big(\int |\,\|x\| - \mathrm{Med}_\mu(\|x\|)|^p\, d\mu\Big)^{1/p} \le \widetilde{c}(\alpha)\beta \sup_{\|u\|_*\le 1}\Big(\int |\langle u, x\rangle|^p\, d\widetilde{\mu}\Big)^{1/p}.$$

2. One may consider convex versions of properties CI and IC. We say that a symmetric probability measure μ satisfies the *convex infimum convolution inequality with constant β* if the pair $(\mu, \Lambda^\star_\mu(\cdot/\beta))$ has convex property (τ), i.e. the inequality (1) holds for all convex functions f and with $\varphi(x) = \Lambda^\star_\mu(x/\beta)$. Analogously μ satisfies the *convex concentration inequality with constant β* if (16) holds for all convex Borel sets A. We do not know if convex IC implies convex CI, but for α-regular measures it implies a weaker version of convex CI, namely

$$\mu(A) \ge 1/2 \ \Rightarrow\ \mu(A + c_1(\alpha)\beta\mathcal{Z}_p(\widetilde{\mu})) \ge 1 - 2e^{-p},$$

and this property yields CWSM($c_2(\alpha)\beta$).

From the results of [20] one may easily deduce that the uniform distribution on $\{-1,1\}^n$ satisfies convex IC(C) with a universal constant C.

3. Property IC may also be investigated for nonsymmetric measures. However, in this case the natural choice of the cost function is $\Lambda^{\star}_{\widetilde{\mu}}(x/\beta)$, where $\widetilde{\mu}$ is the convolution of μ and the symmetric image of μ.

4. We do not know if the infimum convolution property (at least for α-regular measures) implies Cheeger's inequality. If so, we would have equivalence of IC and CI + Cheeger. By Corollary 3.14 this is the case for log-concave measures.

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Rafał Latała
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: rlatala@mimuw.edu.pl
and
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
P.O. Box 21
00-956 Warszawa 10, Poland

Jakub Onufry Wojtaszczyk
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: onufry@mimuw.edu.pl

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