STUDIA MATHEMATICA 165 (1) (2004)

Weak convergence of summation processes in Besov spaces

by

BRUNO MOREL (Lille)

Abstract. We prove invariance principles for partial sum processes in Besov spaces. This functional framework allows us to give a unified treatment of the step process and the smoothed process in the same parametric scale of function spaces. Our functional central limit theorems in Besov spaces hold for i.i.d. sequences and also for a large class of weakly dependent sequences.

1. Introduction. Let (ξ_n) be a sequence of stochastic processes with index set T. The distribution of ξ_n may be considered as a probability measure μ_n on some suitable topological function space S. When $T = [0,1]$, the classical S for limit-distribution theorems such as invariance principles is $C[0,1]$ for continuous processes or the Skorokhod space $D[0,1]$ for processes having only discontinuities of the first kind. As already pointed out by Lamperti [16], taking S as small as possible allows more continuous functionals on S, which are the real center of interest, as far as statistical applications are concerned. So the smaller the space S, the stronger the corresponding limit theorem. An obvious bound in this shrinkage is given by the maximal smoothness shared by the paths of ξ_n and of the limiting process ξ.

With this general motivation, in this paper we discuss some extensions of Donsker's invariance principle. In the whole paper we interchangeably use the terms *weak convergence* and *convergence in distribution*; both mean weak convergence of distribution measures. We consider two most classical partial sum processes built on a given sequence (X_k) of random variables: ζ_n is the random step function defined by

$$\zeta_n(t) := n^{-1/2} \sum_{k=1}^{n} X_k \mathbb{1}_{\{k/n \le t\}}, \qquad 0 \le t \le 1,$$

and ξ_n is the random polygonal line interpolating ζ_n at the points $t = k/n$,

$$\xi_n(t) := n^{1/2} \sum_{k=1}^{n} X_k |[(k-1)/n, k/n] \cap [0,t]|, \qquad 0 \le t \le 1,$$

2000 *Mathematics Subject Classification*: Primary 60F17; Secondary 46E35, 60G17.

Key words and phrases: association, Besov spaces, compactness, weak convergence, invariance principle, summation processes, strong mixing, tightness.

where $|A|$ denotes the (one-dimensional) Lebesgue measure of the Borel set A. This unusual formulation of ξ_n enables a convenient extension to the case $T = [0,1]^d$. It is well known that if (X_k) is i.i.d. with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$, then ζ_n weakly converges in $D[0,1]$ to the Wiener process W, while ξ_n weakly converges in $C[0,1]$ to the same limit. In what follows, these convergences are referred to as *Donsker's invariance principles* or *functional central limit theorems* (FCLT) in the spaces $D[0,1]$ or $C[0,1]$.

Donsker's result has been widely extended to various kinds of dependent sequences (X_k). For simplicity we only consider the strong mixing and association. In the strong mixing case, the functional central limit theorem was successively studied by Davydov [8], Oodaïra and Yoshihara [20], and Doukhan, Massart and Rio [12]. In the associated case, it was studied by Newman and Wright [19]. All these invariance principles rely on moment inequalities for partial sums.

On the other hand, improvement of Donsker's theorem by a suitable restriction of S in order to fit the regularity of the paths goes back to Lamperti [16]. Since both the smooth summation process ξ_n and the Wiener process W have (almost surely) paths in Hölder spaces $C^\alpha[0,1]$, $0 < \alpha < 1/2$, it is natural to extend Donsker's result to these spaces. Lamperti [16] obtains an invariance principle for the smoothed summation process ξ_n in Hölder spaces $C^\alpha[0,1]$, $0 < \alpha < 1/2 - 1/\gamma$, for i.i.d. variables having a finite moment of order $\gamma > 2$. This assumption, stronger than the usual square integrability of X_1, cannot be relaxed as shown by the following example. By considering an i.i.d. sequence $(X_k;\, k \geq 1)$ of symmetric random variables such that $P(X_1 \geq u) = 1/(2u^p)$, $u \geq 1$, Lamperti [16] noticed that the corresponding (ξ_n) is not tight in $C^\alpha[0,1]$ for $\alpha = 1/2 - 1/p$. Recently Račkauskas and Suquet [22] proved more precisely that if ξ_n satisfies the invariance principle in $C^\alpha[0,1]$ for some $0 < \alpha < 1/2$, then necessarily $\sup_{t>0} t^p P(|X_1| > t) < \infty$, for any $p < 1/(1/2 - \alpha)$. Lamperti's FCLT was extended by Hamadouche [15] to various kinds of weak dependent sequences. In the i.i.d. case, Erickson [13] proves the convergence of the smooth process ξ_n when $T = [0,1]^d$ in a more precise scale of function spaces than the Hölder scale. However he considers the d-dimensional indexed step process ζ_n in the Skorokhod space.

Although the smoothed process ξ_n has been studied in the one-parameter family of Hölder spaces for a long time, only few attempts were made to establish the convergence of the step process ζ_n in a parametric family of function spaces rather than in the Skorokhod space. Boufoussi, Chassaing and Roynette [4] use isomorphisms between Besov spaces and certain sequence spaces to obtain an invariance principle for X_1 belonging to the domain of attraction of a stable law with parameter α, $1 < \alpha < 2$. The three-parameter scale of Besov spaces is involved in interpolation theory, approximation problems, partial differential equations and wavelet theory.

It extends classical families of function spaces such as Hölder spaces, Sobolev spaces and most of their generalizations. Gaussian processes in Besov spaces are discussed by Ciesielski, Kerkyacharian and Roynette [7]. The Besov regularity of the Wiener process was obtained by Roynette [23]. Connection between Besov spaces and wavelets have also been used in statistics (see Donoho *et al.* [10] and the references therein).

In this paper, we prove an invariance principle in Besov spaces for both step and smoothed processes with index set $T = [0,1]^d$. As in the above literature, our method relies on moment inequalities of order $\gamma > 2$. Considering Besov spaces allows us to give a unified proof for step and smoothed summation processes in the same parametric scale of function spaces. Another feature of our approach consists in dealing directly with Besov spaces through their definition by a modulus of smoothness instead of considering isomorphisms with sequence spaces. Continuous embeddings between Besov spaces and Hölder spaces allow us to derive Lamperti's FCLT in Hölder spaces and its extensions to dependent sequences and to dimension $d > 1$ from our main theorem under the same assumptions as in the above references. This means that there is no loss in replacing Hölder spaces by Besov spaces for the study of the smoothed summation process ξ_n. The special case $d = 1$ is amazing because the optimal result in Besov spaces for the step process ζ_n is obtained under a moment assumption of order two. Stronger moment assumptions give no stronger invariance principle in contrast to the case $d > 1$.

A definition and some preliminary analytical results on Besov spaces are stated in Section 2. We recall some continuous embeddings and give a compactness criterion. Section 3 deals with weak convergence. We establish a characterization of tightness together with two practical sufficient conditions. The main theorem is given in Section 4, whereas its proof is postponed to Section 5. This general invariance principle in Besov spaces holds under independence, strong mixing or association, as shown in Section 4.

2. Besov spaces. All the functions considered in this paper are real-valued and defined on $[0,1]^d$. For $x = (x_1, \ldots, x_d)$ let $|x| := \max_{1\le k\le d} |x_k|$. Fix $(p,s,q) \in [1,\infty)\times(0,\infty)\times[1,\infty)$ and denote by $m = [s+1]$ the smallest integer such that $m > s$. Introduce the following modulus of smoothness for $f \in L^p = L^p([0,1]^d)$ and $\delta > 0$:

$$\omega_{p,s,q}(f,\delta) := \biggl(\int\limits_{0<|h|<\delta} \frac{\|\Delta_h^m f\|_p^q}{|h|^{sq}} \frac{dh}{|h|^d} \biggr)^{1/q},$$

where Δ_h^m is the finite difference operator defined by $\Delta_h^1 f(x) := f(x+h) - f(x)$ and $\Delta_h^{m+1} := \Delta_h^m \Delta_h^1$. In the above definition we should take care of the domain for the p-norm; our convention is

$$\|\Delta_h^m f\|_p := \Big(\int_{D_{m,h}} |\Delta_h^m f(t)|^p \, dt \Big)^{1/p},$$

where $D_{m,h}$ is the subset of $[0,1]^d$ on which $\Delta_h^m f(t)$ is defined. With the above notations, the Besov space $B_p^{s,q} = B_p^{s,q}([0,1]^d)$ is defined by

$$B_p^{s,q} := \{f \in L^p : \omega_{p,s,q}(f,1) < \infty\}$$

and endowed with the norm

$$\|f\|_{p,s,q} := \|f\|_p + \omega_{p,s,q}(f,1).$$

Below we recall some classical results on Besov spaces. The reader is referred to Triebel [25], Bergh–Löfström [1] and Peetre [21] for further information, more general definitions ($0 < p, q \leq \infty$, $s \in \mathbb{R}$) and proofs.

With the above notations $(B_p^{s,q}, \| \ \|_{p,s,q})$ is a separable Banach space (p and q are both finite). For $1 \leq p_1, p_2, q_1, q_2 < \infty$, $s_1, s_2 > 0$, the following continuous embeddings hold:

$$B_{p_1}^{s_1,q_1} \hookrightarrow B_{p_2}^{s_2,q_1}, \qquad p_2 > p_1,\ s_1 - d/p_1 \geq s_2 - d/p_2.$$

It is worth noticing that $B_p^{s,q}$ contains only continuous functions when $s > d/p$. Recall that membership of f in Hölder spaces C^α, $\alpha > 0$, is defined by

$$\sum_{L(k) \leq [\alpha]} \sup_x |D^k f(x)| + \sum_{L(k)=[\alpha]} \sup_{x \neq y} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^{\alpha-[\alpha]}} < \infty,$$

where $[\alpha]$ denotes the integer part of α and $k = (k_1, \ldots, k_d)$ denotes a multi-integer with length $L(k) := k_1 + \cdots + k_d$. The following continuous embeddings hold:

$$B_p^{s,q} \hookrightarrow C^\alpha, \qquad s - d/p > \alpha.$$

For noninteger α, C^α can be seen as $B_\infty^{\alpha,\infty}$, although C^1 is strictly contained in the Zygmund space $B_\infty^{1,\infty}$. This will allow us to compare our results in Besov spaces with those obtained in Hölder spaces. For step functions, can we compare the Skorokhod topology with a Besov one? Notice first that the condition $s < 1/p$ characterizes membership of step functions in separable Besov spaces. As the following two examples show in the case $d = 1$, it is not difficult to construct a sequence of step functions which converges in each $B_p^{s,q}$, $0 < s < 1/p$, but not in the Skorokhod space, and another one which converges in the Skorokhod sense (and even uniformly), but in no $B_p^{s,q}$ with $0 < s < 1/p$.

Example 2.1. *The sequence of functions defined by*

$$f_n := \mathbb{1}_{[\frac{1}{2}, \frac{1}{2} + \frac{1}{n})}$$

converges to 0 *in* $B_p^{s,q}$ *for every* (p,s,q) *with* $1 \le p < \infty$, $s < 1/p$ *and* $1 \le q < \infty$, *although it does not converge for the Skorokhod topology.*

Proof. Let $1 \le p < \infty$. Elementary calculations lead to $\|f_n\|_p = n^{-1/p}$, then to $\|\Delta_h^1 f_n\|_p = \min(2|h|, 2/n)^{1/p}$ and lastly to $\omega_{p,s,q}(f_n, 1) \le Cn^{s-1/p}$. Thus we obtain the convergence of f_n to 0 in $B_p^{s,q}$ for any $s < 1/p$ and any q. On the other hand should f_n be convergent for the Skorokhod topology, the limit would be 0 too (by uniqueness of the limit in L^p) and because this limit is continuous the convergence would be uniform, which is absurd. ∎

Example 2.2. *The sequence of functions defined by*

$$g_n := \frac{1}{\ln n} \sum_{k=0}^{n-1} \mathbb{1}_{[\frac{2k}{2n}, \frac{2k+1}{2n})}$$

converges uniformly to 0, *but does not converge in* $B_p^{s,q}$ *for any* (p,s,q) *with* $1 \le p < \infty$, $0 < s < 1/p$ *and* $1 \le q < \infty$.

Proof. The convergence of g_n to 0 is uniform because $\|g_n\|_\infty = (\ln n)^{-1}$, but elementary calculations lead to

$$\|\Delta_h^1 g_n\|_p \ge (\ln n)^{-1} |nh|^{1/p} \quad \text{for } |h| \le 1/2n.$$

Therefore

$$\|g_n\|_{p,s,q} \ge \omega_{p,s,q}(g_n, 1) \ge \omega_{p,s,q}(g_n, 1/2n) \ge Cn^s (\ln n)^{-1}$$

and $\|g_n\|_{p,s,q}$ goes to infinity for each $s > 0$. ∎

Therefore neither Besov nor Skorokhod topology is stronger than the other, as far as step functions are concerned. However, both Skorokhod and $B_p^{s,q}$ convergences imply L^p convergence.

After these topological remarks, we give a compactness criterion that provides the tightness criteria stated in the next section. This criterion is a corollary of the Riesz–Fréchet–Kolmogorov theorem (see e.g. Brézis [5, Theorem IV.25]), which ensures that a subset K of L^p $(1 \le p < \infty)$ is relatively compact if and only if it is bounded in L^p and satisfies

$$\lim_{h \to 0} \sup_{f \in K} \|\Delta_h^1 f\|_p = 0. \tag{1}$$

Proposition 2.3. *A subset* K *of* $B_p^{s,q}$ $(1 \le p < \infty,\ s > 0,\ 1 \le q < \infty)$ *is relatively compact if and only if it is bounded in* L^p *and satisfies*

$$\lim_{a \to 0} \sup_{f \in K} \omega_{p,s,q}(f, a) = 0. \tag{2}$$

Proof. The boundedness in L^p is clearly necessary. Let $\varepsilon > 0$ and let $g_1, \dots, g_m$ be an ε-net for $\| \ \|_{p,s,q}$. Since (2) is obvious for a finite subset, by using the elementary inequality

$$\omega_{p,s,q}(f, a) \le \omega_{p,s,q}(g, a) + \|f - g\|_{p,s,q}$$

we see that $\omega_{p,s,q}(f,a) < 2\varepsilon$ uniformly for $f \in K$ for small enough a. Conversely, the conditions are sufficient. Let $\varepsilon > 0$ and $b > 0$ be such that $\omega_{p,s,q}(f,b) \le \varepsilon$ for all $f \in K$. Let

$$c_b := \Big(\int\limits_{b \le |h| < 1} \frac{2^q}{|h|^{sq}} \frac{dh}{|h|^d} \Big)^{1/q}.$$

Because (2) implies (1), K is relatively compact in L^p by the Riesz–Fréchet–Kolmogorov theorem. Thus K can be covered by a finite number of balls whose $\| \, \|_p$-radii are less than $\varepsilon/(1+c_b)$. The elementary inequalities

$$\|\Delta_h^m f\|_p \le 2^m \|f\|_p, \qquad \omega_{p,s,q}(f-g,a) \le \omega_{p,s,q}(f,a) + \omega_{p,s,q}(g,a)$$

give immediately

$$\|f-g\|_{p,s,q} \le \|f-g\|_p + \Big(\int\limits_{b \le |h| < 1} \frac{2^{mq}\|f-g\|_p^q}{|h|^{sq}} \frac{dh}{|h|^d} \Big)^{1/q} + \omega_{p,s,q}(f,b) + \omega_{p,s,q}(g,b),$$

which shows that K can be covered by a finite number of balls whose $\| \, \|_{p,s,q}$-radii are less than 3ε. ■

3. Weak convergence in Besov spaces. In Section 2, the Besov space $B_p^{s,q}$ ($1 \le p < \infty$, $s > 0$, $1 \le q < \infty$) is presented as a separable Banach space equipped with a modulus of smoothness that provides a nice compactness criterion. Let us recall a well-known consequence of the Prokhorov theorem (see for instance Ledoux–Talagrand [17, Section 2.1]). In order to prove the weak convergence of a sequence (ξ_n) of random elements in a separable Banach space B to a limit ξ, it suffices to check the weak convergence of $(f(\xi_n))$ to $f(\xi)$ in $\mathbb{R}$ for any f in a dense subset of the topological dual B' and the tightness of (ξ_n) in B. Real-valued stochastic processes indexed by $[0,1]^d$ with paths in $B_p^{s,q}$ can be seen as random elements in $B := B_p^{s,q}$.

Now we present a necessary and sufficient condition for tightness, deduced from the preliminary framework presented in Section 2; then we deduce two sufficient conditions which will be used in Section 4 to establish invariance principles.

THEOREM 3.1. *A sequence (ξ_n) of random variables with values in $B_p^{s,q}$ ($1 \le p < \infty$, $s > 0$, $1 \le q < \infty$) is tight if and only if*

$$\lim_{M\to\infty} \sup_n \mathbb{P}\{\|\xi_n\|_p > M\} = 0 \tag{3}$$

and for each $\varepsilon > 0$,

$$\lim_{a\to 0} \sup_n \mathbb{P}\{\omega_{p,s,q}(\xi_n,a) > \varepsilon\} = 0. \tag{4}$$

Proof. This theorem is analogous to the classical one in $C([0,1])$ (see for example Theorem 8.2 in Billingsley [2, p. 55]). The proof is exactly the same, except that the Arzelà–Ascoli theorem is replaced by the compactness criterion given by Proposition 2.3. ■

COROLLARY 3.2. *Let $p \in [1,\infty)$ and (ξ_n) be a sequence of stochastic processes indexed by $[0,1]^d$. Suppose there exists a constant $c_1 > 0$ such that*

$$\sup_n \mathbb{E}|\xi_n(t)|^p \le c_1, \quad t \in [0,1]^d, \tag{5}$$

and there exist constants $\delta, c_2 > 0$ and an integer $m \ge \delta/p$ such that

$$\sup_n \mathbb{E}|\Delta_h^m \xi_n(t)|^p \le c_2|h|^\delta, \quad h \in [0,1]^d,\ t \in D_{m,h}. \tag{6}$$

Then (ξ_n) is tight in $B_p^{s,p}$, $0 < s < \delta/p$.

Proof. We only use the "if" part of Theorem 3.1. By the Markov inequality and Fubini theorem, we deduce (3) from (5) and (4) from (6). ■

Boufoussi, Chassaing and Roynette [4] obtain an analogous result by a different approach. Note that (3) and (4) imply $\mathbb{E}\|\xi_n\|_{p,s,p}^p < \infty$ for $0 < s < \delta/p$. Consequently, ξ_n has paths in $B_p^{s,p}$, $0 < s < \delta/p$, almost surely. This first corollary is obtained under assumptions which are similar to Kolmogorov's conditions for membership in Hölder spaces. It is worth noticing that we can take $\delta > 0$ instead of $\delta \ge 1$. Therefore condition (6) does not assume the continuity of the paths. However, (6) seems sometimes too strong a requirement, especially when considering jump processes like partial sums. The next corollary explains how the moment inequality (6) can be replaced by a weaker assumption, which is just (6) for $|h|$ not too small, as soon as we monitor the convergence in probability of the corresponding modulus of smoothness.

COROLLARY 3.3. *Let $p \in [1,\infty)$ and (ξ_n) be a sequence of stochastic processes indexed by $[0,1]^d$. Suppose there exists a constant $c_1 > 0$ such that*

$$\sup_n \mathbb{E}|\xi_n(t)|^p \le c_1, \quad t \in [0,1]^d. \tag{7}$$

Moreover assume that for a sequence (a_n) decreasing to 0, some constants $\delta, c_2 > 0$ and an integer $m \ge \delta/p$,

$$\sup_n \mathbb{E}|\Delta_h^m \xi_n(t)|^p \le c_2|h|^\delta, \quad |h| \ge a_n,\ t \in D_{m,h}, \tag{8}$$

and for each $0 < s < \delta/p$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{\omega_{p,s,p}(\xi_n, a_n) > \varepsilon\} = 0. \tag{9}$$

Then (ξ_n) is tight in $B_p^{s,p}$, $0 < s < \delta/p$.

Proof. Let $b_n = \min(a, a_n)$ and notice that

$$\mathbb{P}\{\omega_{p,s,p}(\xi_n, a) > \varepsilon\} \le \mathbb{P}\{\omega_{p,s,p}(\xi_n, b_n) > \varepsilon/2\} + \mathbb{P}\{\omega_{p,s,p}(\xi_n, b_n, a_n) > \varepsilon/2\},$$

with

$$\omega_{p,s,q}(f,\delta,\eta) := \Big(\int_{\delta\le|h|<\eta} \frac{\|\Delta_h^m f\|_p^q}{|h|^{sq}}\,\frac{dh}{|h|^d}\Big)^{1/q}.$$

Then an easy adaptation of the proof of Corollary 3.2 gives the result. ∎

Introducing a decreasing sequence (a_n) goes back to Davydov [9], who studied the convergence of discontinuous processes to continuous ones in the Skorokhod space $D[0,1]$. Hamadouche [15] uses such a sequence to prove a tightness condition in Hölder spaces C^α, $0 < \alpha < 1$. Hamadouche's result can be obtained as a consequence of Corollary 3.3 via the continuous embeddings recalled in Section 2.

4. Invariance principle for summation processes. We generalize to dimension d the definition of summation processes given in Section 1. Consider a strictly stationary sequence $(X_j;\, j \in \mathbb{Z}^d)$ and denote by Q_n the d-dimensional discrete cube $Q_n := \{1,\dots,n\}^d$. The *jump summation processes* are defined by

$$\zeta_n(t) := n^{-d/2}\sum_{j\in Q_n} X_j \mathbb{1}_{\{j/n\leqslant t\}}, \qquad t\in[0,1]^d,$$

where $x \leqslant y$ means $x_1 \le y_1, \dots, x_d \le y_d$ and $j/n := (j_1/n,\dots,j_d/n)$. The *smoothed summation processes* are defined by

$$\xi_n(t) := n^{d/2}\sum_{j\in Q_n} X_j |R_{n,j}\cap[0,t]|, \qquad t\in[0,1]^d,$$

where $|A|$ denotes the (d-dimensional) Lebesgue measure of A and the elementary cube $R_{n,j}$, $j \in Q_n$, is defined by $R_{n,j} := [(j_1-1)/n,\ j_1/n] \times \dots \times [(j_d-1)/n, j_d/n]$. Denote by $W = (W(t), t \in [0,1]^d)$ a standard d-dimensional Brownian sheet, i.e. a centered Gaussian process with covariance $\mathbb{E}[W_tW_u] = \min(t_1,u_1)\cdots\min(t_d,u_d)$. Let $r \ge 1$. Since $\mathbb{E}|W_t - W_u|^{2r} \le C_r|t-u|^r$, we get the finiteness of $\mathbb{E}\|W\|_{2r,s,2r}$ by Fubini's theorem. Therefore W lies almost surely in $B_p^{s,p}$, $1 \le p < \infty$, $0 < s < 1/2$. By continuous embeddings (see Section 2) we can deduce that W lies in $B_p^{s,q}$, $1 \le p, q < \infty$, $0 < s < 1/2$. Roynette [23] provides an extensive study of membership in Besov spaces for Brownian motion ($d = 1$). For our purpose, it is sufficient to notice that the bound for membership of W in *separable* Besov spaces is $s < 1/2$. The analogous bound for ζ_n (resp. ξ_n) is $s < 1/p$ (resp. $s \le 1$).

Theorem 4.1 (Main theorem). *Let $(X_j;\, j \in \mathbb{Z}^d)$ be a strictly stationary sequence satisfying the following conditions*:

$$\sum_{k\in\mathbb{Z}^d} |\mathrm{Cov}(X_0, X_k)| < \infty; \tag{10}$$

for each $g \in C^\infty([0,1]^d)$, there is a Gaussian random variable G such that

$$n^{-d/2} \sum_{j\in Q_n} g(j/n) X_j \xrightarrow[n\to\infty]{d} G; \tag{11}$$

and there exist $\gamma > 2$, $c > 0$ such that for all finite discrete parallelepipeds J,

$$\mathbb{E}\Big|\sum_{j\in J} X_j\Big|^\gamma \le c|J|^{\gamma/2}. \tag{12}$$

Then

(a) (ζ_n) *converges in distribution to σW in $B_\gamma^{s,\gamma}$*, $0 < s < 1/\gamma$,
(b) (ξ_n) *converges in distribution to σW in $B_\gamma^{s,\gamma}$*, $0 < s < 1/2$,

where $\sigma^2 := \sum_{j\in\mathbb{Z}^d} \mathrm{Cov}(X_0, X_j)$.

In (12), $|J|$ denotes the *cardinality* of the finite index set J. In what follows we shall use freely the same notation for cardinality and Lebesgue measure, the context helping to dispell initial doubts on the meaning of the formulas. We postpone the proof of the main theorem to Section 5. Since $\gamma > 2$, the convergence result obtained for ξ_n is stronger than for ζ_n. This is not surprising, because ξ_n is a smoothed version of ζ_n. The case $\gamma = 2$ is excluded in Theorem 4.1, because hypothesis (10) implies the inequality given in (12) with $\gamma = 2$. However, a careful reading of the proof of Theorem 4.1 (see Section 5) ensures that it is still valid in this case. Therefore we may refer further to the case $\gamma = 2$, although it is not explicitly included in the main theorem. Now we can deduce from Theorem 4.1 the following corollary via the embeddings recalled in Section 2.

Corollary 4.2. *With the above notations and under assumptions of Theorem* 4.1,

(a) (ζ_n) *converges in distribution to σW in $B_p^{s,q}$ for*

$$1 \le p < \infty,\ 0 < s < \min(1/2, 1/p, d/p - (d-1)/\gamma),\ 1 \le q < \infty,$$

(b) (ξ_n) *converges in distribution to σW in $B_p^{s,q}$ for*

$$1 \le p < \infty,\ 0 < s < \min(1/2, 1/2 + d/p - d/\gamma),\ 1 \le q < \infty,$$

(c) *if $\gamma > 2d$, then (ξ_n) converges in distribution to σW in C^α for $0 < \alpha < 1/2 - d/\gamma$.*

Proof. Consequence of the continuous embeddings recalled in Section 2. ∎

Result (c) in this corollary is the d-dimensional generalization of Lamperti's [16] invariance principle in Hölder spaces (Lamperti obtains the case $d = 1$ for i.i.d. variables). It requires $\gamma > 2d$ and can be deduced from (b). Note that (b) holds even if $\gamma \le 2d$. Result (a) provides the natural analogue of (b) for the step process ζ_n. When $d = 1$, the case $\gamma = 2$ (see remarks after Theorem 4.1) gives the convergence of the step process ζ_n in $B_p^{s,q}$, $1 \le p < \infty$, $0 < s < \min(1/p, 1/2)$, $1 \le q < \infty$. This is the optimal result in separable Besov spaces, since $s < 1/p$ (resp. $s < 1/2$) is a necessary condition for membership of ζ_n (resp. W) in $B_p^{s,q}$ with p and q both finite. Therefore a stronger assumption (12) gives no stronger conclusion for the step process ζ_n, in contrast to what happens for the polygonal process ξ_n. This amazing phenomenon occurs only in dimension one.

The rest of this section is devoted to some examples of strictly stationary sequences satisfying the assumptions under which Theorem 4.1 is proved. We begin with the case of independent variables. Afterwards we study two kinds of weak dependence: strong mixing and association.

Corollary 4.3. *Let $(X_j;\, j \in \mathbb{Z}^d)$ be an i.i.d. sequence of random variables with $\mathbb{E}X_0 = 0$ and $\mathbb{E}|X_0|^\gamma < \infty$ for some $\gamma > 2$. Then the conclusion of Theorem* 4.1 *holds with $\sigma^2 = \mathbb{E}X_0^2$.*

Proof. Evidently, the covariance series (10) converges absolutely since it reduces to one term. Assumption (11) is a consequence of the Lindeberg central limit theorem. Moreover (12) is the classical Marcinkiewicz–Zygmund inequality for i.i.d. variables. ∎

Let us now consider strongly mixing variables. We refer to Doukhan [11] for all the results on mixing random variables. We only consider strongly mixing or α-mixing random variables. Let us just recall that most of the classical mixing assumptions imply strong mixing. For a sequence $X = (X_j;\, j \in \mathbb{Z}^d)$ the *strong mixing coefficient* $\alpha_X(r, u, v)$ is defined by

$$\alpha_X(r,u,v) := \sup\{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \sigma(X, \Lambda_1),\, B \in \sigma(X,\Lambda_2),\\ d(\Lambda_1,\Lambda_2) \ge r,\, |\Lambda_1| \le u,\, |\Lambda_2| \le v\},$$

where Λ_i, $i = 1, 2$, are subsets of $\mathbb{Z}^d$ and $\sigma(X, \Lambda_i)$ denotes the σ-algebra generated by $(X_j;\, j \in \Lambda_i)$. We can derive an invariance principle for summation processes built on strongly mixing variables.

Corollary 4.4. *Let $(X_j;\, j \in \mathbb{Z}^d)$ be a strictly stationary strongly mixing sequence of random variables. If there exist $\gamma > 2$ and $\varepsilon > 0$ such that $\mathbb{E}|X_0|^{\gamma+\varepsilon} < \infty$ and*

$$\sum_{r=0}^{\infty} (r+1)^{d(c-u+1)-1} \alpha_X(r,u,v)^{\varepsilon/(c+\varepsilon)} < \infty, \tag{13}$$

where c is an even integer, $c \geq \gamma$ and $u + v \leq c$, then the conclusion of Theorem 4.1 *holds.*

Proof. The verification of assumption (12) reduces to establishing a Marcinkiewicz–Zygmund inequality for strongly mixing variables. This can be found in Doukhan [11, Section 1.4.1]. On the other hand (10) and (11) are implied by the central limit theorem given in Guyon [14, Theorem 3.2] under weaker assumptions than Doukhan's. Therefore Theorem 4.1 holds under Doukhan's assumptions. ∎

Let us now recall that a sequence $(X_j;\, j \in \mathbb{Z}^d)$ is called *associated* if for any finite subset S of $\mathbb{Z}^d$ and any functions f and g mapping $\mathbb{R}^S$ into $\mathbb{R}$ and coordinatewise nondecreasing,

$$\operatorname{Cov}\bigl(f(X_j;\, j \in S), g(X_j;\, j \in S)\bigr) \geq 0.$$

The corresponding invariance principle deduced from the main theorem is the following.

Corollary 4.5. *Let $(X_j;\, j \in \mathbb{Z}^d)$ be a strictly stationary associated sequence of random variables. If there exist $\gamma > 2$ and $\varepsilon > 0$ such that $\mathbb{E}|X_0|^{\gamma+\varepsilon} < \infty$ and*

$$U(n) := \sum_{|k| \geq n} \operatorname{Cov}(X_0, X_k) = O(n^{-d(\gamma-2)(\gamma+\varepsilon)/(2\varepsilon)}), \tag{14}$$

where $|k| = \max(|k_1|, \ldots, |k_d|)$, then the conclusion of Theorem 4.1 *holds.*

Proof. Condition (14) implies the finiteness of

$$\sigma^2 = \sum_{k \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_k).$$

We refer to Bulinskiĭ [6, Corollary 1] for the proof of the Marcinkiewicz–Zygmund inequality (12) under (14). In order to prove (11) we use Newman's theorem for triangular arrays of absolutely continuous functions of associated variables (see [18, Theorem 16 and subsequent remarks on extension to triangular arrays]). ∎

As usual the moment assumption $\mathbb{E}|X_0|^{\gamma+\varepsilon} < \infty$ with $\varepsilon > 0$ on weakly dependent sequences is stronger than the corresponding moment assumption $\mathbb{E}|X_0|^{\gamma} < \infty$ on independent sequences. The other condition (14) plays the same role as (13) in the strong mixing case: it allows one to estimate the lack of independence between variables. When $d = 1$, Bulinskiĭ's condition (14) is exactly Birkel's condition (see Birkel [3])

$$u(n) := 2 \sum_{k=n}^{\infty} \operatorname{Cov}(X_0, X_k) = O(n^{-(\gamma-2)(\gamma+\varepsilon)/(2\varepsilon)}). \tag{15}$$

For strongly mixing variables, Doukhan's condition (13) is essentially the d-dimensional extension of Yokoyama's [26] condition

$$\sum_{r=0}^{\infty}(r+1)^{\gamma/2-1}\alpha_X(r)^{\varepsilon/(\gamma+\varepsilon)}<\infty, \tag{16}$$

where $(X_j;\, j\in\mathbb{Z})$ is a strictly stationary α-mixing sequence with $\mathbb{E}|X_0|^{\gamma+\varepsilon}<\infty$ and $\alpha_X(r)$ is defined by

$$\alpha_X(r):=\sup_{k\in\mathbb{Z}}\{|\mathbb{P}(A\cap B)-\mathbb{P}(A)\mathbb{P}(B)|:\\ A\in\sigma(X_j;\, j\le k),\, B\in\sigma(X_j;\, j\ge r+k)\}.$$

Hamadouche [15] obtains the FCLT in C^α, $\alpha<1/2-1/\gamma$, for the smoothed summation processes ξ_n under Yokoyama's condition (16) (resp. Birkel's condition (15)) in the strongly mixing (resp. associated) case. Theorem 4.1 gives a suitable extension of this result to Besov spaces and to the jump process ζ_n. As explained in Section 2, it is fruitless to compare results for step processes in the Skorokhod space with those in Besov spaces (see e.g. Examples 2.1 and 2.2). However Oodaïra and Yoshihara [20] obtain an invariance principle in $D([0,1])$ for the jump summation process ζ_n built on a strictly stationary strongly mixing sequence satisfying the usual assumption $\mathbb{E}|X_0|^{\gamma+\varepsilon}<\infty$ and

$$\sum_{r=0}^{\infty}\alpha_X(r)^{\varepsilon/(\gamma+\varepsilon)}<\infty. \tag{17}$$

Let us point out that under Oodaïra and Yoshihara's condition (17) we obtain the convergence of the step process ζ_n in $B_2^{s,2}[0,1]$ (see remark on the case $\gamma=2$ after Theorem 4.1) and consequently in any $B_p^{s,q}[0,1]$ for

$$1\le p<\infty,\qquad 0<s<\min(1/2,1/p),\qquad 1\le q<\infty.$$

Oodaïra and Yoshihara's condition is relaxed by Doukhan, Massart and Rio [12] to obtain the FCLT in $D[0,1]$ for strongly mixing variables under a quantile assumption rather than a moment assumption. Their method seems difficult to extend to the general case considered in Theorem 4.1.

5. Proof of the main theorem. The proof of Theorem 4.1 is divided into four lemmas. The first one deals with triangular arrays and requires only the sequence $(X_j;\, j\in\mathbb{Z}^d)$ to be weakly stationary (i.e. the covariances are invariant under translations).

LEMMA 5.1. *Let $(X_j;\, j\in\mathbb{Z}^d)$ be a weakly stationary sequence.*

(a) *Suppose that*

$$\sum_{k\in\mathbb{Z}^d}|\mathrm{Cov}(X_0,X_k)|<\infty.$$

Then

$$\lim_{n\to\infty} \operatorname{Var}\Big(n^{-d/2}\sum_{j\in Q_n} X_j\Big) = \sum_{k\in\mathbb{Z}^d} \operatorname{Cov}(X_0, X_k) =: \sigma^2. \tag{18}$$

(b) *Suppose that the array of real numbers* $(a_{n,j};\ j \in Q_n)$ *satisfies*

$$\lim_{n\to\infty} \sum_{j\in Q_n} a_{n,j}^2 = \tau^2 < \infty, \tag{19}$$

and for each $k \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \sum_{j\in Q_{n,k}} (a_{n,j} - a_{n,j+k})^2 = 0, \tag{20}$$

where $Q_{n,k}$ *denotes the intersection of* Q_n *with its translate* $-k+Q_n$. *Then*

$$\lim_{n\to\infty} \operatorname{Var}\Big(\sum_{j\in Q_n} a_{n,j}X_j\Big) = \sigma^2\tau^2. \tag{21}$$

Proof. For x real, let $x^+ := \max(x, 0)$. The index set $Q_{n,k}$ is either a (discrete) parallelepiped, or empty. Elementary computations show that its cardinality is always equal to

$$|Q_{n,k}| = \prod_{l=1}^{d} (n - |k_l|)^+.$$

Using the weak stationarity, we get

$$\begin{aligned}\operatorname{Var}\Big(n^{-d/2}\sum_{j\in Q_n} X_j\Big) &= \sum_{k\in\mathbb{Z}^d} n^{-d}|Q_{n,k}|\operatorname{Cov}(X_0, X_k)\\ &= \sum_{k\in\mathbb{Z}^d}\prod_{l=1}^{d}\Big(1-\frac{|k_l|}{n}\Big)^+ \operatorname{Cov}(X_0, X_k),\end{aligned}$$

whence (18) follows, by applying the dominated convergence theorem with respect to the counting measure on $\mathbb{Z}^d$. The proof of (21) uses the same arguments, upon observing that

$$\begin{aligned}\operatorname{Var}\Big(\sum_{j\in Q_n} a_{n,j}X_j\Big) &= \sum_{k\in\mathbb{Z}^d} \operatorname{Cov}(X_0, X_k) \sum_{i\in Q_{n,k}} a_{n,i}a_{n,i+k}\\ &= \sum_{k\in\mathbb{Z}^d} \operatorname{Cov}(X_0, X_k) \sum_{i\in Q_{n,k}} (a_{n,i}^2 + (a_{n,i+k} - a_{n,i})a_{n,i}). \ \blacksquare\end{aligned}$$

The second lemma gives the weak convergence against a dense subset of the dual.

LEMMA 5.2. *With the notations of Theorem* 4.1 *and under the assumptions* (10) *and* (11), *for all* $f \in C^\infty[0,1]^d$,

$$\int_{[0,1]^d} f(t)\zeta_n(t)\,dt \xrightarrow[n\to\infty]{d} \int_{[0,1]^d} f(t)\sigma W(t)\,dt,$$

$$\int_{[0,1]^d} f(t)\xi_n(t)\,dt \xrightarrow[n\to\infty]{d} \int_{[0,1]^d} f(t)\sigma W(t)\,dt.$$

Proof. Observe first that the limit is a centered Gaussian random variable whose variance is $\sigma^2\|g\|_2^2$, with

$$g(x) := \int_{[0,1]^d} \mathbb{1}_{\{x\leqslant t\}} f(t)\,dt.$$

Using the definition of the jump processes and exchanging summations gives

$$\int_{[0,1]^d} f(t)\zeta_n(t)\,dt = n^{-d/2}\sum_{j\in Q_n} g(j/n)X_j =: S_n.$$

Since (11) ensures the convergence of S_n, by a classical renormalization argument it suffices to check the convergence of $\mathrm{Var}(S_n)$ to $\sigma^2\|g\|_2^2$. We apply Lemma 5.1 to the triangular array

$$a_{n,j} := n^{-d/2} g(j/n).$$

By Riemann summation,

$$\lim_{n\to\infty} n^{-d}\sum_{j\in Q_n} g(j/n)^2 = \|g\|_2^2.$$

Therefore (19) is satisfied with $\tau^2 := \|g\|_2^2$. Moreover g is Lipschitz, so for any $k \in \mathbb{Z}^d$,

$$n^{-d}\sum_{j\in Q_{n,k}} (g(j/n) - g((j+k)/n))^2 \le C^2\,\frac{|k|^2}{n^2}.$$

Thus (20) is fulfilled and by Lemma 5.1 we get

$$\lim_{n\to\infty} \mathrm{Var}(S_n) = \sigma^2\tau^2 = \sigma^2\|g\|_2^2.$$

The proof for ξ_n can be reduced to that for ζ_n by another application of Lemma 5.1. Using the definition of the jump processes, we get

$$\int_{[0,1]^d} f(t)\xi_n(t)\,dt = n^{d/2}\sum_{j\in Q_n} X_j \int_{R_{n,j}} g(x)\,dx =: S_n'. \tag{22}$$

Define $a_{n,j}' := n^{d/2}\int_{R_{n,j}} g(x)\,dx$ and $b_{n,j} := a_{n,j}' - a_{n,j}$. Notice that

$$|b_{n,j}| \le n^{d/2}\int_{R_{n,j}} |g(j/n) - g(x)|\,dx \le n^{d/2} n^{-d}\,\frac{C}{n} = Cn^{-d/2-1},$$

and for each given $k \in \mathbb{Z}^d$,

$$|b_{n,j} - b_{n,j+k}|^2 \le 2|b_{n,j+k}|^2 + 2|b_{n,j}|^2 \le 4C^2 n^{-(d+2)},$$

so applying Lemma 5.1 to $b_{n,j}$ with $\tau = 0$, we obtain

$$\lim_{n\to\infty} \operatorname{Var}(S_n - S'_n) = 0.$$

Therefore S'_n converges in distribution to the same limit as S_n. ■

Since $C^\infty([0,1]^d)$ is a dense subset of $(B_p^{s,q})'$, $1 < p < \infty$, $s > 0$, $1 < q < \infty$ (see Triebel [24, Sections 2.3.3 and 2.11.1]), Lemma 5.2 gives the convergence against any element of the dual for both (ζ_n) and (ξ_n). Now we use Corollary 3.3 to check the tightness of (ζ_n). The boundedness condition (7) and the moment inequality (8) are consequences of the next lemma.

LEMMA 5.3. *With the notations of Theorem* 4.1 *and under hypothesis* (12), *there exists* $c_1 > 0$ *such that for all* $n \ge 1$ *and* $t \in [0,1]^d$,

$$\mathbb{E}|\zeta_n(t)|^\gamma \le c_1,$$

and there exists $c_2 > 0$ *such that for all* $n \ge 1$ *and* $t, u \in [0,1]^d$,

$$\mathbb{E}|\zeta_n(t) - \zeta_n(u)|^\gamma \le c_2(|t-u| + 1/n)^{\gamma/2}.$$

Proof. Recall that

$$\mathbb{E}|\zeta_n(t)|^\gamma = n^{-d\gamma/2}\Big|\sum_j X_j \mathbb{1}_{\{j/n \le t\}}\Big|^\gamma,$$

and apply the Marcinkiewicz–Zygmund inequality (12) with $J_t = \mathbb{Z}_+^d \cap [0, nt]$ to obtain

$$\mathbb{E}|\zeta_n(t)|^\gamma \le c n^{-d\gamma/2} |J_t|^{\gamma/2}.$$

Since $|J_t| \le n^d$ uniformly in t, $c_1 = c$ is suitable.

Let us now consider

$$\mathbb{E}|\zeta_n(t) - \zeta_n(u)|^\gamma = n^{-d\gamma/2}\mathbb{E}\Big|\sum_j X_j(\mathbb{1}_{\{j/n\le t\}} - \mathbb{1}_{\{j/n \le u\}})\Big|^\gamma,$$

and set $a_j := \mathbb{1}_{\{j/n\le t\}} - \mathbb{1}_{\{j/n\le u\}}$. Observe that

- $a_j = 1$ when $j \in J_+ := \mathbb{Z}_+^d \cap ([0,nt] \setminus [0,nu])$,
- $a_j = -1$ when $j \in J_- := \mathbb{Z}_+^d \cap ([0,nu] \setminus [0,nt])$,
- $a_j = 0$ otherwise.

By the triangle inequality,

$$\mathbb{E}^{1/\gamma}|\zeta_n(t) - \zeta_n(u)|^\gamma \le n^{-d/2}\Big(\mathbb{E}^{1/\gamma}\Big|\sum_{j\in J_+} X_j\Big|^\gamma + \mathbb{E}^{1/\gamma}\Big|\sum_{j\in J_-} X_j\Big|^\gamma\Big).$$

Then, by the Marcinkiewicz–Zygmund inequality,

$$\mathbb{E}^{1/\gamma}|\zeta_n(t) - \zeta_n(u)|^\gamma \le c n^{-d/2}(|J_+|^{1/2} + |J_-|^{1/2}).$$

Elementary calculations provide the estimate

$$\max(|J_+|,|J_-|) \le dn^{d-1}(n|t-u|+1),$$

which leads to

$$\mathbb{E}|\zeta_n(t)-\zeta_n(u)|^\gamma \le c_2(|t-u|+1/n)^{\gamma/2},$$

where $c_2=(2c\sqrt{d})^\gamma$. ∎

So we have (8) for $p=\gamma$, $\delta=\gamma/2$, and $a_n=1/n$. Note that we only need $\delta=1$ for the step process ζ_n. Since $\gamma>2$, we have a better moment inequality than required but this will be helpful further to check the tightness in $B_\gamma^{s,\gamma}$, $0<s<1/2$, of the smooth process ξ_n. It now remains to check (9). This is the purpose of the next lemma.

LEMMA 5.4. *With the notations of Theorem* 4.1 *and under hypothesis* (12), *there exist a random variable* Y_n *and a constant* $C>0$ *such that for all* $n\ge 1$,

$$\mathbb{E}|Y_n|^\gamma \le Cn^{1-\gamma/2}, \tag{23}$$

$$\|\Delta_h^1\zeta_n\|_\gamma \le Y_n|h|^{1/\gamma}, \quad |h|\le 1/n. \tag{24}$$

Proof. By considering $h=h_1e_1+\cdots+h_de_d$, where $\{e_1,\dots,e_d\}$ denotes the canonical basis of $\mathbb{R}^d$, it suffices to prove (24) for each e_kh_k, $1\le k\le d$, and the result follows from the triangle inequality. Thus we assume that $h=h_1e_1$. Then, by symmetry, the assumption $h_1\ge 0$ is not a restriction. Therefore we suppose that $0<h_1\le 1/n$. By definition,

$$\zeta_n(t+h_1e_1)-\zeta_n(t)=n^{-d/2}\sum_j X_j\mathbb{1}_{\{t_1<j_1/n\le t_1+h\}}\mathbb{1}_{\{j_2/n\le t_2\}}\cdots\mathbb{1}_{\{j_d/n\le t_d\}}.$$

Then we use the following splitting:

$$\int_{D_h}|\zeta_n(t+h_1e_1)-\zeta_n(t)|^\gamma\,dt=\sum_{k\in K}\int_{T_k}|\zeta_n(t+h_1e_1)-\zeta_n(t)|^\gamma\,dt,$$

where

$$K:=\{k=(k_2,\dots,k_d):1\le k_2\le n,\dots,1\le k_d\le n\}$$

and

$$T_k:=(0,1-h_1]\times((k_2-1)/n,k_2/n]\times\cdots\times((k_d-1)/n,k_d/n].$$

From the definition of T_k, we get

$$\int_{T_k}\Big|\zeta_n(t+h_1e_1)-\zeta_n(t)\Big|^\gamma dt = n^{-d\gamma/2}\int_{T_k}\Big|\sum_{1\le j\le k}\sum_{1\le j_1\le n}X_{j_1,j_2,\dots,j_d}\mathbb{1}_{\{t_1<j_1/n\le t_1+h_1\}}\Big|^\gamma dt,$$

and so because the intervals $[j/n - h_1, j_1/n)$ are disjoint,

$$\int_{T_k} |\zeta_n(t + h_1 e_1) - \zeta_n(t)|^\gamma \, dt$$
$$= n^{-d\gamma/2} \int_{T_k} \sum_{1 \le j_1 \le n} \Big| \sum_{1 \le j \le k} X_{j_1, j_2, \ldots, j_d} \Big|^\gamma \mathbb{1}_{\{t_1 < j_1/n \le t_1 + h_1\}} \, dt.$$

Therefore

$$\int_{T_k} |\zeta_n(t + h_1 e_1) - \zeta_n(t)|^\gamma \, dt = n^{-d\gamma/2} \sum_{1 \le j_1 \le n} \Big| \sum_{1 \le j \le k} X_{j_1, j_2, \ldots, j_d} \Big|^\gamma |h| n^{1-d},$$

and by summing over k we obtain

$$\|\Delta_h^1 \zeta_n\|_\gamma^\gamma \le |Y_n|^\gamma |h_1|,$$

where

$$|Y_n|^\gamma := n^{1-d-d\gamma/2} \sum_{k \in K} \sum_{1 \le j_1 \le n} \Big| \sum_{1 \le j \le k} X_{j_1, j_2, \ldots, j_d} \Big|^\gamma.$$

Now the Marcinkiewicz–Zygmund inequality (12) implies

$$\mathbb{E}|Y_n|^\gamma \le c n^{1-d-d\gamma/2} \sum_{1 \le j_1 \le n} \sum_{k \in K} \Big(\prod_{i=2}^d k_i \Big)^{\gamma/2}.$$

The elementary estimate

$$\sum_{1 \le j_1 \le n} \sum_{k \in K} \Big(\prod_{i=2}^d k_i \Big)^{\gamma/2} = O(n^{d + (d-1)\gamma/2})$$

finally leads to $\mathbb{E}|Y_n|^\gamma \le C n^{1-\gamma/2}$. ∎

The next step consists in showing that (9) holds via Lemma 5.4. Let $0 < s < 1/\gamma$. By the Fubini theorem, we get

$$\mathbb{E}\omega_{\gamma,s,\gamma}(\zeta_n, 1/n)^\gamma = \int_{0 < |h| < 1/n} |h|^{-s\gamma} \mathbb{E}\|\Delta_h^1 \zeta_n\|_\gamma^\gamma \frac{dh}{|h|^d}.$$

Hence by (24),

$$\mathbb{E}\omega_{\gamma,s,\gamma}(\zeta_n, 1/n)^\gamma \le \mathbb{E}|Y_n|^\gamma \int_{0 < |h| < 1/n} |h|^{1-s\gamma} \frac{dh}{|h|^d}. \tag{25}$$

For $0 < s < 1/\gamma$ the integral in (25) is $O(n^{s\gamma - 1})$, so by combining (25) and (23) we obtain

$$\mathbb{P}\{\omega_{\gamma,s,\gamma}(\zeta_n, 1/n) > \varepsilon\} \le C \varepsilon^{-\gamma} n^{\gamma(s-1/2)}.$$

Therefore (9) holds for $p = \gamma$, $0 < s < 1/\gamma$, $q = \gamma$.

Combining this result with the consequences of Lemma 5.3 ensures by Corollary 3.3 that (ζ_n) is tight in $B_\gamma^{s,\gamma}$, $0 < s < 1/\gamma$. Since Lemma 5.2 allows us to identify the limit, (a) in the main theorem is proved.

Now we use Corollary 3.2 to check the tightness of (ξ_n). For the same reasons as (ζ_n), (ξ_n) satisfies the boundedness condition (7) for $p = \gamma$ under (12). Unlike (ζ_n), (ξ_n) is smooth enough to satisfy the moment inequality (6) for all $t, u \in [0,1]^d$. If $|t-u| \le 1/n$, then t and u are in the same cube $R_{n,j}$ or at worst in adjacent cubes. Elementary techniques show that due to (12), we have

$$\mathbb{E}|\xi_n(t) - \xi_n(u)|^\gamma \le C_d |t-u|^{\gamma/2},$$

where C_d depends on the dimension d but not on n. If $|t-u| \ge 1/n$, let k and l in $\mathbb{Z}^d$ be such that t and k/n (respectively u and l/n) are in the same cube. Then by the triangle inequality,

$$\begin{aligned}\mathbb{E}^{1/\gamma}|\xi_n(t) - \xi_n(u)|^\gamma \le \mathbb{E}^{1/\gamma}|\xi_n(t) - \xi_n(k/n)|^\gamma + \mathbb{E}^{1/\gamma}|\xi_n(l/n) - \xi_n(k/n)|^\gamma \\ + \mathbb{E}^{1/\gamma}|\xi_n(l/n) - \xi_n(u)|^\gamma.\end{aligned}$$

The first and third terms can be bounded using the case $|t-u| \le 1/n$. For the second, notice that $\xi_n(k/n) = \zeta_n(k/n)$ and $\xi_n(l/n) = \zeta_n(l/n)$, then apply Lemma 5.3. Combine all the preceding results to obtain

$$\mathbb{E}|\xi_n(t) - \xi_n(u)|^\gamma \le c_d |t-u|^{\gamma/2}, \quad t, u \in [0,1]^d.$$

Therefore by Corollary 3.2, (ξ_n) is tight in $B_\gamma^{s,\gamma}$, $0 < s < 1/2$. Just as for (ζ_n), Lemma 5.2 allows us to identify the limit. We thus deduce (b) in the main theorem.

Acknowledgments. This work is a part of the author's thesis under the supervision of Professor Ch. Suquet. The author is grateful to the referee for his careful reading and helpful suggestions.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces: an Introduction*, Springer, Berlin, 1976.

[2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[3] T. Birkel, *Moment bounds for associated sequences*, Ann. Probab. 16 (1988), 1184–1193.

[4] B. Boufoussi, P. Chassaing and B. Roynette, *A Kolmogorov criterion and an invariance principle in Besov spaces*, Tech. Rept. No. 24, Prépublications de l'Institut Élie Cartan, Nancy, 1993.

[5] H. Brézis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1983.

[6] A. Bulinskiĭ, *Inequalities for the moments of sums of associated multi-indexed variables*, Theory Probab. Appl. 38 (1994), 342–349.

[7] Z. Ciesielski, G. Kerkyacharian et B. Roynette, *Quelques espaces fonctionnels associés à des processus gaussiens*, Studia Math. 107 (1993), 171–204.

[8] Y. Davydov, *Convergence of distributions generated by stationary stochastic processes*, Theory Probab. Appl. 13 (1968), 691–696.
[9] —, *Weak convergence of discontinuous processes to continuous ones*, in: Probability Theory and Mathematical Statistics, I. Ibragimov and A. Zaitsev (eds.), Gordon and Breach, 1996, 15–16.
[10] D. Donoho, I. Johnstone, G. Kerkyacharian and D. Picard, *Wavelet shrinkage: asymptopia? (with discussion)*, J. Roy. Statist. Soc. Ser. B 57 (1995), 301–369.
[11] P. Doukhan, *Mixing: Properties and Examples*, Lecture Notes in Statist. 85, Springer, Berlin, 1994.
[12] P. Doukhan, P. Massart and E. Rio, *The functional central limit theorem for strongly mixing processes*, Ann. Inst. H. Poincaré Probab. Statist. 30 (1994), 63–82.
[13] R. Erickson, *Lipschitz smoothness and convergence with an application to the central limit theorem for summation processes*, Ann. Probab. 9 (1981), 831–851.
[14] X. Guyon, *Champs aléatoires sur un réseau*, Masson, Paris, 1993.
[15] D. Hamadouche, *Invariance principles in Hölder spaces*, Portugal. Math. 57 (2000), 127–151.
[16] J. Lamperti, *On convergence of stochastic processes*, Trans. Amer. Math. Soc. 104 (1962), 430–435.
[17] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*: *Isoperimetry and Processes*, Springer, Berlin, 1991.
[18] C. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in: IMS Lecture Notes Monogr. Ser. 5, Inst. Math. Statist., Hayward, CA, 1984, 127–140.
[19] C. Newman and A. Wright, *An invariance principle for certain dependent sequences*, Ann. Probab. 9 (1981) 671–675.
[20] H. Oodaïra and K. Yoshihara, *Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition*, Kosadai Math. Sem. Rep. 24 (1972), 259–269.
[21] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Ser. 1, Duke Univ., Durham, 1976.
[22] A. Račkauskas and Ch. Suquet, *On the Hölderian functional central limit theorem for i.i.d. random elements in Banach spaces*, in: Limit Theorems in Probability and Statistics, Vol. II (Balatonlelle, 1999), János Bolyai Math. Soc., Budapest, 2002, 485–498.
[23] B. Roynette, *Mouvement brownien et espaces de Besov*, Stochast. Stochast. Rep. 43 (1993), 221–260.
[24] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[25] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.
[26] R. Yokoyama, *Moment bounds for stationary mixing sequences*, Z. Wahrsch. Verw. Gebiete 52 (1980), 45–57.

Laboratoire de Statistique et de Probabilités
F.R.E. CNRS 2222
Bât. M2, U.F.R. de Mathématiques
Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq Cedex, France
E-mail: bruno.morel@ifrance.com

Received February 14, 2000
Revised version April 26, 2004 (4474)