The weak type inequality for the Walsh system

by

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma^#$ is
bounded from the Hardy space $H_{1/2}$ to weak-$L_{1/2}$ and is not bounded from $H_{1/2}$ to $L_{1/2}$.

1. Introduction. The first result on a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [6] showed that the maximal
operator $\sigma^* f$ is of weak type $(1, 1)$, from which the a.e. convergence follows
by standard arguments. Schipp’s result implies by interpolation also the boundedness of $\sigma^*: L_p \to L_p \ (1 < p \leq \infty)$. This fails to hold for $p = 1$
but Fujii [2] proved that $\sigma^*$ is bounded from the dyadic Hardy space $H_1$
to $L_1$ (see also Simon [8]). Fujii’s theorem was extended by Weisz [11], who
proved that the maximal operator of the Fejér means of the one-dimensional
Walsh–Fourier series is bounded from the martingale Hardy space $H_p(I)$ to
$L_p(I)$ for $p > 1/2$. Simon [9] gave an example to show that this does not
hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [14] proved that
$\sigma^*$ is bounded from the Hardy space $H_{1/2}(I)$ to weak-$L_{1/2}(I)$.

For the two-dimensional Walsh–Fourier series Weisz [12] proved that the
maximal operator

$$\sigma^* f = \sup_{n \geq 1} \frac{1}{n} \left| \sum_{j=1}^n S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space $H_p$
to $L_p$ for $p > 2/3$, and Goginava [4] generalized this result to $d$-dimensional
Walsh–Fourier series. The a.e. convergence of the arithmetic means of square
partial sums of double Vilenkin–Fourier series was studied by Gát [3].

The main aim of this paper is to prove that the maximal operator of the
Marcinkiewicz–Fejér means of the double Walsh–Fourier series is bounded
from the dyadic Hardy space $H_{1/2}$ to weak-$L_{1/2}$ and is not bounded from

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$H_{1/2}$ to $L_{1/2}$ provided that the supremum in the maximal operator is taken over spatial indices.

2. Definitions and notation. Let $\mathbb{P}$ denote the set of positive integers, and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote by $Z_2$ the discrete cyclic group of order 2, that is, $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $Z_2$ is such that the measure of a singleton is $1/2$. Let $G$ be the complete direct product of a countably infinite number of copies of the compact group $Z_2$. Elements of $G$ are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbb{N}$). The group operation on $G$ is coordinatewise addition, and the measure (denoted by $\mu$) and the topology are the product measure and topology. The compact Abelian group $G$ is called the Walsh group. A base of neighborhoods of $x \in G$ can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{ y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots) \}$$

for $n \in \mathbb{N}$. These sets are called the dyadic intervals. Let $0 = (0, 0, \ldots) \in G$ denote the null element of $G$, $I_n := I_n(0)$ ($n \in \mathbb{N}$), $\bar{I}_n := G \setminus I_n$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ with the $n$th coordinate 1, and the other zeros. Define

$$x_{i,j} := \sum_{s=i}^{j} x_s e_s, \quad x_{i,i-1} = 0.$$

For $k \in \mathbb{N}$ and $x \in G$ set

$$r_k(x) := (-1)^{x_k},$$

the $k$th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ ($i \in \mathbb{N}$), i.e. $n$ is expressed in the number system of base 2. Define $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, \ n \in \mathbb{P}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \in \bar{I}_n. \end{cases}$$
The rectangular partial sums of the 2-dimensional Walsh–Fourier series are defined as follows:
\[ S_{M,N}f(x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i, j) w_i(x^1)w_j(x^2), \]
where the number
\[ \hat{f}(i, j) = \int_{G \times G} f(x^1, x^2)w_i(x^1)w_j(x^2) \, d\mu(x^1, x^2) \]
is said to be the \((i, j)\)th Walsh–Fourier coefficient of the function \(f\).

The norm (or quasinorm) of the space \(L_p(G \times G)\) is defined by
\[ \|f\|_p := \left( \int_{G \times G} |f(x^1, x^2)|^p \, d\mu(x^1, x^2) \right)^{1/p} \quad (0 < p < \infty). \]
The space weak-\(L_p(G \times G)\) consists of all measurable functions \(f\) for which
\[ \|f\|_{\text{weak-}L_p(G \times G)} := \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\})^{1/p} < \infty. \]

The \(\sigma\)-algebra generated by the dyadic 2-dimensional cube \(I_k(x^1) \times I_k(x^2)\) of measure \(2^{-k} \times 2^{-k}\) will be denoted by \(F_k\) \((k \in \mathbb{N})\).

Denote by \(f = (f(n), n \in \mathbb{N})\) a one-parameter martingale with respect to \((F_n, n \in \mathbb{N})\) (for details, see e.g. [10, 13]). The maximal function of the martingale \(f\) is defined by
\[ f^* = \sup_{n \in \mathbb{N}} |f(n)|. \]

In case \(f \in L_1(G \times G)\), the maximal function can also be given by
\[ f^*(x^1, x^2) = \sup_{n \geq 1} \frac{1}{\mu(I_n(x^1) \times I_n(x^2))} \left| \int_{I_n(x^1) \times I_n(x^2)} f(u^1, u^2) \, d\mu(u^1, u^2) \right|, \]
\((x^1, x^2) \in G \times G\).

For \(0 < p < \infty\) the Hardy martingale space \(H_p(G \times G)\) consists of all martingales for which
\[ \|f\|_{H_p} := \|f^*\|_p < \infty. \]

If \(f \in L_1(G \times G)\) then it is easy to show that the sequence \((S_{2^n, 2^n}(f) : n \in \mathbb{N})\) is a martingale. If \(f\) is a martingale, that is, \(f = (f(0), f(1), \ldots)\), then the Walsh–Fourier coefficients must be defined in a somewhat different way:
\[ \hat{f}(i, j) = \lim_{k \to \infty} \int_{G \times G} f^{(k)}(x^1, x^2)w_i(x^1)w_j(x^2) \, d\mu(x^1, x^2). \]
The Walsh–Fourier coefficients of \(f \in L_1(G \times G)\) are the same as those of the martingale \((S_{2^n, 2^n}(f) : n \in \mathbb{N})\) obtained from \(f\).
For $n = 1, 2, \ldots$ and a martingale $f$ the Marcinkiewicz-Fejér means of order $2^n$ of the 2-dimensional Walsh–Fourier series of the function $f$ are given by

$$\sigma_{2^n} f(x^1, x^2) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} S_{j,j} f(x^1, x^2).$$

For the martingale $f$ we consider the maximal operator

$$\sigma\# f = \sup_n |\sigma_{2^n} f(x^1, x^2)|.$$

The 2-dimensional Marcinkiewicz–Fejér kernel of order $2^n$ of the 2-dimensional Walsh–Fourier series is defined by

$$K_{2^n}(x^1, x^2) := \frac{1}{2^n} \sum_{k=0}^{2^n-1} D_k(x^1)D_k(x^2).$$

It is easy to show that

$$\sigma_{2^n} f(x^1, x^2) = \int_{G \times G} f(t^1, t^2) K_{2^n}(x^1 + t^1, x^2 + t^2) d\mu(t^1, t^2).$$

A bounded measurable function $a$ is a $p$-atom if there exists a dyadic 2-dimensional cube $I \times I$ such that

(a) $\int_{I \times I} a \, d\mu = 0$;

(b) $\|a\|_\infty \leq \mu(I \times I)^{-1/p}$;

(c) supp $a \subset I \times I$.

3. Formulation of main results

**Theorem 1.** The maximal operator $\sigma\#$ is bounded from the Hardy space $H_{1/2}(G \times G)$ to weak-$L_{1/2}(G \times G)$.

**Theorem 2.** The maximal operator $\sigma\#$ is not bounded from $H_{1/2}(G \times G)$ to $L_{1/2}(G \times G)$.

**Corollary 1.** Let $p > 1/2$. Then $\sigma\#$ is bounded from the Hardy space $H_p(G \times G)$ to $L_p(G \times G)$.

**Corollary 2.** Let $0 < p < 1/2$. Then $\sigma\#$ is not bounded from $H_p(G \times G)$ to weak-$L_p(G \times G)$.

4. Auxiliary propositions. We shall need the following lemmas (see [5, 13]).

**Lemma 1** (Weisz). Suppose that an operator $V$ is sublinear and, for some $0 < p < 1$,

$$\sup_{\varrho > 0} \varrho^p \mu \{ x \in (G \times G) \setminus (I \times I) : |Va(x)| > \varrho \} \leq c_p < \infty$$
for every $p$-atom $a$, where $I$ denote the support of the atom. If $V$ is bounded from $L_{p_1}$ to $L_{p_1}$ for a fixed $1 < p_1 \leq \infty$, then

$$\|Vf\|_{\text{weak}-L_p(G \times G)} \leq c_p \|f\|_{H_p}.$$ 

**Lemma 2** (Nagy). Let $A, m, n \in \mathbb{N}, m \leq n < A$, and $(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1})$. Then

$$K_{2A}(x^1, x^2) = \begin{cases} 0 & \text{if } \exists i \in B_1, x^1_i \neq x^2_i, \\ 0 & \text{if } \forall i \in B_1, x^1_i = x^2_i, \exists s \in B_2, x^1 - e_m - e_s \notin I_{n+1}, x^1_s = 1, \\ 2^{s+m-2} & \text{if } \forall i \in B_1, x^1_i = x^2_i, \exists s \in B_2, x^1 - e_s - e_m \in I_{n+1}, x^1_s = 1, \\ 2^{2m-1} & \text{if } x^1 - e_m \in I_{n+1}, \forall i \in B_1, x^1_i = x^2_i, \end{cases}$$

where $B_1 = \{ n + 1, \ldots, A - 1 \}, B_2 = \{ m + 1, \ldots, n \}$.

**Lemma 3** (Nagy). Let $A, s, l \in \mathbb{N}, (x^1, x^2) \in I_A \times (I_l \setminus I_{l+1})$ and $l < s + l < A$. Then

$$K_{2A}(x^1, x^2) = \begin{cases} 0 & \text{if } \exists s, l < s + l < A, x^2 - x^2_s e_l - e_{s+l} \notin I_A, x^2_{s+l} \neq 0, \\ 2^{2l+s-2} & \text{if } \exists s, l < s + l < A, x^2 - x^2_s e_l - e_{s+l} \in I_A, x^2_{s+l} = 0, \\ 2^{l-2} n(A, l) & \text{if } x^2 - x^2_s e_l \in I_A, \end{cases}$$

where $n(A, l) = [-2^{l-A}(2^A - 2^{l-1} + 1/2) - (2^A - 2)]$.

**Lemma 4** ([4]). Let $(x^1, x^2) \in \bar{I}_N \times \bar{I}_N$. Then

$$\int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) d\mu(t^1, t^2) \leq \frac{c}{2^{3N}} \left| \sum_{j=1}^{2N-1} D_j(x^1)D_j(x^2) \right|, \quad A \geq N.$$ 

**Lemma 5** ([4]). Let $(x^1, x^2) \in I_N \times I_N$. Then

$$\int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) d\mu(t^1, t^2) \leq \frac{c}{2^{3N}} \left\{ \sum_{j=1}^{2N-1} D_j(x^1)D_j(x^2) + 2^N \left| \sum_{j=1}^{2N-1} D_j(x^2) \right| \right\}, \quad A \geq N.$$
Lemma 6. Let \( (x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) \), \( n \geq m \), \( m, n = 0, \ldots, N-1 \), \( A > N \). Then

\[
\int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) \, d\mu(t^1, t^2) \leq \frac{c 2^{m-n}}{2^{3N}} \sum_{r=m+1}^{n+1} 2^r D_{2n+1}(x^1 + e_m + e_r) D_{2N}(x^2 + e_n + x_{n+1,N-1}^1).
\]

Proof. From Lemma 2 and by (1) we can write the following estimate:

\[
\left| \sum_{j=1}^{2N-1} D_j(x^1) D_j(x^2) \right| \leq c 2^{m-n} \sum_{r=m+1}^{n+1} 2^r D_{2n+1}(x^1 + e_m + e_r) D_{2N}(x^2 + e_n + x_{n+1,N-1}^1).
\]

Applying Lemma 4 we complete the proof.

Lemma 7. Let \( (x^1, x^2) \in I_N \times (I_l \setminus I_{l+1}) \), \( l = 0, \ldots, N-1 \), \( A > N \). Then

\[
\int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) \, d\mu(t^1, t^2) \leq \frac{c 2^l}{2^{3N}} \sum_{m=l+1}^{N} 2^m D_{2N}(x^2 + e_l + e_m).
\]

Proof. Since (see [7] and (1))

\[
\left| \sum_{j=1}^{2N-1} D_j(x^2) \right| \leq c \sum_{j=0}^{N} 2^j D_{2N}(x^2 + e_j) = c 2^l D_{2N}(x^2 + e_l)
\]

and (see Lemma 3)

\[
\left| \sum_{j=1}^{2N-1} D_j(x^1) D_j(x^2) \right| \leq c 2^l \sum_{m=l+1}^{N-1} 2^m D_{2N}(x^2 + e_m),
\]

from Lemma 4 we obtain

\[
\int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) \, d\mu(t^1, t^2) \leq \frac{c 2^l}{2^{3N}} \left\{ \sum_{m=l+1}^{N-1} 2^m D_{2N}(x^2 + e_l + e_m) + 2^N D_{2N}(x^2 + e_l) \right\}
\leq \frac{c 2^l}{2^{3N}} \sum_{m=l+1}^{N} 2^m D_{2N}(x^2 + e_l + e_m).
5. Proofs of main results

Proof of Theorem 1. We shall apply Lemma 1; we may suppose that $a \in L_{\infty}$ is a 1/2-atom with support $I_N \times I_N$. Since $\sigma_{2A}a(x^1, x^2) = 0$ for $A \leq N$, we may assume that $A > N$.

Suppose that $\varrho = c2^\lambda$ for some $\lambda \in \mathbb{N}$.

It is evident that

\begin{equation}
(2) \quad \mu\{(x^1, x^2) \in \overline{I_N} \times \overline{I_N} : |\sigma^#a(x^1, x^2)| > c2^\lambda\} = \mu\{(x^1, x^2) \in I_N \times I_N : |\sigma^#a(x^1, x^2)| > c2^\lambda\} + \mu\{(x^1, x^2) \in \overline{I_N} \times I_N : |\sigma^#a(x^1, x^2)| > c2^\lambda\} + \mu\{(x^1, x^2) \in I_N \times \overline{I_N} : |\sigma^#a(x^1, x^2)| > c2^\lambda\}.
\end{equation}

Let $(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1})$, $0 \leq m \leq n < N$. Then from Lemma 6 we have

\begin{equation}
(3) \quad \sigma^#a(x^1, x^2) \leq c\|a\|_{\infty} \sup_{A \geq N} \int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) d\mu(t^1, t^2)
\leq \frac{c2^4N2^{m-n}}{2^{3N}} \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1 + e_m + e_r) D_{2^N}(x^2 + e_n + x^1_{n+1,N-1})
= c2^{N+m-n} \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1 + e_m + e_r) D_{2^N}(x^2 + e_n + x^1_{n+1,N-1}).
\end{equation}

Define

\[ \sigma_1^#(x^1, x^2) := c2^{N+m-n} \times \sum_{r=m+1}^{n+1} 2^r D_{2^{n+1}}(x^1 + e_m + e_r) D_{2^N}(x^2 + e_n + x^1_{n+1,N-1}). \]

It is evident (see (1)) that $\sigma_1^#(x^1, x^2) \neq 0$ implies that

\[ x^1 \in I_N(\overline{0}, x^1_m = 1, \overline{0}, x^1_{n} = 1, \overline{0}, x^1_{n+1}, \ldots, x^1_{N-1}) \]

and

\[ x^2 \in I_N(\overline{0}, x^2_n = 1, x^1_{n+1}, \ldots, x^1_{N-1}) \]

for some $l$ with $m < l \leq n+1$, where $\overline{0}$ denotes a string of zeros. Consequently,

\[ \sigma_1^#(x^1, x^2) \leq c2^{2N+m+l}. \]
Suppose $2N + m + l \leq \lambda$. Then
\[
\sigma_1^#(x^1, x^2) \leq c2^\lambda \quad \text{and} \quad \mu\{\sigma_1^# > c2^\lambda\} = 0.
\]
Hence, we can suppose that
\[
m + l > \lambda - 2N.
\]
We have
\[
E := \sum_{n=0}^{N-1} \sum_{m=0}^{n} \mu\{(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) : \sigma_1^#(x^1, x^2) > c2^\lambda\}
\leq c \sum_{n=0}^{N-1} \sum_{m=0}^{n} \sum_{l=m+1, m+l>\lambda-2N} \sum_{x_{n+1}^1=0}^{1} \ldots \sum_{x_{N-1}^1=0}^{1} \mu\{(x^1, x^2) \in I_N(\overline{0}, x_m^1 = 1, \overline{0}, x_1^1 = 1, \overline{0}, x_{n+1}^1, \ldots, x_{N-1}^1) \times I_N(\overline{0}, x_n^2 = 1, x_{n+1}^1, \ldots, x_{N-1}^1)\}.
\]
Define
\[
A := \{(l, m) : m + l > \lambda - 2N\}, \quad B := \{(l, m) : 0 \leq l \leq n, 0 \leq m \leq l\}.
\]
Suppose $\lambda - 2N \leq 0$. Then it is evident that
\[
A \cap B = \{(l, m) : 0 \leq l \leq n, 0 \leq m \leq l\}.
\]
Hence, we can write
\[
E \leq c \sum_{n=0}^{N-1} \sum_{l=0}^{n} \sum_{m=0}^{l} 2^{N-n} \leq \frac{c}{2^N} \sum_{n=0}^{N-1} \sum_{m=0}^{n} 2^m \leq \frac{c}{2^N} \leq \frac{c}{2\lambda/2}.
\]
Suppose $\lambda - 2N > 0$ and $0 \leq n < (\lambda - 2N)/2$. Then it is easy to show that
\[
A \cap B = \emptyset.
\]
Suppose $\lambda - 2N > 0$ and $(\lambda - 2N)/2 \leq n < \lambda - 2N$. Then we can write
\[
A \cap B = \{(l, m) : (\lambda - 2N)/2 \leq l \leq n, \lambda - 2N - l \leq m \leq l\}.
\]
Consequently,
\[
E \leq c \sum_{n=\lceil\lambda/2\rceil-N}^{\lfloor\lambda\rfloor-2N} \sum_{l=\lfloor\lambda/2\rfloor-N}^{\lfloor\lambda\rfloor-2N} \sum_{m=\lfloor\lambda\rfloor-2N-l}^{n} 2^{N-n} \leq \frac{c}{2^N} \sum_{n=\lceil\lambda/2\rceil-N}^{\lfloor\lambda\rfloor-2N} \frac{(n - (\lambda/2 - N))^2}{2^n} \leq \frac{c}{2^{N\lambda/2-N}} \leq \frac{c}{2\lambda/2},
\]
Suppose $\lambda - 2N > 0$ and $\lambda - 2N \leq n < N$. Then it is evident that

$$A \cap B = \{(l, m) : (\lambda - 2N)/2 \leq l \leq \lambda - 2N, \lambda - 2N - l \leq m \leq l\}$$

$$\cup \{(l, m) : \lambda - 2N < l \leq n, 0 \leq m \leq l\}.$$ 

Consequently, we can write

$$E \leq c \sum_{n=[\lambda]-2N}^{N-1} \sum_{l=[\lambda/2]-N}^{[\lambda]-2N-l} \sum_{m=[\lambda]-2N-l}^{l} \frac{2^{N-n}}{2^{2N}}$$

$$+ c \sum_{n=[\lambda]-2N}^{N-1} \sum_{l=[\lambda]-2N}^{n} \sum_{m=0}^{l} \frac{2^{N-n}}{2^{2N}}$$

$$\leq \frac{c}{2^N} \frac{(\lambda/2 - N)^2}{2^{\lambda-2N}} \leq \frac{c}{2^{\lambda/2}}.$$ 

Combining (3)–(6) we obtain

$$\sum_{n=0}^{N-1} \sum_{m=0}^{n} \mu\{(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) : \sigma^# a(x^1, x^2) > c2^\lambda\}$$

$$\leq c/2^{\lambda/2}.$$ 

Analogously, we can prove that

$$\sum_{n=0}^{N-1} \sum_{m=n}^{N-1} \mu\{(x^1, x^2) \in (I_m \setminus I_{m+1}) \times (I_n \setminus I_{n+1}) : \sigma^# a(x^1, x^2) > c2^\lambda\}$$

$$\leq c/2^{\lambda/2}.$$ 

From (7) and (8) we get

$$\mu\{(x^1, x^2) \in \tilde{I}_N \times \tilde{I}_N : |\sigma^# a(x^1, x^2)| > c2^\lambda\} \leq c/2^{\lambda/2}.$$ 

Let $(x^1, x^2) \in I_N \times (I_l \setminus I_{l+1})$. Then from Lemma 7 we have

$$\sigma^# a(x^1, x^2) \leq c2^{4N} \sup_{A \supset N} \int_{I_N \times I_N} K_{2A}(x^1 + t^1, x^2 + t^2) d\mu(t^1, t^2)$$

$$\leq \frac{c2^{4N+l}}{2^{3N}} \sum_{m=l+1}^{N} 2^m D_{2N}(x^2 + e_l + e_m)$$

$$= c2^{N+l} \sum_{m=l+1}^{N} 2^m D_{2N}(x^2 + e_l + e_m).$$

Define

$$\sigma^#_2(x^1, x^2) := c2^{N+l} \sum_{m=l+1}^{N} 2^m D_{2N}(x^2 + e_l + e_m).$$
From (1) we can write
\[
D_{2N}(x^2 + e_l + e_m) = \begin{cases} 
2^n, & x^2 \in I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0}), \\
0, & x^2 \notin I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0}). 
\end{cases}
\]

Therefore
\[
\sigma_2^\#(x^1, x^2) \neq 0
\]
implies that
\[
x^2 \in I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0})
\]
for some \( m \) with \( l < m \leq N \). Consequently,
\[
\sigma_2^\#(x^1, x^2) \leq c2^{l+2N+m}.
\]

Suppose \( l + 2N + m \leq \lambda \). Then
\[
\sigma_2^\#(x^1, x^2) \leq c2^\lambda \quad \text{and} \quad \mu\{\sigma_2^\# > c2^\lambda\} = 0.
\]

Hence, we can suppose that
\[
l + 2N + m > \lambda.
\]

Define
\[
T := \{(m, l) : l + m > \lambda - 2N\}, \quad S := \{(m, l) : 0 \leq l \leq m < N\}.
\]

Suppose \( \lambda - 2N \leq 0 \). Then it is evident that
\[
T \cap S = \{(m, l) : 0 \leq m < N, 0 \leq l \leq m\}.
\]

Hence
\[
(11) \quad \sum_{l=0}^{N-1} \sum_{m=l+1}^{N-1} \mu\{(x^1, x^2) \in I_N \times (I_l \setminus I_{l+1}) : \sigma^\# a(x^1, x^2) > c2^\lambda\} \\
\leq \sum_{l=0}^{N-1} \sum_{m=l+1}^{N-1} \mu\{(x^1, x^2) \in I_N \times I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0}) : \\
\sigma^\# a(x^1, x^2) > c2^\lambda\} \\
\leq c \sum_{l=0}^{N-1} \sum_{m=l+1}^{N-1} \frac{1}{2^{2N}} \leq \frac{cN^2}{2^{2N}} < \frac{c}{2^{\lambda/2}}.
\]

Suppose \( 2N < \lambda \leq 3N \). Then it is easy to show that
\[
T \cap S = \{(m, l) : \lambda/2 - N \leq m < \lambda - 2N, \lambda - 2N - m \leq l \leq m\} \\
\cup \{(m, l) : \lambda - 2N \leq m < N, 0 \leq l \leq m\}.
\]
Consequently,

\begin{align}
\sum_{l=0}^{N-1} \mu \{ (x^1, x^2) \in I_N \times (I_l \setminus I_{l+1}) : \sigma^# a(x^1, x^2) > c2^\lambda \} \\
\leq c \sum_{m=\lceil\lambda/2\rceil-N}^{\lceil\lambda/2\rceil-2N-N} \sum_{l=\lceil\lambda/2\rceil-2N-m}^{m} \mu \{ (x^1, x^2) \in I_N \times I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0}) : \sigma^# a(x^1, x^2) > c2^\lambda \} \\
+ \sum_{m=\lceil\lambda/2\rceil-2N}^{N-1} \sum_{l=0}^{m} \mu \{ (x^1, x^2) \in I_N \times I_N(\bar{0}, x_l = 1, \bar{0}, x_m = 1, \bar{0}) : \sigma^# a(x^1, x^2) > c2^\lambda \} \\
\leq c \sum_{m=\lceil\lambda/2\rceil-N}^{\lceil\lambda/2\rceil-2N-N} \sum_{l=\lceil\lambda/2\rceil-2N-m}^{m} \frac{1}{2^{2N}} + \sum_{m=\lceil\lambda/2\rceil-2N}^{N-1} \sum_{l=0}^{m} \frac{1}{2^{2N}} \\
\leq c \sum_{m=\lceil\lambda/2\rceil-N}^{\lceil\lambda/2\rceil-2N-N} \frac{m - (\lambda/2 - N)}{2^{2N}} + \frac{cN^2}{2^{2N}} \leq \frac{cN^2}{2^{2N}} \leq \frac{c\lambda^2}{2^{(2/3)\lambda}} \leq \frac{c}{2^{\lambda/2}}.
\end{align}

Suppose \( \lambda > 3N \). Then

\[ T \cap S = \{(m, l) : \lambda/2 - N \leq m < N, \lambda - 2N - m \leq l \leq m \}. \]

Consequently,

\begin{align}
\sum_{l=0}^{N-1} \mu \{ (x^1, x^2) \in I_N \times (I_l \setminus I_{l+1}) : \sigma^# a(x^1, x^2) > c2^\lambda \} \\
\leq c \sum_{m=\lceil\lambda/2\rceil-N}^{\lceil\lambda/2\rceil-2N-N} \sum_{l=\lceil\lambda/2\rceil-2N-m}^{m} \frac{1}{2^{2N}} \leq \frac{c(2N - \lambda/2)^2}{2^{2N-\lambda/2}} \frac{1}{2^{\lambda/2}} \leq \frac{c}{2^{\lambda/2}}.
\end{align}

Combining (11)-(13) we obtain

\begin{align}
\mu \{ (x^1, x^2) \in I_N \times \bar{I}_N : |\sigma^# a(x^1, x^2)| > c2^\lambda \} \leq c/2^{\lambda/2}.
\end{align}

Analogously, we can prove that

\begin{align}
\mu \{ (x^1, x^2) \in \bar{I}_N \times I_N : |\sigma^# a(x^1, x^2)| > c2^\lambda \} \leq c/2^{\lambda/2}.
\end{align}

From (9), (14) and (15) we obtain

\[ \mu \{ (x^1, x^2) \in \bar{I}_N \times \bar{I}_N : |\sigma^# a(x^1, x^2)| > c2^\lambda \} \leq c/2^{\lambda/2}. \]

Theorem 1 is proved.

**Proof of Theorem 2.** Let \( A \in \mathbb{P} \) and

\[ f_A(x^1, x^2) := (D_{2A+1}(x^1) - D_{2A}(x^1))(D_{2A+1}(x^2) - D_{2A}(x^2)). \]
It is evident that

$$\hat{f}_A(i, k) = \begin{cases} 1 & \text{if } i, k = 2^A, \ldots, 2^{A+1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write that

$$S_{k,k}(f_A; x^1, x^2) = \begin{cases} 0 & \text{if } k = 0, \ldots, 2^A, \\ (D_k(x^1) - D_{2^A}(x^1))(D_k(x^2) - D_{2^A}(x^2)) & \text{if } k = 2^A + 1, \ldots, 2^{A+1} - 1, \\ f_A(x^1, x^2) & \text{if } k \geq 2^{A+1}. \end{cases}$$

We have

$$f^*_A(x^1, x^2) = \sup_k |S_{2k,2k}(f_A; x^1, x^2)| = |f_A(x^1, x^2)|,$$

$$\|f_A\|_{H_p} = \|f^*_A\|_p = \|D_{2^A}\|_p^2 = 2^{2A(1-1/p)}.$$

Since

$$D_{k+2^A} - D_{2^A} = w_{2^A} D_k, \quad k = 1, \ldots, 2^A,$$

from (16) we obtain

$$\sigma^# f_A(x^1, x^2) = \sup_n |\sigma_{2^n}(f_A; x^1, x^2)| \geq \sigma_{2^{A+1}}(f_A; x^1, x^2)$$

$$= \frac{1}{2^{A+1}} \left| \sum_{k=0}^{2^{A+1}-1} S_{k,k}(f_A; x^1, x^2) \right|$$

$$= \frac{1}{2^{A+1}} \left| \sum_{k=2^A+1}^{2^{A+1}-1} (D_k(x^1) - D_{2^A}(x^1))(D_k(x^2) - D_{2^A}(x^2)) \right|$$

$$= \frac{1}{2^{A+1}} \left| \sum_{k=1}^{2^{A-1}} (D_{k+2^A}(x^1) - D_{2^A}(x^1))(D_{k+2^A}(x^2) - D_{2^A}(x^2)) \right|$$

$$= \frac{1}{2^{A+1}} \left| \sum_{k=1}^{2^{A-1}} D_k(x^1)D_k(x^2) \right| = \frac{1}{2} |K_{2^A}(x^1, x^2)|.$$

Let

$$(x^1, x^2) \in I_A(\overline{0}, x^1_m = 1, \overline{0}, x^1_n = 1, x^1_{n+1}, \ldots, x^1_{A-1}) \times I_A(\overline{0}, x^2_n = 1, x^1_{n+1}, \ldots, x^1_{A-1}).$$

Then from Lemma 2 we obtain

$$|K_{2^A}(x^1, x^2)| = 2^{m+n-2}.$$
Hence we can write
\[
\int_{G \times G} |K_{2A}(x^1, x^2)|^{1/2} d\mu(x^1, x^2)
\]
\[
\geq \sum_{m=0}^{A-1} \sum_{n=m+1}^{A-1} \sum_{x_{n+1}=0}^{1} \cdots \sum_{x_{A-1}=0}^{1} \int_{G \times G}
\]
\[
|K_{2A}(x^1, x^2)|^{1/2} d\mu(x^1, x^2)
\]
\[
= \sum_{m=0}^{A-1} \sum_{n=m+1}^{A-1} 2^{(m+n-2)/2} \sum_{x_{n+1}=0}^{1} \cdots \sum_{x_{A-1}=0}^{1} \int_{G \times G}
\]
\[
1_{I_A(0,x^1_{n}=1,0,x^1_{n+1}=1,x^1_{n+1},...,x^1_{A-1})} (x^1) \times 1_{I_A(0,x^2_{n}=1,0,x^2_{n+1},...,x^2_{A-1})} (x^2)
\]
\[
\geq c \sum_{m=0}^{A-1} 2^{m/2} \sum_{n=m}^{A-1} 2^{n/2} \frac{1}{2^{2A}} 2^{A-n} \geq \frac{cA}{2^{A}}.
\]
and
\[
\frac{\|\sigma^# f\|_{1/2}}{\|f\|_{1/2}} \geq \frac{cA^2}{2^{2A}2^{A(1-2)}} \geq cA^2 \to \infty \text{ as } A \to \infty.
\]

Theorem 2 is proved.

Since \(\sigma^#\) is bounded from \(L_\infty(G \times G)\) to \(L_\infty(G \times G)\) the validity of Corollaries 3 and 4 follows by interpolation (see Weisz [13]) from Theorems 1 and 2.

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**References**


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