STUDIA MATHEMATICA 185 (1) (2008)

The weak type inequality for the Walsh system

by

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma^{\#}$ is bounded from the Hardy space $H_{1/2}$ to weak-$L_{1/2}$ and is not bounded from $H_{1/2}$ to $L_{1/2}$.

1. Introduction. The first result on a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [6] showed that the maximal operator $\sigma^* f$ is of weak type $(1,1)$, from which the a.e. convergence follows by standard arguments. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to L_1 (see also Simon [8]). Fujii's theorem was extended by Weisz [11], who proved that the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_p(I)$ to $L_p(I)$ for $p > 1/2$. Simon [9] gave an example to show that this does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [14] proved that σ^* is bounded from the Hardy space $H_{1/2}(I)$ to weak-$L_{1/2}(I)$.

For the two-dimensional Walsh–Fourier series Weisz [12] proved that the maximal operator

$$\sigma^* f = \sup_{n\geq 1} \frac{1}{n}\Big|\sum_{j=1}^{n} S_{j,j}(f)\Big|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to L_p for $p > 2/3$, and Goginava [4] generalized this result to d-dimensional Walsh–Fourier series. The a.e. convergence of the arithmetic means of square partial sums of double Vilenkin–Fourier series was studied by Gát [3].

The main aim of this paper is to prove that the maximal operator of the Marcinkiewicz–Fejér means of the double Walsh–Fourier series is bounded from the dyadic Hardy space $H_{1/2}$ to weak-$L_{1/2}$ and is not bounded from

2000 *Mathematics Subject Classification*: Primary 42C10.

Key words and phrases: Walsh function, Hardy space, maximal operator.

This paper was written during the author's visit at the College of Nyíregyháza, Hungary.

$H_{1/2}$ to $L_{1/2}$ provided that the supremum in the maximal operator is taken over spatial indices.

2. Definitions and ntation. Let $\mathbb{P}$ denote the set of positive integers, and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is, $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is such that the measure of a singleton is 1/2. Let G be the complete direct product of a countably infinite number of copies of the compact group Z_2. Elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0,1\}$ ($k \in \mathbb{N}$). The group operation on G is coordinatewise addition, and the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the *Walsh group*. A base of neighborhoods of $x \in G$ can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $n \in \mathbb{N}$. These sets are called the *dyadic intervals*. Let $0 = (0, 0, \dots) \in G$ denote the null element of G, $I_n := I_n(0)$ ($n \in \mathbb{N}$), $\bar{I}_n := G \setminus I_n$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$ with the nth coordinate 1, and the other zeros. Define

$$x_{i,j} := \sum_{s=i}^{j} x_s e_s, \qquad x_{i,i-1} = 0.$$

For $k \in \mathbb{N}$ and $x \in G$ set

$$r_k(x) := (-1)^{x_k},$$

the kth Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ ($i \in \mathbb{N}$), i.e. n is expressed in the number system of base 2. Define $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The *Walsh–Paley system* is defined as the sequence of *Walsh–Paley functions*

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G,\ n \in \mathbb{P}).$$

The *Walsh–Dirichlet kernel* is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \in \bar{I}_n. \end{cases} \tag{1}$$

The rectangular partial sums of the 2-dimensional Walsh–Fourier series are defined as follows:

$$S_{M,N}f(x^1,x^2) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \widehat{f}(i,j)w_i(x^1)w_j(x^2),$$

where the number

$$\widehat{f}(i,j) = \int_{G\times G} f(x^1,x^2)w_i(x^1)w_j(x^2)\,d\mu(x^1,x^2)$$

is said to be the (i,j)th *Walsh–Fourier coefficient* of the function f.

The norm (or quasinorm) of the space $L_p(G\times G)$ is defined by

$$\|f\|_p := \Big(\int_{G\times G} |f(x^1,x^2)|^p\,d\mu(x^1,x^2)\Big)^{1/p} \qquad (0<p<\infty).$$

The space weak-$L_p(G\times G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G\times G)} := \sup_{\lambda>0} \lambda\mu(|f|>\lambda)^{1/p} < \infty.$$

The σ-algebra generated by the dyadic 2-dimensional cube $I_k(x^1)\times I_k(x^2)$ of measure $2^{-k}\times 2^{-k}$ will be denoted by F_k $(k\in\mathbb{N})$.

Denote by $f=(f^{(n)}, n\in\mathbb{N})$ a one-parameter martingale with respect to $(F_n, n\in\mathbb{N})$ (for details, see e.g. [10, 13]). The maximal function of the martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} |f^{(n)}|.$$

In case $f\in L_1(G\times G)$, the maximal function can also be given by

$$f^*(x^1,x^2) = \sup_{n\geq 1} \frac{1}{\mu(I_n(x^1)\times I_n(x^2))}\Big|\int_{I_n(x^1)\times I_n(x^2)} f(u^1,u^2)\,d\mu(u^1,u^2)\Big|, \qquad (x^1,x^2)\in G\times G.$$

For $0<p<\infty$ the *Hardy martingale space* $H_p(G\times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f\in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^n}(f) : n\in\mathbb{N})$ is a martingale. If f is a martingale, that is, $f=(f^{(0)}, f^{(1)},\dots)$, then the Walsh–Fourier coefficients must be defined in a somewhat different way:

$$\widehat{f}(i,j) = \lim_{k\to\infty}\int_{G\times G} f^{(k)}(x^1,x^2)w_i(x^1)w_j(x^2)\,d\mu(x^1,x^2).$$

The Walsh–Fourier coefficients of $f\in L_1(G\times G)$ are the same as those of the martingale $(S_{2^n,2^n}(f) : n\in\mathbb{N})$ obtained from f.

For $n = 1, 2, \dots$ and a martingale f the *Marcinkiewicz-Fejér means* of order 2^n of the 2-dimensional Walsh–Fourier series of the function f are given by

$$\sigma_{2^n} f(x^1, x^2) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} S_{j,j} f(x^1, x^2).$$

For the martingale f we consider the maximal operator

$$\sigma^{\#} f = \sup_n |\sigma_{2^n} f(x^1, x^2)|.$$

The 2-*dimensional Marcinkiewicz–Fejér kernel* of order 2^n of the 2-dimensional Walsh–Fourier series is defined by

$$K_{2^n}(x^1, x^2) := \frac{1}{2^n} \sum_{k=0}^{2^n-1} D_k(x^1) D_k(x^2).$$

It is easy to show that

$$\sigma_{2^n} f(x^1, x^2) = \int_{G\times G} f(t^1, t^2) K_{2^n}(x^1 + t^1, x^2 + t^2)\, d\mu(t^1, t^2).$$

A bounded measurable function a is a *p-atom* if there exists a dyadic 2-dimensional cube $I \times I$ such that

(a) $\int_{I\times I} a\, d\mu = 0$;
(b) $\|a\|_\infty \le \mu(I \times I)^{-1/p}$;
(c) $\operatorname{supp} a \subset I \times I$.

3. Formulation of main results

THEOREM 1. *The maximal operator $\sigma^{\#}$ is bounded from the Hardy space $H_{1/2}(G \times G)$ to weak-$L_{1/2}(G \times G)$.*

THEOREM 2. *The maximal operator $\sigma^{\#}$ is not bounded from $H_{1/2}(G\times G)$ to $L_{1/2}(G \times G)$.*

COROLLARY 1. *Let $p > 1/2$. Then $\sigma^{\#}$ is bounded from the Hardy space $H_p(G \times G)$ to $L_p(G \times G)$.*

COROLLARY 2. *Let $0 < p < 1/2$. Then $\sigma^{\#}$ is not bounded from $H_p(G \times G)$ to weak-$L_p(G \times G)$.*

4. Auxiliary propositions. We shall need the following lemmas (see [5, 13]).

LEMMA 1 (Weisz). *Suppose that an operator V is sublinear and, for some $0 < p < 1$,*

$$\sup_{\varrho>0} \varrho^p\, \mu\{x \in (G \times G) \setminus (I \times I) : |Va(x)| > \varrho\} \le c_p < \infty$$

for every p-atom a, where I denote the support of the atom. If V is bounded from L_{p_1} *to* L_{p_1} *for a fixed* $1 < p_1 \le \infty$, *then*

$$\|Vf\|_{\textit{weak-}L_p(G\times G)} \le c_p \|f\|_{H_p}.$$

LEMMA 2 (Nagy). *Let* $A, m, n \in \mathbb{N}$, $m \le n < A$, *and* $(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1})$. *Then*

$$K_{2^A}(x^1,x^2) = \begin{cases} 0 & \textit{if } \exists i \in B_1,\, x_i^1 \ne x_i^2, \\ 0 & \textit{if } \forall i \in B_1,\, x_i^1 = x_i^2, \exists s \in B_2,\, x^1 - e_m - e_s \notin I_{n+1},\, x_s^1 = 1, \\ 2^{s+m-2} & \textit{if } \forall i \in B_1,\, x_i^1 = x_i^2,\, \exists s \in B_2,\, x^1 - e_s - e_m \in I_{n+1},\, x_s^1 = 1, \\ 2^{2m-1} & \textit{if } x^1 - e_m \in I_{n+1},\, \forall i \in B_1,\, x_i^1 = x_i^2, \end{cases}$$

where $B_1 = \{n+1, \dots, A-1\}$, $B_2 = \{m+1, \dots, n\}$.

LEMMA 3 (Nagy). *Let* $A, s, l \in \mathbb{N}, (x^1, x^2) \in I_A \times (I_l \setminus I_{l+1})$ *and* $l < s + l < A$. *Then*

$$K_{2^A}(x^1,x^2) = \begin{cases} 0 & \textit{if } \exists s, l < s+l < A,\, x^2 - x_l^2 e_l - e_{s+l} \notin I_A,\, x_{s+l}^2 \ne 0, \\ 2^{2l+s-2} & \textit{if } \exists s, l < s+l < A,\, x^2 - x_l^2 e_l - e_{s+l} \in I_A,\, x_{s+l}^2 \ne 0, \\ 2^{l-2} n(A,l) & \textit{if } x^2 - x_l^2 e_l \in I_A, \end{cases}$$

where $n(A,l) = [-2^{l-A}(2^A - 2^{l-1} + 1/2) - (2^A - 2)]$.

LEMMA 4 ([4]). *Let* $(x^1, x^2) \in \bar{I}_N \times \bar{I}_N$. *Then*

$$\int_{I_N \times I_N} K_{2^A}(x^1+t^1, x^2+t^2)\, d\mu(t^1,t^2) \le \frac{c}{2^{3N}} \Big| \sum_{j=1}^{2^N-1} D_j(x^1) D_j(x^2) \Big|, \qquad A \ge N.$$

LEMMA 5 ([4]). *Let* $(x^1, x^2) \in I_N \times \bar{I}_N$. *Then*

$$\int_{I_N \times I_N} K_{2^A}(x^1+t^1, x^2+t^2)\, d\mu(t^1,t^2) \le \frac{c}{2^{3N}} \Big\{ \Big| \sum_{j=1}^{2^N-1} D_j(x^1) D_j(x^2) \Big| + 2^N \Big| \sum_{j=1}^{2^N-1} D_j(x^2) \Big| \Big\}, \qquad A \ge N.$$

Lemma 6. *Let* $(x^1,x^2)\in(I_m\setminus I_{m+1})\times(I_n\setminus I_{n+1})$, $n\ge m$, $m,n=0,\dots,N-1$, $A>N$. *Then*

$$\int\limits_{I_N\times I_N} K_{2^A}(x^1+t^1,x^2+t^2)\,d\mu(t^1,t^2)$$
$$\le\frac{c2^{m-n}}{2^{3N}}\sum_{r=m+1}^{n+1}2^rD_{2^{n+1}}(x^1+e_m+e_r)D_{2^N}(x^2+e_n+x^1_{n+1,N-1}).$$

Proof. From Lemma 2 and by (1) we can write the following estimate:

$$\Big|\sum_{j=1}^{2^N-1}D_j(x^1)D_j(x^2)\Big|$$
$$\le c2^{m-n}\sum_{r=m+1}^{n+1}2^rD_{2^{n+1}}(x^1+e_m+e_r)D_{2^N}(x^2+e_n+x^1_{n+1,N-1}).$$

Applying Lemma 4 we complete the proof.

Lemma 7. *Let* $(x^1,x^2)\in I_N\times(I_l\setminus I_{l+1})$, $l=0,\dots,N-1$, $A>N$. *Then*

$$\int\limits_{I_N\times I_N} K_{2^A}(x^1+t^1,x^2+t^2)\,d\mu(t^1,t^2)\le\frac{c2^l}{2^{3N}}\sum_{m=l+1}^{N}2^mD_{2^N}(x^2+e_l+e_m).$$

Proof. Since (see [7] and (1))

$$\Big|\sum_{j=1}^{2^N-1}D_j(x^2)\Big|\le c\sum_{j=0}^{N}2^jD_{2^N}(x^2+e_j)=c2^lD_{2^N}(x^2+e_l)$$

and (see Lemma 3)

$$\Big|\sum_{j=1}^{2^N-1}D_j(x^1)D_j(x^2)\Big|\le c2^l\sum_{m=l+1}^{N-1}2^mD_{2^N}(x^2+e_m),$$

from Lemma 4 we obtain

$$\int\limits_{I_N\times I_N} K_{2^A}(x^1+t^1,x^2+t^2)\,d\mu(t^1,t^2)$$
$$\le\frac{c2^l}{2^{3N}}\Big\{\sum_{m=l+1}^{N-1}2^mD_{2^N}(x^2+e_l+e_m)+2^ND_{2^N}(x^2+e_l)\Big\}$$
$$\le\frac{c2^l}{2^{3N}}\sum_{m=l+1}^{N}2^mD_{2^N}(x^2+e_l+e_m).$$

5. Proofs of main results

Proof of Theorem 1. We shall apply Lemma 1; we may suppose that $a \in L_\infty$ is a 1/2-atom with support $I_N \times I_N$. Since $\sigma_{2^A} a(x^1, x^2) = 0$ for $A \le N$, we may assume that $A > N$.

Suppose that $\varrho = c2^\lambda$ for some $\lambda \in \mathbb{N}$.

It is evident that

$$(2) \quad \begin{aligned} &\mu\{(x^1,x^2) \in \overline{I_N \times I_N} : |\sigma^\# a(x^1,x^2)| > c2^\lambda\} \\ &\qquad = \mu\{(x^1,x^2) \in \bar{I}_N \times \bar{I}_N : |\sigma^\# a(x^1,x^2)| > c2^\lambda\} \\ &\qquad\quad + \mu\{(x^1,x^2) \in I_N \times \bar{I}_N : |\sigma^\# a(x^1,x^2)| > c2^\lambda\} \\ &\qquad\quad + \mu\{(x^1,x^2) \in \bar{I}_N \times I_N : |\sigma^\# a(x^1,x^2)| > c2^\lambda\}. \end{aligned}$$

Let $(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1})$, $0 \le m \le n < N$. Then from Lemma 6 we have

$$(3) \quad \begin{aligned} &\sigma^\# a(x^1,x^2) \\ &\le c\|a\|_\infty \sup_{A \ge N} \int_{I_N \times I_N} K_{2^A}(x^1+t^1, x^2+t^2)\, d\mu(t^1,t^2) \\ &\le \frac{c2^{4N}2^{m-n}}{2^{3N}} \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1+e_m+e_r) D_{2^N}(x^2+e_n+x^1_{n+1,N-1}) \\ &= c2^{N+m-n} \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1+e_m+e_r) D_{2^N}(x^2+e_n+x^1_{n+1,N-1}). \end{aligned}$$

Define

$$\begin{aligned} \sigma_1^\#(x^1,x^2) &:= c2^{N+m-n} \\ &\quad \times \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1+e_m+e_r) D_{2^N}(x^2+e_n+x^1_{n+1,N-1}). \end{aligned}$$

It is evident (see (1)) that $\sigma_1^\#(x^1,x^2) \neq 0$ implies that

$$x^1 \in I_N(\overline{0}, x^1_m = 1, \overline{0}, x^1_l = 1, \overline{0}, x^1_{n+1}, \ldots, x^1_{N-1})$$

and

$$x^2 \in I_N(\overline{0}, x^2_n = 1, x^1_{n+1}, \ldots, x^1_{N-1})$$

for some l with $m < l \le n+1$, where $\overline{0}$ denotes a string of zeros. Consequently,

$$\sigma_1^\#(x^1,x^2) \le c2^{2N+m+l}.$$

Suppose $2N+m+l\leq\lambda$. Then

$$\sigma_1^{\#}(x^1,x^2)\leq c2^{\lambda}\quad\text{and}\quad\mu\{\sigma_1^{\#}>c2^{\lambda}\}=0.$$

Hence, we can suppose that

$$m+l>\lambda-2N.$$

We have

$$\begin{aligned}E:=&\sum_{n=0}^{N-1}\sum_{m=0}^{n}\mu\{(x^1,x^2)\in(I_m\setminus I_{m+1})\times(I_n\setminus I_{n+1}):\sigma_1^{\#}(x^1,x^2)>c2^{\lambda}\}\\ &\leq c\sum_{n=0}^{N-1}\sum_{m=0}^{n}\sum_{l=m+1,m+l>\lambda-2N}^{n+1}\sum_{x_{n+1}^1=0}^{1}\cdots\sum_{x_{N-1}^1=0}^{1}\\ &\qquad\mu\{(x^1,x^2)\in I_N(\overline{0},x_m^1=1,\overline{0},x_l^1=1,\overline{0},x_{n+1}^1,\dots,x_{N-1}^1)\\ &\qquad\times I_N(\overline{0},x_n^2=1,x_{n+1}^1,\dots,x_{N-1}^1)\}.\end{aligned}$$

Define

$$A:=\{(l,m):m+l>\lambda-2N\},\qquad B:=\{(l,m):0\leq l\leq n,\ 0\leq m\leq l\}.$$

Suppose $\lambda-2N\leq 0$. Then it is evident that

$$A\cap B=\{(l,m):0\leq l\leq n,\ 0\leq m\leq l\}.$$

Hence, we can write

$$E\leq c\sum_{n=0}^{N-1}\sum_{l=0}^{n}\sum_{m=0}^{l}\frac{2^{N-n}}{2^{2N}}\leq\frac{c}{2^N}\sum_{n=0}^{N-1}\frac{n^2}{2^n}\leq\frac{c}{2^N}\leq\frac{c}{2^{\lambda/2}}.\tag{4}$$

Suppose $\lambda-2N>0$ and $0\leq n<(\lambda-2N)/2$. Then it is easy to show that

$$A\cap B=\emptyset.$$

Suppose $\lambda-2N>0$ and $(\lambda-2N)/2\leq n<\lambda-2N$. Then we can write

$$A\cap B=\{(l,m):(\lambda-2N)/2\leq l\leq n,\ \lambda-2N-l\leq m\leq l\}.$$

Consequently,

$$\begin{aligned}E&\leq c\sum_{n=[\lambda/2]-N}^{[\lambda]-2N}\sum_{l=[\lambda/2]-N}^{n}\sum_{m=[\lambda]-2N-l}^{l}\frac{2^{N-n}}{2^{2N}}\\ &\leq\frac{c}{2^N}\sum_{n=[\lambda/2]-N}^{[\lambda]-2N}\frac{(n-(\lambda/2-N))^2}{2^n}\leq\frac{c}{2^N2^{\lambda/2-N}}\leq\frac{c}{2^{\lambda/2}}.\end{aligned}\tag{5}$$

Suppose $\lambda - 2N > 0$ and $\lambda - 2N \le n < N$. Then it is evident that

$$A \cap B = \{(l,m) : (\lambda - 2N)/2 \le l \le \lambda - 2N,\ \lambda - 2N - l \le m \le l\} \cup \{(l,m) : \lambda - 2N < l \le n,\ 0 \le m \le l\}.$$

Consequently, we can write

$$E \le c \sum_{n=[\lambda]-2N}^{N-1} \sum_{l=[\lambda/2]-N}^{[\lambda]-2N} \sum_{m=[\lambda]-2N-l}^{l} \frac{2^{N-n}}{2^{2N}} + c \sum_{n=[\lambda]-2N}^{N-1} \sum_{l=[\lambda]-2N}^{n} \sum_{m=0}^{l} \frac{2^{N-n}}{2^{2N}} \le \frac{c}{2^N} \frac{(\lambda/2 - N)^2}{2^{\lambda-2N}} \le \frac{c}{2^{\lambda/2}}. \tag{6}$$

Combining (3)–(6) we obtain

$$\sum_{n=0}^{N-1} \sum_{m=0}^{n} \mu\{(x^1,x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) : \sigma^{\#} a(x^1,x^2) > c2^{\lambda}\} \le c/2^{\lambda/2}. \tag{7}$$

Analogously, we can prove that

$$\sum_{n=0}^{N-1} \sum_{m=n}^{N-1} \mu\{(x^1,x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) : \sigma^{\#} a(x^1,x^2) > c2^{\lambda}\} \le c/2^{\lambda/2}. \tag{8}$$

From (7) and (8) we get

$$\mu\{(x^1,x^2) \in \bar{I}_N \times \bar{I}_N : |\sigma^{\#} a(x^1,x^2)| > c2^{\lambda}\} \le c/2^{\lambda/2}. \tag{9}$$

Let $(x^1,x^2) \in I_N \times (I_l \setminus I_{l+1})$. Then from Lemma 7 we have

$$\sigma^{\#} a(x^1,x^2) \le c2^{4N} \sup_{A>N} \int_{I_N \times I_N} K_{2^A}(x^1+t^1, x^2+t^2)\, d\mu(t^1,t^2) \le \frac{c2^{4N+l}}{2^{3N}} \sum_{m=l+1}^{N} 2^m D_{2^N}(x^2+e_l+e_m) = c2^{N+l} \sum_{m=l+1}^{N} 2^m D_{2^N}(x^2+e_l+e_m). \tag{10}$$

Define

$$\sigma_2^{\#}(x^1,x^2) := c2^{N+l} \sum_{m=l+1}^{N} 2^m D_{2^N}(x^2+e_l+e_m).$$

From (1) we can write

$$D_{2^N}(x^2+e_l+e_m)=\begin{cases}2^n, & x^2\in I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0}),\\ 0, & x^2\notin I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0}).\end{cases}$$

Therefore

$$\sigma_2^{\#}(x^1,x^2)\neq 0$$

implies that

$$x^2\in I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0})$$

for some m with $l<m\le N$. Consequently,

$$\sigma_2^{\#}(x^1,x^2)\le c2^{l+2N+m}.$$

Suppose $l+2N+m\le\lambda$. Then

$$\sigma_2^{\#}(x^1,x^2)\le c2^{\lambda}\quad\text{and}\quad\mu\{\sigma_2^{\#}>c2^{\lambda}\}=0.$$

Hence, we can suppose that

$$l+2N+m>\lambda.$$

Define

$$T:=\{(m,l):l+m>\lambda-2N\},\qquad S:=\{(m,l):0\le l\le m<N\}.$$

Suppose $\lambda-2N\le 0$. Then it is evident that

$$T\cap S=\{(m,l):0\le m<N,\ 0\le l\le m\}.$$

Hence

$$\begin{aligned}(11)\qquad &\sum_{l=0}^{N-1}\mu\{(x^1,x^2)\in I_N\times(I_l\setminus I_{l+1}):\sigma^{\#}a(x^1,x^2)>c2^{\lambda}\}\\ &\le\sum_{l=0}^{N-1}\sum_{m=l+1}^{N-1}\mu\{(x^1,x^2)\in I_N\times I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0}):\\ &\qquad\qquad\sigma^{\#}a(x^1,x^2)>c2^{\lambda}\}\\ &\le c\sum_{l=0}^{N-1}\sum_{m=l+1}^{N-1}\frac{1}{2^{2N}}\le\frac{cN^2}{2^{2N}}<\frac{c}{2^N}<\frac{c}{2^{\lambda/2}}.\end{aligned}$$

Suppose $2N<\lambda\le 3N$. Then it is easy to show that

$$\begin{aligned}T\cap S=&\{(m,l):\lambda/2-N\le m<\lambda-2N,\lambda-2N-m\le l\le m\}\\ &\cup\{(m,l):\lambda-2N\le m<N,\ 0\le l\le m\}.\end{aligned}$$

Consequently,

$$(12)\qquad \sum_{l=0}^{N-1} \mu\{(x^1,x^2)\in I_N\times(I_l\setminus I_{l+1}) : \sigma^{\#}a(x^1,x^2) > c2^{\lambda}\}$$

$$\le c \sum_{m=[\lambda/2]-N}^{[\lambda]-2N} \sum_{l=[\lambda]-2N-m}^{m} \mu\{(x^1,x^2)\in I_N\times I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0}) : \sigma^{\#}a(x^1,x^2)>c2^{\lambda}\}$$

$$+ \sum_{m=[\lambda]-2N}^{N-1} \sum_{l=0}^{m} \mu\{(x^1,x^2)\in I_N\times I_N(\overline{0},x_l=1,\overline{0},x_m=1,\overline{0}) : \sigma^{\#}a(x^1,x^2)>c2^{\lambda}\}$$

$$\le c \sum_{m=[\lambda/2]-N}^{[\lambda]-2N} \sum_{l=[\lambda]-2N-m}^{m} \frac{1}{2^{2N}} + \sum_{m=[\lambda]-2N}^{N-1} \sum_{l=0}^{m} \frac{1}{2^{2N}}$$

$$\le c \sum_{m=[\lambda/2]-N}^{[\lambda]-2N} \frac{m-(\lambda/2-N)}{2^{2N}} + \frac{cN^2}{2^{2N}} \le \frac{cN^2}{2^{2N}} \le \frac{c\lambda^2}{2^{(2/3)\lambda}} \le \frac{c}{2^{\lambda/2}}.$$

Suppose $\lambda > 3N$. Then

$$T\cap S = \{(m,l) : \lambda/2 - N \le m < N,\ \lambda - 2N - m \le l \le m\}.$$

Consequently,

$$(13)\qquad \sum_{l=0}^{N-1} \mu\{(x^1,x^2)\in I_N\times(I_l\setminus I_{l+1}) : \sigma^{\#}a(x^1,x^2) > c2^{\lambda}\}$$

$$\le c \sum_{m=[\lambda/2]-N}^{N-1} \sum_{l=[\lambda]-2N-m}^{m} \frac{1}{2^{2N}} \le \frac{c(2N-\lambda/2)^2}{2^{2N-\lambda/2}} \frac{1}{2^{\lambda/2}} \le \frac{c}{2^{\lambda/2}}.$$

Combining (11)–(13) we obtain

$$(14)\qquad \mu\{(x^1,x^2)\in I_N\times \bar{I}_N : |\sigma^{\#}a(x^1,x^2)| > c2^{\lambda}\} \le c/2^{\lambda/2}.$$

Analogously, we can prove that

$$(15)\qquad \mu\{(x^1,x^2)\in \bar{I}_N\times I_N : |\sigma^{\#}a(x^1,x^2)| > c2^{\lambda}\} \le c/2^{\lambda/2}.$$

From (9), (14) and (15) we obtain

$$\mu\{(x^1,x^2)\in \overline{I_N\times I_N} : |\sigma^{\#}a(x^1,x^2)| > c2^{\lambda}\} \le c/2^{\lambda/2}.$$

Theorem 1 is proved.

Proof of Theorem 2. Let $A\in\mathbb{P}$ and

$$f_A(x^1,x^2) := (D_{2^{A+1}}(x^1) - D_{2^A}(x^1))(D_{2^{A+1}}(x^2) - D_{2^A}(x^2)).$$

It is evident that

$$\widehat{f}_A(i,k)=\begin{cases}1 & \text{if } i,k=2^A,\dots,2^{A+1}-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then we can write that

$$(16)\qquad S_{k,k}(f_A;x^1,x^2)$$
$$\begin{cases}0 & \text{if } k=0,\dots,2^A,\\ (D_k(x^1)-D_{2^A}(x^1))(D_k(x^2)-D_{2^A}(x^2)) & \text{if } k=2^A+1,\dots,2^{A+1}-1,\\ f_A(x^1,x^2) & \text{if } k\geq 2^{A+1}.\end{cases}$$

We have

$$f_A^*(x^1,x^2)=\sup_k |S_{2^k,2^k}(f_A;x^1,x^2)|=|f_A(x^1,x^2)|,$$
$$\|f_A\|_{H_p}=\|f_A^*\|_p=\|D_{2^A}\|_p^2=2^{2A(1-1/p)}.$$

Since

$$D_{k+2^A}-D_{2^A}=w_{2^A}D_k,\qquad k=1,\dots,2^A,$$

from (16) we obtain

$$\begin{aligned}\sigma^{\#}f_A(x^1,x^2)&=\sup_n|\sigma_{2^n}(f_A;x^1,x^2)|\geq\sigma_{2^{A+1}}(f_A;x^1,x^2)\\ &=\frac{1}{2^{A+1}}\Big|\sum_{k=0}^{2^{A+1}-1}S_{k,k}(f_A;x^1,x^2)\Big|\\ &=\frac{1}{2^{A+1}}\Big|\sum_{k=2^A+1}^{2^{A+1}-1}(D_k(x^1)-D_{2^A}(x^1))(D_k(x^2)-D_{2^A}(x^2))\Big|\\ &=\frac{1}{2^{A+1}}\Big|\sum_{k=1}^{2^A-1}(D_{k+2^A}(x^1)-D_{2^A}(x^1))(D_{k+2^A}(x^2)-D_{2^A}(x^2))\Big|\\ &=\frac{1}{2^{A+1}}\Big|\sum_{k=1}^{2^A-1}D_k(x^1)D_k(x^2)\Big|=\frac{1}{2}\,|K_{2^A}(x^1,x^2)|.\end{aligned}$$

Let

$$\begin{aligned}(x^1,x^2)\in I_A(\overline{0},x_m^1=1,\overline{0},x_n^1=1,x_{n+1}^1,\dots,x_{A-1}^1)\\ \times I_A(\overline{0},x_n^2=1,x_{n+1}^1,\dots,x_{A-1}^1).\end{aligned}$$

Then from Lemma 2 we obtain

$$|K_{2^A}(x^1,x^2)|=2^{m+n-2}.$$

Hence we can write

$$\int\limits_{G\times G} |K_{2^A}(x^1,x^2)|^{1/2}\,d\mu(x^1,x^2)$$

$$\geq \sum_{m=0}^{A-1}\sum_{n=m+1}^{A-1}\sum_{x^1_{n+1}=0}^{1}\cdots\sum_{x^1_{A-1}=0}^{1}\int\limits_{I_A(\overline{0},x^1_m=1,\overline{0},x^1_n=1,x^1_{n+1},\dots,x^1_{A-1})\times I_A(\overline{0},x^1_n=1,x^1_{n+1},\dots,x^1_{A-1})} |K_{2^A}(x^1,x^2)|^{1/2}\,d\mu(x^1,x^2)$$

$$=\sum_{m=0}^{A-1}\sum_{n=m+1}^{A-1}2^{(m+n-2)/2}\sum_{x^1_{n+1}=0}^{1}\cdots\sum_{x^1_{A-1}=0}^{1}\int\limits_{G\times G}1_{I_A(\overline{0},x^1_m=1,\overline{0},x^1_n=1,x^1_{n+1},\dots,x^1_{A-1})}(x^1) \times 1_{I_A(\overline{0},x^2_n=1,x^1_{n+1},\dots,x^1_{A-1})}(x^2)\,d\mu(x^1,x^2)$$

$$\geq c\sum_{m=0}^{A-1}2^{m/2}\sum_{n=m}^{A-1}2^{n/2}\frac{1}{2^{2A}}2^{A-n}\geq\frac{cA}{2^A},$$

and

$$\frac{\|\sigma^{\#}f_A\|_{1/2}}{\|f_A\|_{1/2}}\geq\frac{cA^2}{2^{2A}2^{2A(1-2)}}\geq cA^2\to\infty\quad\text{ as } A\to\infty.$$

Theorem 2 is proved.

Since $\sigma^{\#}$ is bounded from $L_\infty(G\times G)$ to $L_\infty(G\times G)$ the validity of Corollaries 3 and 4 follows by interpolation (see Weisz [13]) from Theorems 1 and 2.

Acknowledgments. The author wishes to express his gratitude to the referees for helpful remarks.

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Received November 25, 2005
Revised version January 20, 2008 (5813)