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Weierstrass division theorem in quasianalytic local rings

by

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Abstract. The main result of this paper is the following: if the Weierstrass division theorem is valid in a quasianalytic differentiable system, then this system is contained in the system of analytic germs. This result has already been known for particular examples, such as the quasianalytic Denjoy–Carleman classes.

Introduction. The Weierstrass division theorem is one of the central theorems in local real analytic geometry. It was successfully used to study the geometry of semianalytic and subanalytic sets by Łojasiewicz [4], [5]. It was also used in [3] with methods of model theory to give a proof of Gabrielov's theorem concerning the complement of a subanalytic set. In this paper, we consider the problem of extending the Weierstrass division theorem to rings of germs, at the origin of $\mathbb{R}^n$, of smooth quasianalytic functions (without flat functions). The first result on this problem was obtained by Childress [2] in a specific situation. He showed that, for the particular case of the *quasianalytic Denjoy–Carleman class*, if the Weierstrass division theorem is true in a fixed quasianalytic class, then the class is analytic. In this paper we extend the result of Childress to any quasianalytic local ring of germs of smooth functions. More precisely, let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be such that each $\mathcal{C}_n$ is a quasianalytic local subring of the ring of germs, at the origin of $\mathbb{R}^n$, of C^∞ functions. For each $n \in \mathbb{N}$, $\mathcal{C}_n$ contains the local ring of germs of Nash functions and is closed under taking partial derivatives. We also suppose that the system $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is closed under composition. We prove that if the Weierstrass division theorem holds for $\mathcal{C}_3$, then for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is contained in the ring of germs, at the origin of $\mathbb{R}^n$, of real analytic functions.

1. Quasianalytic differentiable systems. Let $X = (X_1, \ldots, X_n)$ be an n-tuple of distinct indeterminates with $n \in \mathbb{N}$. The ring of formal series in $X_1, \ldots, X_n$ over a field $\mathbb{K}$ will be denoted by $\mathbb{K}[[X]]$, and the subring of

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$\mathbb{R}[[X]]$ of formal series which converge in some neighborhood of the origin in $\mathbb{R}^n$ will be denoted by $\mathbb{R}\langle X\rangle$.

Denote by $\mathcal{A}_n$ (resp. $\mathcal{E}_n$) the ring of real-analytic (resp. smooth) function germs at the origin of $\mathbb{R}^n$, and by $\mathcal{N}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of Nash functions. Clearly, $\mathcal{N}_n \subseteq \mathcal{A}_n \subseteq \mathcal{E}_n$ for all $n \in \mathbb{N}$, and $\mathcal{A}_n$ is isomorphic to $\mathbb{R}\langle X_1, \ldots, X_n\rangle$

Definition 1.1. A *differentiable system* is a sequence

$$\mathcal{C} = \{\mathcal{C}_n;\, n \in \mathbb{N}\}$$

such that, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is a local subring of $\mathcal{E}_n$ and the following hold:

(C1) $\mathcal{N}_n \subseteq \mathcal{C}_n \subseteq \mathcal{E}_n$;
(C2) if $\varphi_1, \ldots, \varphi_n \in \mathcal{C}_m$ are such that $\varphi_1(0) = \cdots = \varphi_n(0) = 0$, then for every $f \in \mathcal{C}_n$ the composition $f(\varphi_1, \ldots, \varphi_n)$ belongs to $\mathcal{C}_m$;
(C3) $\partial f/\partial x_i \in \mathcal{C}_n$ for every $f \in \mathcal{C}_n$ and each $i = 1, \ldots, n$.

Let

$$\widehat{\cdot} : \mathcal{C}_n \to \mathbb{R}[[X]]$$

be the map which associates to each $f \in \mathcal{C}_n$ its Taylor expansion. We consider the following condition:

(C4) $\widehat{\cdot}$ is an injective homomorphism.

Definition 1.2. A differentiable system is called *quasianalytic* if the condition (C4) holds.

We say that the Weierstrass division theorem holds in the differentiable quasianalytic system $(\mathcal{C}_n)_n$ if the following condition is satisfied, for each $n \in \mathbb{N}$:

(W_n) If $f \in \mathcal{C}_n$ and $\widehat{f}(0, X_n) \in \mathbb{R}[[X_n]]$ is nonzero of order d, then for every $g \in \mathcal{C}_n$ there are $q \in \mathcal{C}_n$ and $r_i \in \mathcal{C}_{n-1}$, $i = 1, \ldots, d-1$, such that

$$g = qf + (r_{d-1}X_n^{d-1} + \cdots + r_0).$$

The following theorem is an extension of the main result of Childress' paper [2].

Theorem 1. *Assume that* (W_3) *holds in the differentiable quasianalytic system* $(\mathcal{C}_n)_{n\in\mathbb{N}}$. *Then* $\mathcal{C}_n \subseteq \mathcal{A}_n$ *for every* $n \in \mathbb{N}$.

2. Proof of Theorem 1. For the proof of Theorem 1 we need a few lemmas.

Lemma 2.1. *Under the hypothesis of Theorem* 1, *we have*

$$\mathcal{C}_1 \subseteq \mathcal{A}_1.$$

Proof. Let $f \in \mathcal{C}_1$. Introduce a new variable T and put

$$g(X_1, T) = f(X_1 + T).$$

We have $g \in \mathcal{C}_2$. By (W_3), there are $q \in \mathcal{C}_3$ and $r_0, r_1 \in \mathcal{C}_2$ such that

$$g(X_1, T) = (T^2 + Y_1^2)q(X_1, Y_1, T) + r_1(X_1, Y_1)T + r_0(X_1, Y_1). \tag{2.1}$$

This yields the formal equation

$$\widehat{g}(X_1, T) = (T^2 + Y_1^2)\widehat{q}(X_1, Y_1, T) + \widehat{r}_1(X_1, Y_1)T + \widehat{r}_0(X_1, Y_1) \tag{2.2}$$

and then, after setting $T = iY_1$,

$$\widehat{f}(X_1 + iY_1) = \widehat{r}_0(X_1, Y_1) + iY_1\widehat{r}_1(X_1, Y_1). \tag{2.3}$$

We put

$$u(X_1, Y_1) = r_0(X_1, Y_1) \quad \text{and} \quad v(X_1, Y_1) = Y_1 r_1(X_1, Y_1). \tag{2.4}$$

Note that, by (2.1) with $Y_1 = T = 0$,

$$f(X_1) = g(X_1, 0) = r_0(X_1, 0) = u(X_1, 0).$$

Hence, if u is proved to be analytic, f will be analytic as well.

From (2.3), we have the Cauchy–Riemann equalities

$$\frac{\partial \widehat{u}}{\partial X_1} = \frac{\partial \widehat{v}}{\partial Y_1} \quad \text{and} \quad \frac{\partial \widehat{v}}{\partial X_1} = -\frac{\partial \widehat{u}}{\partial Y_1}.$$

Then, by quasianalyticity, these equalities are satisfied by the functions u and v themselves.

The function given by $F(X_1 + iY_1) = u(X_1, Y_1) + iv(X_1, Y_1)$ is then holomorphic in a neighborhood of the origin in $\mathbb{C}$. Then u and v belong to $\mathcal{A}_2$, and hence $f \in \mathcal{A}_1$. ∎

Let $f \in \mathbb{R}[[X_1, \ldots, X_n]]$ and $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$. If $\xi \in \mathbb{S}^{n-1}$, write $f_\xi(t) = f(t\xi) \in \mathbb{R}[[t]]$.

Lemma 2.2 ([1]). *Let $f \in \mathbb{R}[[X_1, \ldots, X_n]]$. Assume that $f_\xi(t) \in \mathbb{R}\langle t\rangle$ for each $\xi \in \mathbb{S}^{n-1}$. Then $f \in \mathbb{R}\langle X_1, \ldots, X_n\rangle$.*

Proof of Theorem 1. Assume that (W_3) holds for the system $(\mathcal{C}_n)_n$. Let $f \in \mathcal{C}_n$. For each $\xi \in \mathbb{S}^{n-1}$, we have $f_\xi \in \mathcal{C}_1$. By Lemma 2.1, $f_\xi \in \mathbb{R}\langle t\rangle$ for each $\xi \in \mathbb{S}^{n-1}$. Hence by Lemma 2.2, $f \in \mathbb{R}\langle X_1, \ldots, X_n\rangle \simeq \mathcal{A}_n$. ∎

3. On the Taylor map.

3. On the Taylor map. Let $(\mathcal{C}_n)_n$ be a quasianalytic system. The purpose of this section is to prove the following:

Theorem 2. *For every $n \geq 3$, the injection $\widehat{\cdot} : \mathcal{C}_n \to \mathbb{R}[[X]]$ is not surjective.*

Lemma 3.1. *Assume that $\widehat{\cdot} : \mathcal{C}_3 \to \mathbb{R}[[X_1, X_2, X_3]]$ is surjective. Then* (W_3) *holds for the system $(\mathcal{C}_n)_n$.*

Proof. Let $f \in \mathcal{C}_3$ with $\widehat{f}(0,0,X_3) \in \mathbb{R}[[X_3]]$ nonzero of order d. Then for every $g \in \mathcal{C}_3$ there are $q' \in \mathbb{R}[[X_1,X_2,X_3]]$ and $r'_i \in \mathbb{R}[[X_1,X_2]]$, $i = 1,\dots,d-1$, such that

$$\widehat{g} = q'\widehat{f} + (r'_{d-1}X_3^{d-1} + \cdots + r'_0).$$

By hypothesis, there exists $q \in \mathcal{C}_3$ such that $\widehat{q} = q'$ and there are $r_{d-1},\dots,r_0$ in $\mathcal{C}_2$ such that $\widehat{r_i} = r'_i$ for all $i = 0,\dots,d-1$. Since the system is quasianalytic, we have

$$g = qf + (r_{d-1}X_3^{d-1} + \cdots + r_0),$$

which proves the lemma. ■

Proof of Theorem 2. If $\widehat{\cdot} : \mathcal{C}_3 \to \mathbb{R}[[X_1,X_2,X_3]]$ were surjective, then, by Lemma 3.1, (W_3) would hold and, by Theorem 1, $\mathcal{C}_n \subset \mathcal{A}_n$ for each $n \in \mathbb{N}$, which contradicts the hypothesis of surjectivity. ■

We do not know if the conclusion of Theorem 2 holds for the case $1 \le n \le 2$.

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