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Biorthogonal systems in Banach spaces

by

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Abstract. We give biorthogonal system characterizations of Banach spaces that fail the Dunford–Pettis property, contain an isomorphic copy of c_0, or fail the hereditary Dunford–Pettis property. We combine this with previous results to show that each infinite-dimensional Banach space has one of three types of biorthogonal systems.

1. Introduction. When we first encounter an arbitrary Banach space, we usually search for some kind of fundamental structure in the space to make our understanding of it more complete. Very often, if a space has (or fails) a certain property, we can find a fundamental structure within the space that reflects the property (or failure thereof). Of course, in this case, we would like to find a strong structure, like a Schauder basis or finite-dimensional decomposition (FDD), in the space. However, this is not always possible, as even a separable Banach space need not contain a Schauder basis [8]. For this reason it is interesting to consider weaker structures than FDD's and Schauder bases which exist in every separable Banach space and try to prove that a separable Banach space has a certain property if and only if there is a structure in the space which reflects the property.

One useful basis-like structure that has been considered for a long time is that of fundamental total biorthogonal system. Markushevich [11] showed in 1943 that each separable Banach space contains a fundamental total biorthogonal system. The main theorems of the present paper give a biorthogonal system characterization of spaces failing the Dunford–Pettis property and spaces containing an isomorphic copy of c_0. Combining this with work already done in the field yields a theorem about the existence of biorthogonal systems in any given infinite-dimensional Banach space.

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2. Notation and motivation. Throughout this paper, $\mathfrak{X}$ denotes an arbitrary (infinite-dimensional real) Banach space. If $\mathfrak{X}$ is a Banach space, then $\mathfrak{X}^*$ is its topological dual space, $B(\mathfrak{X})$ is its (closed) unit ball, and $S(\mathfrak{X})$ is its unit sphere. If X is a subset of $\mathfrak{X}$, then $\mathrm{sp}\{X\}$ is the linear span of X while $[X]$ is the closed linear span of X. The Kronecker delta δ_{nm} takes the value 1 when $n = m$ and 0 when $n \neq m$.

Definition 2.1. For a subset X of $\mathfrak{X}$ and a subset Z of $\mathfrak{X}^*$:

(1) the *annihilator* of X is $X^\perp = \{x^* \in \mathfrak{X}^* : x^*(x) = 0 \text{ for all } x \in X\}$,
(2) the *preannihilator* of Z is $Z^\top = \{x \in \mathfrak{X} : x^*(x) = 0 \text{ for all } x^* \in Z\}$,
(3) X is *fundamental* if $[X] = \mathfrak{X}$, or equivalently, $X^\perp = \{0\}$,
(4) Z is *total* if the weak*-closure of $\mathrm{sp}\{Z\}$ is $\mathfrak{X}^*$, or equivalently, $Z^\top = \{0\}$,
(5) for a fixed $\tau \geq 1$, Z *τ-norms* X (or X is τ-normed by Z) if

$$\|x\| \leq \tau \sup_{z \in Z \setminus \{0\}} \frac{z(x)}{\|z\|}$$

for each $x \in X$,
(6) Z *norms* X if Z 1-norms X.

It is easy to see that if Z τ-norms $\mathfrak{X}$ for a $\tau \geq 1$ then Z is total.

Definition 2.2. A system $\{x_n, x_n^*\}_{n=1}^\infty$ in $X \times Z$ is

(1) a *biorthogonal system* if $x_n^*(x_m) = \delta_{nm}$,
(2) *M-bounded* if $\{x_n\}$ and $\{x_n^*\}$ are bounded and $\sup_n \|x_n\|\,\|x_n^*\| \leq M$,
(3) *bounded* if it is M-bounded for some (finite) M,
(4) *fundamental* if $\{x_n\}$ is fundamental,
(5) *total* if $\{x_n^*\}$ is total.

A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space $\mathfrak{X}$ is called *semi-normalized* if there are constants $0 < \alpha \leq \beta < \infty$ such that $\alpha \leq \|x_n\| \leq \beta$ for each $n \in \mathbb{N}$. Recall that $\{x_n\}_{n=1}^\infty$ is a *basic sequence* if each x_n is non-zero and there exists a finite constant $K > 0$ such that

$$\Big\|\sum_{j=1}^{m} a_j x_j\Big\| \leq K \Big\|\sum_{j=1}^{n} a_j x_j\Big\| \tag{2.1}$$

for all choices $\{a_j\}_{j \in \mathbb{N}}$ and any integers $m < n$. When this is the case, the smallest K for which (2.1) holds is called the *basis constant* of $\{x_n\}_{n=1}^\infty$ and there exists a biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ such that $\|x_n^*\| \leq 2K/\|x_n\|$.

Operators between Banach spaces are assumed to be bounded and linear. All notation and terminology, not otherwise explained, are as in [6] or [10].

Our motivation begins with the following structure theorem of E. Odell [12]:

THEOREM 2.3. *Every infinite-dimensional Banach space contains a subspace isomorphic to c_0, a subspace isomorphic to ℓ_1 or a subspace that fails the Dunford–Pettis property.*

Our goal is to find a biorthogonal system version of this theorem in which the conditions imposed on the biorthogonal systems directly reflect the property they characterize. Luckily, some of the work, the ℓ_1 case, has already been done for us. In fact, our results are inspired by this previous work. In 2000, S. J. Dilworth, M. Girardi, and W. B. Johnson characterized spaces containing isomorphic copies of ℓ_1 using biorthogonal systems.

THEOREM 2.4 ([7]). *The following statements are equivalent*:

(1) $\ell_1 \hookrightarrow \mathfrak{X}$.

(2) *There is a bounded wc_0^*-stable biorthogonal system in $\mathfrak{X} \times \mathfrak{X}^*$.*

And in the case that $\mathfrak{X}$ is separable:

(3) *There is a bounded fundamental total wc_0^*-stable biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$.*

Furthermore for each $\varepsilon > 0$: if (2) holds then the system can be taken to be $(1+\varepsilon)$-bounded; if (3) holds then the system can be taken to be $[(1+\sqrt{2})+\varepsilon]$-bounded and so that $[x_n^*]$ $(2+\varepsilon)$-norms $\mathfrak{X}$.

Recall that $\{x_n, x_n^*\}$ is a *wc_0^*-stable biorthogonal system* if, for each isomorphic embedding T of $\mathfrak{X}$ into some $\mathcal{Y}$, there exists a lifting $\{y_n^*\}$ of $\{x_n^*\}$ (i.e., $T^* y_n^* = x_n^*$ for each n) such that $\{y_n^*\}$ is a semi-normalized weakly null sequence in $\mathcal{Y}^*$ (or equivalently, such that $\{Tx_n, y_n^*\}$ in $\mathcal{Y} \times \mathcal{Y}^*$ is a wc_0^*-biorthogonal system).

They also characterized Banach spaces that have the *Schur property* (i.e. weak and strong sequential convergence in $\mathfrak{X}$ coincide) via biorthogonal systems. In the next section we will discuss the Dunford–Pettis property. Recall that the Schur property is related to the Dunford–Pettis property and embeddings of ℓ_1 in the following way (cf. [4, p. 23]): $\mathfrak{X}^*$ fails the Schur property if and only if $\mathfrak{X}$ fails the Dunford–Pettis property or $\ell_1 \hookrightarrow \mathfrak{X}$. This fact provides a link between the above results and the results of the next section that characterize failure of the Dunford–Pettis property.

3. Spaces failing the Dunford–Pettis property. Recall that a Banach space $\mathfrak{X}$ has the *Dunford–Pettis property* (DP) if whenever $\{x_n\}_n \subset \mathfrak{X}$ and $\{x_n^*\}_n \subset \mathfrak{X}^*$ are weakly null sequences, we have $\lim_{n\to\infty} x_n^*(x_n) = 0$. We refer the reader to the excellent survey article [4] for a complete treatment of all things Dunford–Pettis. Further results and additional open questions can be found in [2].

Now suppose $\mathfrak{X}$ is a Banach space that fails the Dunford–Pettis property. Then there exists a weakly null sequence $\{w_k\}_{k\in\mathbb{N}}$ in $\mathfrak{X}$ and a weakly null sequence $\{w_k^*\}_{k\in\mathbb{N}}$ in $\mathfrak{X}^*$ such that $\lim_{k\to\infty}|w_k^*(w_k)|\neq 0$. We may assume, without loss of generality, that there exists $\delta>0$ such that $w_k^*(w_k)>\delta$ for each $k\in\mathbb{N}$. If this is not the case we can pass to a suitable subsequence and adjust signs. Now $\{w_k\}_{k\in\mathbb{N}}$ and $\{w_k^*\}_{k\in\mathbb{N}}$ are semi-normalized so we may renormalize if necessary to get for each $k\in\mathbb{N}$:

(1) $w_k\in S(\mathfrak{X})$,
(2) $w_k^*(w_k)=1$,
(3) $1\le\|w_k^*\|\le M$ for some constant M.

This leads to the following definition.

Definition 3.1. Let $M\ge 1$. Then $\mathfrak{X}$ fails the *M-Dunford–Pettis property* (M-DP) provided there is a weakly null sequence $\{w_k\}_k$ from $S(\mathfrak{X})$ and a weakly null sequence $\{w_k^*\}_k$ from $\mathfrak{X}^*$ such that $w_k^*(w_k)=1$ and $1\le\|w_k^*\|\le M$ for each $k\in\mathbb{N}$.

Note that clearly $\mathfrak{X}$ fails M-DP for some M if and only if $\mathfrak{X}$ fails DP. We only bother to define it here to make the statement of Theorem 3.3 a bit clearer.

Definition 3.2. A biorthogonal system $\{x_n,x_n^*\}$ in $\mathfrak{X}\times\mathfrak{X}^*$ is called a DP-*biorthogonal system* if $\{x_n\}$ and $\{x_n^*\}$ are semi-normalized weakly-null sequences.

Theorem 3.3. *The following statements are equivalent*:

(1) *$\mathfrak{X}$ fails the Dunford–Pettis property.*
(2) *There is a bounded* DP-*biorthogonal system in $\mathfrak{X}\times\mathfrak{X}^*$.*

And in the case that $\mathfrak{X}$ is separable:

(3) *There is a bounded fundamental total* DP-*biorthogonal system $\{x_n,x_n^*\}$ in $\mathfrak{X}\times\mathfrak{X}^*$.*

Furthermore, for an $\mathfrak{X}$ failing the M-Dunford–Pettis property, for each $\varepsilon>0$: if (2) *holds then the system can be taken to be $(M+\varepsilon)$-bounded*; *if* (3) *holds then the system can be taken to be $[M(1+\sqrt{2})^2+\varepsilon]$-bounded and so that $[x_n^*]$ norms $\mathfrak{X}$.*

It is clear that (2) implies (1) as well as (3) implies (1). That (1) implies (2) follows from Theorem 3.6. That (1) implies (3) in the separable case follows from Theorem 3.9.

The following well known basic facts will be used.

Fact 3.4. *If $\{x_n\}_n$ is weakly null and $\underline{\lim}_n\|x_n\|>0$ and $\varepsilon>0$, then $\{x_n\}_n$ has a subsequence which is a basic sequence with basis constant at most $1+\varepsilon$.*

FACT 3.5. *Let $\mathfrak{X}_0$ be a finite-codimensional subspace of $\mathfrak{X}$ and $\{x_n\}_{n\in\mathbb{N}}$ be a weakly null sequence in $\mathfrak{X}$. Then*

$$d(x_n, \mathfrak{X}_0) := \inf_{x_0\in\mathfrak{X}_0} \|x_n - x_0\| \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, if $\{x_n\}_n$ is semi-normalized and $\varepsilon > 0$, there exist n_ε and $\widetilde{x}_{n_\varepsilon} \in \mathfrak{X}_0$ with $\|x_{n_\varepsilon} - \widetilde{x}_{n_\varepsilon}\| < \varepsilon$ and $\|x_{n_\varepsilon}\| = \|\widetilde{x}_{n_\varepsilon}\|$.

We can now give a quantitative proof that (1) implies (2) in Theorem 3.3.

THEOREM 3.6. *Let $\mathfrak{X}$ fail the M-Dunford–Pettis property and $\varepsilon > 0$. Then there is a biorthogonal system $\{x_n, x_n^*\}_{n=1}^\infty$ in $\mathfrak{X} \times \mathfrak{X}^*$ such that*:

(1) *$\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ are weakly null,*
(2) *$\|x_n\| = 1$ for each $n \in \mathbb{N}$,*
(3) *$1 \le \|x_n^*\| \le M + \varepsilon$ for each $n \in \mathbb{N}$,*
(4) *$\{x_n\}_{n=1}^\infty$ is a basic sequence.*

Proof. Since $\mathfrak{X}$ fails the M-Dunford–Pettis property there exist sequences $\{w_k\}_{k\in\mathbb{N}}$ and $\{w_k^*\}_{k\in\mathbb{N}}$ as in Definition 3.1. By Fact 3.4 we may assume $\{w_k\}_{k\in\mathbb{N}}$ is a basic sequence.

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers with $\varepsilon_1 < \varepsilon/2(M+\varepsilon)$ and $\sum_{n\in\mathbb{N}} \varepsilon_n < 1/2K$ where K is the basis constant of $\{w_k\}_{k\in\mathbb{N}}$. We will construct a system $\{x_n, x_n^*\}_{n=1}^\infty$ in $\mathfrak{X}\times\mathfrak{X}^*$ and an increasing sequence $\{k_n\}_{n\ge1}$ of integers such that

(a) $\{x_n, x_n^*\}_{n=1}^\infty$ is biorthogonal,
(b) $\|x_n\| = 1$ for each $n \in \mathbb{N}$,
(c) $1 \le \|x_n^*\| \le M/(1-2\varepsilon_n)$ for each $n \in \mathbb{N}$,
(d) $\|x_n - w_{k_n}\| \le \varepsilon_n/M$ for each $n \in \mathbb{N}$,
(e) $\|x_n^* - w_{k_n}^*\| \le \varepsilon_n + 2M\varepsilon_n/(1-2\varepsilon_n)$ for each $n \in \mathbb{N}$.

Conditions (d) and (e) will give us (1): for $x^* \in \mathfrak{X}^*$,

$$|x^*(x_n)| \le \|x^*\|\,\|x_n - w_{k_n}\| + |x^*(w_{k_n})| \to 0$$

so $\{x_n\}_n$ is weakly null and similarly for $\{x_n^*\}_n$.

Condition (c) gives us (3):

$$1 \le \|x_n^*\| \le \frac{M}{1-2\varepsilon_n} \le \frac{M}{1-2\big(\frac{\varepsilon}{2(M+\varepsilon)}\big)} = M+\varepsilon.$$

Condition (d) gives us (4): we have

$$\sum_n \|w_{k_n} - x_n\| \le \sum_n \varepsilon_n < \frac{1}{2K}.$$

Then $\{x_n\}_n$ is basic (and equivalent to $\{w_{n_k}\}_k$).

Now we construct $\{x_n, x_n^*\}_{n=1}^\infty$ by induction. To start, let $k_1 = 1$ and $x_1 = w_1$ and $x_1^* = w_1^*$. Fix $n > 1$ and assume that a system $\{x_j, x_j^*\}_{j<n}$

along with a sequence $\{k_j\}_{j<n}$ have been constructed to satisfy the above conditions. Let

$$\mathfrak{X}_n = [x_j^*]_{j<n}^{\top}, \qquad \mathcal{Z}_n = [x_j]_{j<n}^{\perp}.$$

Using Fact 3.5, find $k_n > k_{n-1}$ and $x_n \in \mathfrak{X}_n$ and $z_n^* \in \mathcal{Z}_n$ so that

$$d(w_{k_n}, \mathfrak{X}_n) \le \|w_{k_n} - x_n\| < \varepsilon_n/M, \qquad d(w_{k_n}^*, \mathcal{Z}_n) \le \|w_{k_n}^* - z_n^*\| < \varepsilon_n$$

with $\|x_n\| = 1$ and $1 \le \|z_n^*\| \le M$. Note that

$$\begin{aligned}|z_n^*(x_n) - w_{k_n}^*(w_{k_n})| &= |z_n^*(x_n - w_{k_n}) - (w_{k_n}^* - z_n^*)(w_{k_n})| \\ &< M\,\frac{\varepsilon_n}{M} + \varepsilon_n = 2\varepsilon_n,\end{aligned}$$

and so $1 - 2\varepsilon_n < z_n^*(x_n) < 1 + 2\varepsilon_n$. Let

$$x_n^* := \frac{z_n^*}{z_n^*(x_n)}.$$

Thus conditions (a) and (c) hold. As for condition (e):

$$\begin{aligned}\|x_n^* - w_{k_n}^*\| &\le \|w_{k_n}^* - z_n^*\| + \left\| z_n^* - \frac{z_n^*}{z_n^*(x_n)} \right\| \\ &\le \varepsilon_n + \frac{1}{z_n^*(x_n)} |z_n^*(x_n) - 1|\,\|z_n^*\| \le \varepsilon_n + \frac{2\varepsilon_n}{1 - 2\varepsilon}\,M. \ \blacksquare\end{aligned}$$

The constructions of fundamental total biorthogonal systems in the proofs of (1)⇒(3) in Theorems 3.3 and 4.5 use the Haar matrices, which are summarized below.

Remark 3.7. Fix $m \ge 0$ and consider the 2^m-dimensional Hilbert space $\ell_2^{2^m}$, along with its unit vector basis $\{e_j^2\}_{j=1}^{2^m}$.

The Haar basis $\{h_j^m\}_{j=1}^{2^m}$ of $\ell_2^{2^m}$ can be described as follows. For $0 \le n \le m$ and $1 \le k \le 2^n$ let

$$I_k^n = \{j \in \mathbb{N} : 2^{m-n}(k-1) < j \le 2^{m-n}k\}.$$

Thus

$$I_1^0 = \{1, 2, \ldots, 2^m\}, \qquad I_1^1 = \{1, 2, \ldots, 2^{m-1}\}, \qquad I_1^1 = \{1 + 2^{m-1}, \ldots, 2^m\}.$$

In general, the collection $\{I_k^n\}_{k=1}^{2^n}$ of sets along the nth level (disjointly) partitions the set $\{1, 2, \ldots, 2^m\}$ into 2^n sets, each containing 2^{m-n} consecutive integers, and I_k^n is the disjoint union $I_k^n = I_{2k-1}^{n+1} \cup I_{2k}^{n+1}$. Now let

$$h_1^m = 2^{-m/2} \sum_{j \in I_1^0} e_j^2$$

and, for $0 \le n < m$ and $1 \le k \le 2^n$, let $h_{2^n+k}^m$ be supported on I_k^n as

$$h_{2^n+k}^m = 2^{(n-m)/2} \Big[\sum_{j \in I_{2k-1}^{n+1}} e_j^2 - \sum_{j \in I_{2k}^{n+1}} e_j^2 \Big].$$

Note that $\{h_j^m\}_{j=1}^{2^m}$ forms an orthonormal basis for $\ell_2^{2^m}$.

Let $H_m = (a^m_{ij})$ be the $2^m \times 2^m$ Haar matrix that transforms the unit vector basis of $\ell_2^{2^m}$ onto the Haar basis; thus, the jth column vector of H_m is just h^m_j and so H_m is a unitary matrix. For example, for $m = 2$ we have

$$H_2 = \begin{bmatrix} 2^{-1} & +2^{-1} & +2^{-1/2} & 0 \\ 2^{-1} & +2^{-1} & -2^{-1/2} & 0 \\ 2^{-1} & -2^{-1} & 0 & +2^{-1/2} \\ 2^{-1} & -2^{-1} & 0 & -2^{-1/2} \end{bmatrix}.$$

Now if $\{z_j, z^*_j\}_{j=1}^{2^m}$ is a biorthogonal sequence in $\mathfrak{X} \times \mathfrak{X}^*$ and $\{x_i, x^*_i\}_{i=1}^{2^m}$ is such that

$$H_m \begin{bmatrix} z_1 \\ \vdots \\ z_{2^m} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{2^m} \end{bmatrix}, \qquad H_m \begin{bmatrix} z^*_1 \\ \vdots \\ z^*_{2^m} \end{bmatrix} = \begin{bmatrix} x^*_1 \\ \vdots \\ x^*_{2^m} \end{bmatrix}$$

then

$$x_i := \sum_{j=1}^{2^m} a^m_{ij} z_j, \qquad x^*_i := \sum_{j=1}^{2^m} a^m_{ij} z^*_j.$$

It is not hard to see that since H_m is a unitary matrix,

(1) $x^*_i(x_j) = \delta_{ij}$,
(2) $[x_i]_{i=1}^{2^m} = [z_j]_{j=1}^{2^m}$,
(3) $[x^*_i]_{i=1}^{2^m} = [z^*_j]_{j=1}^{2^m}$.

Note that, for each $1 \le i \le 2^m$,

(4) $a^m_{i1} = 2^{-m/2}$,
(5) $\sum_{j=2}^{2^m} |a^m_{ij}| = (1+\sqrt{2})(1 - 2^{-m/2}) \nearrow 1 + \sqrt{2}$ as $m \to \infty$.

It follows that

(6) $\|x_i\| \le 2^{-m/2}\|z_1\| + (1+\sqrt{2}) \max_{1<j\le 2^m} \|z_j\|$,
(7) $\|x^*_i\| \le 2^{-m/2}\|z^*_1\| + (1+\sqrt{2}) \max_{1<j\le 2^m} \|z^*_j\|$,
(8) for each $x^* \in \mathfrak{X}^*$,

$$|x^*(x_i)| \le 2^{-m/2}|x^*(z_1)| + (1+\sqrt{2}) \max_{1<j\le 2^m} |x^*(z_j)|,$$

(9) for each $x^{**} \in \mathfrak{X}^{**}$,

$$|x^{**}(x^*_i)| \le 2^{-m/2}|x^{**}(z^*_1)| + (1+\sqrt{2}) \max_{1<j\le 2^m} |x^{**}(z^*_j)|.$$

The following notation will (hopefully) simplify the proofs of Theorem 3.9 and Theorem 4.8.

DEFINITION 3.8. A sequence $\{J_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ is a *blocking* of $\mathbb{N}$ if $\mathbb{N}$ is the disjoint union $\bigcup_{k=1}^{\infty} J_k$ and

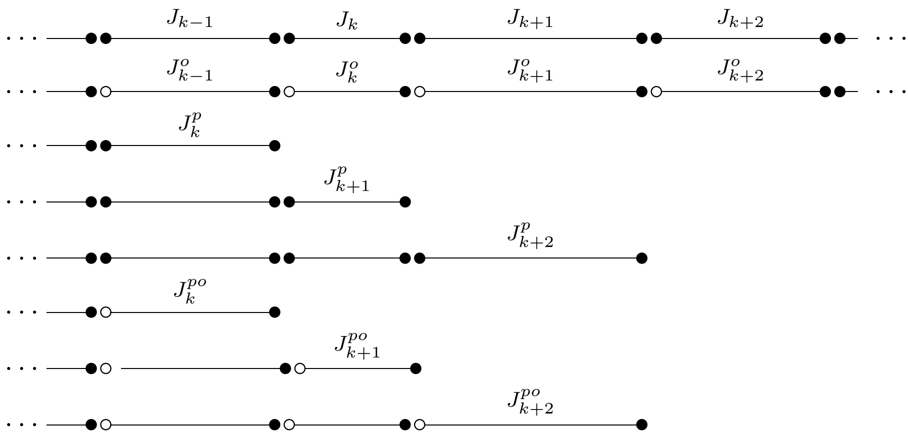
$$\max J_k < \min J_{k+1}$$

for each $k \in \mathbb{N}$. Given a blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$, let $J_0 = \emptyset$ and

$$J_k^p := \bigcup_{0 \le j < k} J_j, \qquad J_k^o := J_k \setminus \{\text{the first element in } J_k\}$$

$$J_k^{po} := \bigcup_{0 \le j < k} J_j^o, \qquad \mathbb{N}^o := \bigcup_{k=1}^{\infty} J_k^o$$

for each $k \in \mathbb{N}$. Pictorially one has:



It follows from the next theorem that (1) implies (3) for separable $\mathfrak{X}$ in Theorem 3.3.

THEOREM 3.9. *Let $\mathfrak{X}$ fail the M-Dunford–Pettis property and $\varepsilon > 0$. If $\{a_n, b_n^*\}_{n\in\mathbb{N}} \subset \mathfrak{X} \times \mathfrak{X}^*$ then there exists an $[M(1+\sqrt{2})^2 + \varepsilon]$-bounded* DP*-biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ such that $[a_n]_{n\in\mathbb{N}} \subset [x_n]_{n\in\mathbb{N}}$ and $[b_n^*]_{n\in\mathbb{N}} \subset [x_n^*]_{n\in\mathbb{N}}$.*

Proof. Without loss of generality, $[a_n]_{n\in\mathbb{N}}$ and $[b_n^*]_{n\in\mathbb{N}}$ are each infinite-dimensional. Since $\mathfrak{X}$ fails the M-Dunford–Pettis property, by Theorem 3.6, there is a biorthogonal system $\{w_n, w_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ with both $\{w_n\}_n$ and $\{w_n^*\}_n$ weakly null, $\|w_n\| = 1$, and $1 \le \|w_n^*\| \le M + \varepsilon$. Fix a sequence $\{\delta_k\}_{k=1}^{\infty}$ of positive numbers decreasing to zero with $\delta_1 < 1/2$ and

$$\frac{M+\varepsilon}{(1+2\varepsilon)M} < 1 - 2\delta_1. \tag{3.2}$$

It suffices to find a system $\{x_n, x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ along with (following the terminology in Definition 3.8) a blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ and an increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$, satisfying

(1) $x_m^*(x_n) = \delta_{mn}$,
(2) $\|x_n\| \le (1+\sqrt{2}) + \varepsilon$,
(3) $\|x_n^*\| \le (1+2\varepsilon)M(1+\sqrt{2}) + \varepsilon$,
(4) for each $x^* \in S(\mathfrak{X}^*)$, if $n \in J_k$, then

$$|x^*(x_n)| \le \delta_k + (1+\sqrt{2}) \max_{j \in J_k^o} (|x^*(w_{i_j})| + \delta_k),$$

(5) for each $x^{**} \in S(X^{**})$, if $n \in J_k$ then

$$|x^{**}(x_n^*)| \le \delta_k \left(\frac{4+2M}{1-2\delta_k} \right) + (1+\sqrt{2}) \max_{j \in J_k^o} |x^{**}(w_{i_j}^*)|,$$

(6) $[a_n]_{n=1}^\infty \subset [x_n]_{n=1}^\infty$,
(7) $[b_n^*]_{n=1}^\infty \subset [x_n^*]_{n=1}^\infty$.

The construction will inductively produce blocks $\{x_n, x_n^*\}_{n \in J_k}$. Let x_0 and x_0^* be the zero vectors. Fix $k \ge 1$. Assume that $\{J_j\}_{0 \le j < k}$ along with $\{x_n, x_n^*\}_{n \in J_k^p}$ and $\{i_n\}_{n \in J_k^{po}}$ have been constructed to satisfy conditions (1) through (5). Now to construct J_k along with $\{x_n, x_n^*\}_{n \in J_k}$ and $\{i_n\}_{n \in J_k^o}$.

Let

$$\mathcal{P}_k := [x_n^*]_{n \in J_k^p}^\top, \qquad \mathcal{Q}_k := [x_n]_{n \in J_k^p}^\perp, \qquad n_k = \max J_k^p.$$

The idea is to find a biorthogonal system $\{z_n, z_n^*\}_{n \in J_k}$ in $\mathcal{P}_k \times \mathcal{Q}_k$ by first finding just one pair $\{z_{1+n_k}, z_{1+n_k}^*\}$ which helps guarantee condition (6) if k is odd and condition (7) if k even; however, $\{z_{1+n_k}, z_{1+n_k}^*\}$ would not necessarily satisfy conditions (2) through (5), and so J_k^o and $\{z_n, z_n^*\}_{n \in J_k^o}$ and $\{i_n\}_{n \in J_k^o}$ are constructed and then the appropriate Haar matrix is applied to $\{z_n, z_n^*\}_{n \in J_k}$ to produce $\{x_n, x_n^*\}_{n \in J_k}$ so that $\{x_n, x_n^*\}_{n \in J_k^p \cup J_k}$ with $\{i_n\}_{n \in J_k^{po} \cup J_k^o}$ satisfy conditions (1) through (5).

The pair $\{z_{1+n_k}, z_{1+n_k}^*\}$ is constructed by a standard Gram–Schmidt biorthogonal procedure. If k is odd, start in $\mathfrak{X}$. Let

$$h_k = \min\{h : a_h \notin [x_n]_{n \le n_k}\}.$$

Set

$$z_{1+n_k} = a_{h_k} - \sum_{n \le n_k} x_n^*(a_{h_k}) x_n,$$

and for any $y_{1+n_k}^*$ in $\mathfrak{X}^*$ such that $y_{1+n_k}^*(z_{1+n_k}) \ne 0$,

$$z_{1+n_k}^* = \frac{y_{1+n_k}^* - \sum_{n \le n_k} y_{1+n_k}^*(x_n) x_n^*}{y_{1+n_k}^*(z_{1+n_k})}.$$

If k is even, start in $\mathfrak{X}^*$. Let

$$h_k = \min\{h : b_h^* \notin [x_n^*]_{n \le n_k}\}.$$

Set

$$z^*_{1+n_k} = b^*_{h_k} - \sum_{n\le n_k} b^*_{h_k}(x_n)x^*_n,$$

and, for any y_{1+n_k} in $\mathfrak{X}$ such that $z^*_{1+n_k}(y_{1+n_k}) \neq 0$,

$$z_{1+n_k} = \frac{y_{1+n_k} - \sum_{n\le n_k} x^*_n(y_{1+n_k})x_n}{z^*_{1+n_k}(y_{1+n_k})}.$$

Clearly $z^*_{1+n_k}(z_{1+n_k}) = 1$, $z_{1+n_k} \in \mathcal{P}_k$ and $z^*_{1+n_k} \in \mathcal{Q}_k$.

Find a natural number m_k larger than one so that

$$2^{-m_k/2}\max(\|z_{1+n_k}\|, \|z^*_{1+n_k}\|) < \min(\varepsilon, \delta_k)$$

and let

$$J_k := \{1+n_k, \dots, 2^{m_k}+n_k\} \quad \text{and so} \quad J^o_k := \{2+n_k, \dots, 2^{m_k}+n_k\}.$$

Let

$$\widetilde{\mathcal{P}}_k := \mathcal{P}_k \cap [z^*_{1+n_k}]^\top, \qquad \widetilde{\mathcal{Q}}_k := \mathcal{Q}_k \cap [z_{1+n_k}]^\perp.$$

The next step is to find a biorthogonal system $\{z_n, z^*_n\}_{n\in J^o_k}$ along with $\{i_n\}_{n\in J^o_k}$ satisfying

$$\{z_n, z^*_n\} \in S(\widetilde{\mathcal{P}}_k) \times ((1+\varepsilon)M)B(\widetilde{\mathcal{Q}}_k) \tag{3.3}$$

and

$$\|w_{i_n} - z_n\| < \delta_k, \qquad \|w^*_{i_n} - z^*_n\| < \delta_k + \frac{2\delta_k(M+\varepsilon)}{1-2\delta_k} \tag{3.4}$$

for each $n \in J^o_k$. Towards this, fix $j \in J^o_k$ and assume that a biorthogonal system $\{z_n, z^*_n\}_{2+n_k\le n<j}$ along with $\{i_n\}_{2+n_k\le n<j}$ have been constructed so that conditions (3.3) and (3.4) hold for $2+n_k \le n < j$. Let

$$\mathfrak{X}_j := \widetilde{\mathcal{P}}_k \cap [z^*_n]^\top_{2+n_k\le n<j}, \qquad \mathcal{Y}_j := \widetilde{\mathcal{Q}}_k \cap [z_n]^\perp_{2+n_k\le n<j}.$$

Then by Fact 3.5 there exists a natural number $i_j > i_{j-1}$ along with $z_j \in \mathfrak{X}_j$ and $\widetilde{z}^*_j \in \mathcal{Y}_j$ such that

$$\begin{aligned} d(w_{i_j}, \mathfrak{X}_j) \le \|w_{i_j} - z_j\| < \frac{\delta_k}{M+\varepsilon}, \quad & d(w^*_{i_j}, \mathcal{Y}_j) \le \|w^*_{i_j} - \widetilde{z}^*_j\| < \delta_k, \\ \|z_j\| = 1, \quad & 1 \le \|\widetilde{z}_j\| \le M+\varepsilon. \end{aligned}$$

Note that $\widetilde{z}^*_j(z_j)$ need not be equal to 1 but it is close to 1 since

$$\begin{aligned} |\widetilde{z}^*_j(z_j) - w^*_{i_j}(w_{i_j})| &= |\widetilde{z}^*_j(z_j) - (w^*_{i_j} - \widetilde{z}^*_j)(w_{i_j}) - \widetilde{z}^*_j(w_{i_j})| \\ &= |\widetilde{z}^*_j(z_j - w_{i_j}) - (w^*_{i_j} - \widetilde{z}^*_j)(w_{i_j})| \\ &\le \|\widetilde{z}^*_j\|\|z_j - w_{i_j}\| + \|w^*_{i_j} - \widetilde{z}^*_j\|\|w_{i_j}\| \\ &< (M+\varepsilon)\frac{\delta_k}{M+\varepsilon} + \delta_k = 2\delta_k \end{aligned} \tag{3.5}$$

and so $1-2\delta_k \le \widetilde{z}_j^*(z_j) \le 1+2\delta_k$. Let

$$z_j^* = \frac{\widetilde{z}_j^*}{\widetilde{z}_j^*(z_j)}$$

so that $z_j^*(z_j)=1$. Now $z_j \in S(\widetilde{\mathcal{P}}_k)$ and $1 \le \|z_j^*\| \le (M+\varepsilon)/(1-2\delta_k)$ and so $z_j^* \in (1+2\varepsilon)MB(\tilde{\mathcal{Q}}_k)$ by (3.2). Note that by (3.5),

$$\begin{aligned}\|w_{i_j}^* - z_j^*\| &\le \|w_{i_j}^* - \widetilde{z}_j^*\| + \|\widetilde{z}_j^* - z_j^*\| \\ &\le \delta_k + \|\widetilde{z}_j^*\|\left|1 - \frac{1}{\widetilde{z}_j^*(z_j)}\right| \\ &\le \delta_k + (1+\varepsilon)M\frac{2\delta_k}{1-2\delta_k}.\end{aligned}$$

This completes the inductive construction of $\{z_n, z_n^*\}_{n\in J_k^o}$ and $\{i_n\}_{n\in J_k^o}$.

Now apply the Haar matrix to $\{z_n, z_n^*\}_{n\in J_k}$ to produce $\{x_n, x_n^*\}_{n\in J_k}$. With help from the observations in Remark 3.7, note that $\{x_n, x_n^*\}_{n\in J_k}$ is biorthogonal and is in $\mathcal{P}_k \times \mathcal{Q}_k$. Furthermore, for each n in J_k,

$$\begin{aligned}\|x_n\| &\le 2^{-m_k/2}\|z_{1+n_k}\| + (1+\sqrt{2})\max_{j\in J_k^o}\|z_j\| \le \varepsilon + (1+\sqrt{2}), \\ \|x_n^*\| &\le 2^{-m_k/2}\|z_{1+n_k}^*\| + (1+\sqrt{2})\max_{j\in J_k^o}\|z_j^*\| \le \varepsilon + (1+\varepsilon)M(1+\sqrt{2}).\end{aligned}$$

If $x^* \in S(\mathfrak{X}^*)$ then

$$\begin{aligned}|x^*(x_n)| &\le 2^{-m_k/2}\|z_{1+n_k}\| + (1+\sqrt{2})\max_{j\in J_k^o}|x^*(z_j)| \\ &\le \delta_k + (1+\sqrt{2})\max_{j\in J_k^o}(|x^*(w_{i_j})| + \delta_k)\end{aligned}$$

and for each $x^{**} \in S(\mathfrak{X}^{**})$,

$$\begin{aligned}|x^{**}(x_n^*)| &\le 2^{-m_k/2}\|z_{1+n_k}^*\| + (1+\sqrt{2})\max_{j\in J_k^o}|x^{**}(z_j^*)| \\ &\le \delta_k + (1+\sqrt{2})\max_{j\in J_k^o}\left(|x^{**}(w_{i_j}^*)| + \delta_k + \frac{2\delta_k(1+2\varepsilon)M}{1-2\delta_k}\right),\end{aligned}$$

and this simplifies to give us (5). Thus $\{x_n, x_n^*\}_{n\in J_k^p\cup J_k}$ with $\{i_n\}_{n\in J_k^{po}\cup J_k^o}$ satisfy conditions (1) through (5). If k is odd, then

$$[a_h]_{h\le h_k} \subset [x_n, z_{1+n_k}]_{n\in J_k^p} \subset [x_n]_{n\in J_k^p\cup J_k},$$

while if k is even, then

$$[b_h^*]_{h\le h_k} \subset [x_n^*, z_{1+n_k}^*]_{n\in J_k^p} \subset [x_n^*]_{n\in J_k^p\cup J_k}.$$

Clearly the constructed system $\{x_n, x_n^*\}_{n=1}^\infty$ with the blocking $\{J_k\}_{k=1}^\infty$ of $\mathbb{N}$ and the increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$ satisfy conditions (1) through (7). ■

4. Spaces containing c_0. To motivate the biorthogonal system characterization of spaces containing c_0 we recall some well known facts about such spaces. We will see that c_0 subspaces of $\mathfrak{X}$ correspond essentially to weakly unconditionally Cauchy series in $\mathfrak{X}$ so we briefly recall some essential facts about such series.

DEFINITION 4.1. A series $\sum_n x_n$ is called *weakly unconditionally Cauchy* (wuC) if given any permutation π of $\mathbb{N}$, the sequence $\{\sum_{k=1}^n x_{\pi(k)}\}_n$ is weakly Cauchy. Equivalently, $\sum_n x_n$ is wuC if and only if for each $x^* \in \mathfrak{X}^*$ we have $\sum_n |x^*(x_n)| < \infty$.

Bessaga and Pełczyński [1] tied together wuC series and c_0.

THEOREM 4.2 ([1]). *Let $\mathfrak{X}$ be a Banach space.*

(1) *A basic sequence $\{x_n\}_n$ in $\mathfrak{X}$ with $\sum_n x_n$ wuC and* $\inf_n \|x_n\| > 0$ *is equivalent to the unit vector basis of c_0.*
(2) *In order that each wuC series $\sum_n x_n$ in $\mathfrak{X}$ be unconditionally convergent it is both necessary and sufficient that $\mathfrak{X}$ contains no copy of c_0.*

The following ideas will help us define our c_0-biorthogonal system in a very natural way.

REMARK 4.3. (i) Let $\{x_n, x_n^*\}$ be a biorthogonal system with $\sum_n x_n$ wuC and $\underline{\lim}_n \|x_n\| > 0$. If $\{x_{n_k}\}_k$ is any subsequence of $\{x_n\}_n$, then $\sum_k x_{n_k}$ is wuC and $\underline{\lim}_k \|x_{n_k}\| > 0$ so Fact 3.4 tells us $\{x_{n_k}\}_k$ has a subsequence $\{x_{n_{k_j}}\}_j$ which is basic and $\inf_j \|x_{n_{k_j}}\| > 0$. Then by Theorem 4.2, $\{x_{n_{k_j}}\}_j$ is equivalent to the unit vector basis of c_0. Thus each subsequence of $\{x_n\}_n$ has a further subsequence which is equivalent to the unit vector basis of c_0.

(ii) (cf. [5]) Let T be a bounded linear operator from c_0 to $\mathfrak{X}$ and $x_n = Te_n$ where $\{e_n\}_n$ is the unit vector basis of c_0. Then for $x^* \in \mathfrak{X}^*$,

$$\sum_n |x^*(x_n)| = \sum_n |x^*(Te_n)| = \sum_n |T^*x^*(e_n)| < \infty$$

since $T^*x^* \in \ell_1$. Thus $\sum_n x_n$ is wuC. Conversely if $\sum_n x_n$ is wuC in $\mathfrak{X}$, then define $T : c_0 \to \mathfrak{X}$ by $T(\{t_n\}_n) = \sum_n t_n x_n$. Then T is well defined and has a closed graph so T is bounded. So the bounded linear operators from c_0 to $\mathfrak{X}$ correspond precisely to the wuC series in $\mathfrak{X}$.

(iii) Let $T : c_0 \hookrightarrow \mathfrak{X}$ be an isomorphic embedding and $\{e_n\}_n$ be the unit vector basis of c_0. Since T is an embedding there exist constants C_1 and C_2

such that for any $(\alpha_n)_n \in c_0$ we have

$$C_1\|(\alpha_n)_n\|_{c_0} \leq \|T((\alpha_n)_n)\|_{\mathfrak{X}} \leq C_2\|(\alpha_n)_n\|_{c_0}.$$

Then for each $n \in \mathbb{N}$,

$$C_1 = C_1\|e_n\|_{c_0} \leq \|Te_n\|_{\mathfrak{X}} \leq C_2\|e_n\|_{c_0} = C_2$$

and so $\{Te_n\}_n$ is semi-normalized. By (ii) above, the series $\sum_n Te_n$ is wuC.

Based on this we make the following definition.

DEFINITION 4.4. A biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ is called a *c_0-biorthogonal system* if $\{x_n\}_n$ is normalized and has a subsequence $\{x_{n_j}\}_j$ for which $\sum_j x_{n_j}$ is wuC.

THEOREM 4.5. *The following statements are equivalent*:

(1) *$\mathfrak{X}$ contains an isomorphic copy of c_0.*
(2) *There is a bounded c_0-biorthogonal system in $\mathfrak{X} \times \mathfrak{X}^*$.*

And in the case that $\mathfrak{X}$ is separable:

(3) *There is a bounded fundamental total c_0-biorthogonal system $\{x_n, x_n^*\} \subset \mathfrak{X} \times \mathfrak{X}^*$.*

Furthermore, *for each $\varepsilon > 0$*: *if* (2) *holds then the system can be taken to be $(2 + \varepsilon)$-bounded*; *if clause* (3) *holds then the system can be taken to be $[2(1 + \sqrt{2})^2 + \varepsilon]$-bounded and so that $[x_n^*]$ norms $\mathfrak{X}$.*

That (2) implies (1) as well as (3) implies (1) follow from Remark 4.3. That (1) implies (2) is Theorem 4.6. That (1) implies (3) in the separable case follows from Theorem 4.8.

THEOREM 4.6. *If $\mathfrak{X}$ contains an isomorphic copy of c_0 and $\varepsilon > 0$*, *then there exists a $(2+\varepsilon)$-bounded c_0-biorthogonal system $\{x_n, x_n^*\} \subset S(\mathfrak{X}) \times \mathfrak{X}^*$.*

Proof. Let $T : c_0 \hookrightarrow \mathfrak{X}$ be an isomorphic embedding and $\varepsilon > 0$. Let $\{e_j\}_j$ be the unit vector basis of c_0. Then Remark 4.3 implies that $\sum_j Te_j$ is wuC and $\{Te_j\}_j$ is semi-normalized. Fact 3.4 gives us a subsequence $\{Te_{j_n}\}_n$ of $\{Te_j\}_j$ that is basic with basis constant at most $1 + \varepsilon/2$. Let

$$x_n = \frac{Te_{j_n}}{\|Te_{j_n}\|}.$$

Note that $\{x_n\}_n$ is a normalized basic sequence with basis constant at most $1 + \varepsilon/2$ and $\sum_n x_n$ is wuC. We may pick our biorthogonal functionals accordingly. ■

Notice that the proof of Theorem 4.6 gives us a bit more than a c_0-biorthogonal system: it gives us a biorthogonal system $\{x_n, x_n^*\}$ with the entire series $\sum_n x_n$ wuC.

To construct a fundamental total biorthogonal system in the separable case we need the following lemma.

LEMMA 4.7. *If Y_0 is a finite-codimensional subspace of $\mathfrak{X}^*$ and $\varepsilon > 0$, then there is a finite-codimensional subspace X_0 of $\mathfrak{X}$ that is $(2+\varepsilon)$-normed by Y_0.*

Proof. Let X_0 be the pre-annihilator of any finite-dimensional subspace of $\mathfrak{X}^*$ that $(1+\varepsilon)$-norms the annihilator of Y_0. Then for $f \in S(X_0)$ we have

$$\begin{aligned}\sup_{y^*\in S(Y_0)} |y^*(f)| &= \inf_{y^{**}\in Y_0^\perp} \|f-y^{**}\| \\ &\geq \inf_{y^{**}\in Y_0^\perp} \max[\|f\|-\|y^{**}\|, \sup_{x^*\in S(X_0^\perp)} |(f-y^{**})(x^*)|] \\ &\geq \inf_{y^{**}\in Y_0^\perp} \max\left[1-\|y^{**}\|, \frac{1}{1+\varepsilon}\|y^{**}\|\right] \\ &= \inf_{0\le t<\infty} \max\left[1-t, \frac{t}{1+\varepsilon}\right] = \frac{1}{2+\varepsilon}.\end{aligned}$$

So $\|f\| \leq (2+\varepsilon)\sup_{y^*\in S(Y_0)} |y^*(f)|$ for each $f \in S(X_0)$. Thus X_0 is $(2+\varepsilon)$-normed by Y_0. ∎

The following theorem will give us a fundamental total c_0-biorthogonal system in the separable case.

THEOREM 4.8. *Suppose $\mathfrak{X}$ has a subspace isomorphic to c_0. Let $\varepsilon > 0$ and $\{a_n, b_n^*\} \subset \mathfrak{X}\times\mathfrak{X}^*$. Then there exists a $[2(1+\sqrt{2})^2+\varepsilon]$-bounded c_0-biorthogonal system $\{x_n, x_n^*\} \subset \mathfrak{X}\times\mathfrak{X}^*$ with $[a_n]_n \subseteq [x_n]_n$ and $[b_n^*]_n \subseteq [x_n^*]_n$.*

Proof. Without loss of generality, $[a_n]_{n\in\mathbb{N}}$ and $[b_n^*]_{n\in\mathbb{N}}$ are each infinite-dimensional. Since $c_0 \hookrightarrow \mathfrak{X}$, by Theorem 4.6, there is a $(2+\varepsilon)$-bounded biorthogonal system $\{w_n, w_n^*\}$ in $S(\mathfrak{X})\times\mathfrak{X}^*$ with $\sum_n w_n$ wuC. Fix a sequence $\{\delta_k\}_{k=1}^\infty$ of positive numbers decreasing to zero with $\sum_k \delta_k < \infty$. Again we follow the notation in Definition 3.8. It suffices to find a system $\{x_n, x_n^*\}_{n=1}^\infty$ in $\mathfrak{X}\times\mathfrak{X}^*$ along with a blocking $\{J_k\}_{k=1}^\infty$ of $\mathbb{N}$ and an increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$, satisfying

(a) $x_m^*(x_n) = \delta_{mn}$,
(b) $\|x_n\| \leq (1+\sqrt{2})+\varepsilon$,
(c) $\|x_n^*\| \leq (2+\varepsilon)(1+\sqrt{2})+\varepsilon$,
(d) for each $x^* \in S(\mathfrak{X}^*)$, if $n \in J_k$ then

$$|x^*(x_n)| \leq (2+\sqrt{2})\delta_k + (1+\sqrt{2})\max_{j\in J_k^o} |x^*(w_{i_j})|,$$

(e) $[a_n]_{n=1}^\infty \subset [x_n]_{n=1}^\infty$,
(f) $[b_n^*]_{n=1}^\infty \subset [x_n^*]_{n=1}^\infty$.

The construction will inductively produce blocks $\{x_n, x_n^*\}_{n\in J_k}$. Let x_0 and x_0^* be the zero vectors. Fix $k \geq 1$. Assume that $\{J_j\}_{0\leq j<k}$ along with $\{x_n, x_n^*\}_{n\in J_k^p}$ and $\{i_n\}_{n\in J_k^{po}}$ have been constructed to satisfy conditions (a) through (d). Now to construct J_k along with $\{x_n, x_n^*\}_{n\in J_k}$ and $\{i_n\}_{n\in J_k^o}$.

Let

$$\mathcal{P}_k := [x_n^*]^{\top}_{n\in J_k^p}, \qquad \mathcal{Q}_k := [x_n]^{\perp}_{n\in J_k^p}, \qquad n_k = \max J_k^p.$$

The idea is to find a biorthogonal system $\{z_n, z_n^*\}_{n\in J_k}$ in $\mathcal{P}_k \times \mathcal{Q}_k$ by first finding just one pair $\{z_{1+n_k}, z_{1+n_k}^*\}$ which helps guarantee condition (e) if k is odd and condition (f) if k is even; however, $\{z_{1+n_k}, z_{1+n_k}^*\}$ would not necessarily satisfy conditions (b) through (d) so J_k^o and $\{z_n, z_n^*\}_{n\in J_k^o}$, and $\{i_n\}_{n\in J_k^o}$ are constructed and then the appropriate Haar matrix is applied to $\{z_n, z_n^*\}_{n\in J_k}$ to produce $\{x_n, x_n^*\}_{n\in J_k}$ so that $\{x_n, x_n^*\}_{n\in J_k^p\cup J_k}$ with $\{i_n\}_{n\in J_k^{po}\cup J_k^o}$ satisfy conditions (a) through (d).

The pair $\{z_{1+n_k}, z_{1+n_k}^*\}$ is constructed by a standard Gram-Schmidt biorthogonal procedure. If k is odd, start in $\mathfrak{X}$. Let

$$h_k = \min\{h : a_h \notin [x_n]_{n\leq n_k}\}.$$

Set

$$z_{1+n_k} = a_{h_k} - \sum_{n\leq n_k} x_n^*(a_{h_k})x_n,$$

and for any $y_{1+n_k}^*$ in $\mathfrak{X}^*$ such that $y_{1+n_k}^*(z_{1+n_k}) \neq 0$,

$$z_{1+n_k}^* = \frac{y_{1+n_k}^* - \sum_{n\leq n_k} y_{1+n_k}^*(x_n)x_n^*}{y_{1+n_k}^*(z_{1+n_k})}.$$

If k is even, start in $\mathfrak{X}^*$. Let

$$h_k = \min\{h : b_h^* \notin [x_n^*]_{n\leq n_k}\}.$$

Set

$$z_{1+n_k}^* = b_{h_k}^* - \sum_{n\leq n_k} b_{h_k}^*(x_n)x_n^*,$$

and, for any y_{1+n_k} in $\mathfrak{X}$ such that $z_{1+n_k}^*(y_{1+n_k}) \neq 0$,

$$z_{1+n_k} = \frac{y_{1+n_k} - \sum_{n\leq n_k} x_n^*(y_{1+n_k})x_n}{z_{1+n_k}^*(y_{1+n_k})}.$$

Clearly $z_{1+n_k}^*(z_{1+n_k}) = 1$, $z_{1+n_k} \in \mathcal{P}_k$ and $z_{1+n_k}^* \in \mathcal{Q}_k$.

Find a natural number m_k larger than one so that

$$2^{-m_k/2}\max(\|z_{1+n_k}\|, \|z_{1+n_k}^*\|) < \min(\varepsilon, \delta_k)$$

and let

$$J_k := \{1+n_k, \ldots, 2^{m_k}+n_k\} \quad \text{and so} \quad J_k^o := \{2+n_k, \ldots, 2^{m_k}+n_k\}.$$

Let

$$\widetilde{\mathcal{P}}_k := \mathcal{P}_k \cap [z^*_{1+n_k}]^\top, \qquad \widetilde{\mathcal{Q}}_k := \mathcal{Q}_k \cap [z_{1+n_k}]^\perp.$$

Now we find a biorthogonal system $\{z_n, z^*_n\}_{n\in J^o_k}$ along with $\{i_n\}_{n\in J^o_k}$ satisfying

$$\{z_n, z^*_n\} \in S(\widetilde{\mathcal{P}}_k) \times (2+\varepsilon)B(\widetilde{\mathcal{Q}}_k) \tag{4.6}$$

and

$$\|w_{i_n} - z_n\| < \delta_k \tag{4.7}$$

for each $n \in J^o_k$. Towards this, fix $j \in J^o_k$ and assume that a biorthogonal system $\{z_n, z^*_n\}_{2+n_k\le n<j}$ along with $\{i_n\}_{2+n_k\le n<j}$ have been constructed so that conditions (4.6) and (4.7) hold for $2+n_k \le n < j$. Let

$$\mathfrak{X}_j := \widetilde{\mathcal{P}}_k \cap [z^*_n]^\top_{2+n_k\le n<j}, \qquad \mathcal{Y}_j := \widetilde{\mathcal{Q}}_k \cap [z_n]^\perp_{2+n_k\le n<j}.$$

Apply Lemma 4.7 with $Y_0 = \mathcal{Y}_j$ to get a finite-codimensional subspace X_0 of $\mathfrak{X}$ that is $(2+\varepsilon/2)$-normed by $\mathcal{Y}_j$. Then by Fact 3.5 there exists a natural number $i_j > i_{j-1}$ along with $z_j \in S(\mathfrak{X}_j \cap X_0)$ such that

$$d(w_{i_j}, \mathfrak{X}_j \cap X_0) \le \|z_j - w_{i_j}\| < \delta_k.$$

Since X_0 is $(2+\varepsilon/2)$-normed by $\mathcal{Y}_j$ there is $\widetilde{z}^*_j \in S(\mathcal{Y}_j)$ such that

$$\frac{1}{2+\varepsilon} \le \widetilde{z}^*_j(z_j).$$

Let

$$z^*_j = \frac{1}{\widetilde{z}^*_j(z_j)}\,\widetilde{z}^*_j$$

so that $z^*_j(z_j) = 1$ and note that

$$\|z^*_j\| = \frac{1}{\widetilde{z}^*_j(z_j)}\,\|\widetilde{z}^*_j\| \le 2+\varepsilon.$$

This completes the inductive construction of $\{z_n, z^*_n\}_{n\in J^o_k}$ and $\{i_n\}_{n\in J^o_k}$.

Now apply the Haar matrix to $\{z_n, z^*_n\}_{n\in J_k}$ to produce $\{x_n, x^*_n\}_{n\in J_k}$. With help from the observations in Remark 3.7, note that $\{x_n, x^*_n\}_{n\in J_k}$ is biorthogonal and is in $\mathcal{P}_k \times \mathcal{Q}_k$. Furthermore, for each n in J_k,

$$\|x_n\| \le 2^{-m_k/2}\|z_{1+n_k}\| + (1+\sqrt{2})\max_{j\in J^o_k}\|z_j\| \le \varepsilon + (1+\sqrt{2})$$

$$\|x^*_n\| \le 2^{-m_k/2}\|z^*_{1+n_k}\| + (1+\sqrt{2})\max_{j\in J^o_k}\|z^*_j\| \le \varepsilon + (2+\varepsilon)(1+\sqrt{2}).$$

If $x^* \in S(\mathfrak{X}^*)$ then

$$\begin{aligned}|x^*(x_n)| &\le 2^{-m_k/2}\|z_{1+n_k}\| + (1+\sqrt{2})\max_{j\in J^o_k}|x^*(z_j)| \\ &\le \delta_k + (1+\sqrt{2})\max_{j\in J^o_k}(|x^*(z_j - w_{i_j})| + |x^*(w_{i_j})|)\end{aligned}$$

$$\leq \delta_k + (1+\sqrt{2}) \max_{j\in J_k^o} (\delta_k + |x^*(w_{i_j})|)$$
$$= (2+\sqrt{2})\delta_k + (1+\sqrt{2}) \max_{j\in J_k^o} |x^*(w_{i_j})|.$$

Thus $\{x_n, x_n^*\}_{n\in J_k^p \cup J_k}$ with $\{i_n\}_{n\in J_k^{po}\cup J_k^o}$ satisfy conditions (a) through (d). If k is odd, then

$$[a_h]_{h\leq h_k} \subset [x_n, z_{1+n_k}]_{n\in J_k^p} \subset [x_n]_{n\in J_k^p\cup J_k},$$

while if k is even, then

$$[b_h^*]_{h\leq h_k} \subset [x_n^*, z_{1+n_k}^*]_{n\in J_k^p} \subset [x_n^*]_{n\in J_k^p\cup J_k}.$$

Clearly the constructed system $\{x_n, x_n^*\}_{n=1}^\infty$ with the blocking $\{J_k\}_{k=1}^\infty$ of $\mathbb{N}$ and the increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$ satisfy conditions (a) through (f).

Note that condition (d) tells us that if for each $k\in\mathbb{N}$ we pick any $n_k \in J_k$, then for $x^* \in S(\mathfrak{X}^*)$ we have

$$\sum_k |x^*(x_{n_k})| \leq (2+\sqrt{2})\sum_k \delta_k + (1+\sqrt{2})\sum_k \max_{j\in J_k^o} |x^*(w_{i_j})|$$
$$\leq (2+\sqrt{2})\sum_k \delta_k + (1+\sqrt{2})\sum_j |x^*(w_{i_j})| < \infty.$$

So $\sum_k x_{n_k}$ is wuC. ∎

5. Piecing it all together. Inspired by Theorem 2.3 we might try to combine Theorems 3.3 and 4.5 with the Dilworth–Girardi–Johnson ℓ_1 result (Theorem 2.4) to get the following theorem giving the existence of biorthogonal systems in any Banach space.

FALSE CONJECTURE 5.1. *For any given infinite-dimensional Banach space $\mathfrak{X}$ there exists a bounded biorthogonal system $\{x_n, x_n^*\}$ that is of one of the following three types*:

(1) *a c_0-biorthogonal system*,
(2) *a wc_0^*-stable biorthogonal system*,
(3) *a* DP-*biorthogonal system.*

However, this does not follow directly from the previous results. The trouble lies in part (3). Theorem 2.3 guarantees us that if $\mathfrak{X}$ contains no isomorphic copies of c_0 or ℓ_1, then there is a *subspace* (say $\mathcal{Y}$) of $\mathfrak{X}$ that fails DP. So from Theorem 3.3 we get a DP-biorthogonal system $\{y_n, y_n^*\}$ in $\mathcal{Y}\times\mathcal{Y}^*$. Since $\{y_n\}_n$ is weakly null in $\mathcal{Y}$ it is also weakly null in $\mathfrak{X}$. Unfortunately the fact that $\{y_n^*\}_n$ is weakly null in $\mathcal{Y}^*$ does not necessarily tell us that if we extend each y_n^* to $x_n^* \in \mathfrak{X}^*$, then $\{x_n^*\}_n$ is weakly null in $\mathfrak{X}^*$. Another way to see that part (3) is not correct is to notice that DP does not necessarily pass to closed subspaces. Since it is a $C(K)$ space, ℓ_∞ has

DP; however ℓ_2 does not have DP. So if part (3) were correct it would say that $\mathcal{Y}$ failing DP implies $\mathfrak{X}$ fails DP, which is false. We recall the following related property.

Definition 5.2. A Banach space $\mathfrak{X}$ has the *hereditary Dunford–Pettis property* ($\mathrm{DP_h}$) if every closed subspace of $\mathfrak{X}$ has the Dunford–Pettis property.

For detailed discussions of $\mathrm{DP_h}$ see [2, 3, 4]. In 1987 Cembranos gave the following useful characterization of $\mathrm{DP_h}$.

Theorem 5.3 ([3]). *A Banach space $\mathfrak{X}$ has $\mathrm{DP_h}$ if and only if every normalized weakly null sequence in $\mathfrak{X}$ has a subsequence which is equivalent to the unit vector basis of c_0.*

In 1989 Knaust and Odell [9] gave a quantitative improvement of this result by showing that the equivalence is uniform for all normalized weakly null sequences. Using the hereditary Dunford–Pettis property we can restate Theorem 2.3.

Restatement 5.4. *Every infinite-dimensional Banach space, $\mathfrak{X}$, contains a subspace isomorphic to c_0, a subspace isomorphic to ℓ_1 or $\mathfrak{X}$ fails $\mathrm{DP_h}$.*

In light of this restatement we see that a biorthogonal system characterization of $\mathrm{DP_h}$ is in order. Theorem 5.3 will give it to us.

Definition 5.5. A biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ is called a $\mathrm{DP_h}$*-biorthogonal system* if $\{x_n\}_n$ is semi-normalized, weakly null and for any subsequence $\{x_{n_j}\}_j$ the series $\sum_j x_{n_j}$ is not wuC.

Theorem 5.6. *A Banach space $\mathfrak{X}$ fails $\mathrm{DP_h}$ if and only if for each $\varepsilon > 0$ there is a $(2+\varepsilon)$-bounded $\mathrm{DP_h}$-biorthogonal system $\{x_n, x_n^*\}$ in $S(\mathfrak{X}) \times \mathfrak{X}^*$.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{X}$ fails $\mathrm{DP_h}$ and $\varepsilon > 0$. Then Theorem 5.3 gives us a normalized weakly null sequence $\{x_n\}_n$ with no subsequence equivalent to the unit vector basis of c_0. Without loss of generality $\{x_n\}_n$ is a basic sequence with basis constant at most $2+\varepsilon$. Now if for some subsequence $\{x_{n_j}\}_j$ we have $\sum_j x_{n_j}$ wuC then Theorem 4.2 tells us that $\{x_{n_j}\}_j$ is equivalent to the unit vector basis of c_0, which is a contradiction. Since $\{x_n\}_n$ is basic with basis constant at most $2+\varepsilon$, we may pick a sequence of biorthogonal functionals $\{x_n^*\}_n \subset (2+\varepsilon)B(\mathfrak{X}^*)$.

($\Leftarrow$) Suppose there exists such a biorthogonal system $\{x_n, x_n^*\}$. If $\mathfrak{X}$ has $\mathrm{DP_h}$ then Theorem 5.3 gives us a subsequence $\{x_{n_j}\}_j$ of $\{x_n\}_n$ that is equivalent to the unit vector basis of c_0. But then we would have $\sum_j x_{n_j}$ wuC, which is a contradiction. ∎

Finally, putting this together with Theorems 3.3 and 4.5 and the Dilworth–Girardi–Johnson ℓ_1 result we get a correct theorem.

THEOREM 5.7. *For any given infinite-dimensional Banach space $\mathfrak{X}$ there exists a bounded biorthogonal system $\{x_n, x_n^*\}$ that is of one of the following three types*:

(1) *a c_0-biorthogonal system*,
(2) *a wc_0^*-stable biorthogonal system*,
(3) *a* $\mathrm{DP_h}$*-biorthogonal system.*

Note that this theorem confirms the importance of c_0 in infinite-dimensional Banach spaces. The presence of a c_0-biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X}$ gives us a part of $\mathfrak{X}$ which is particularly c_0-rich in the sense that $[x_n]$ is isomorphic to c_0 by design and, of course, the same is true for any subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. On the other hand, the existence of a $\mathrm{DP_h}$-biorthogonal system $\{x_n, x_n^*\}$ in $\mathfrak{X}$ would signify a part of $\mathfrak{X}$ is completely lacking in c_0 subspaces. In particular, $[x_n]$ is not isomorphic to c_0 and the same is true for any subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ since $\sum_n x_{n_j}$ is not wuC. In the third case if $\mathfrak{X}$ has a wc_0^*-stable biorthogonal system $\{x_n, x_n^*\}$, then $[x_n]$ is not isomorphic to c_0 since the proof in [7] yields $[x_n] \approx \ell_1$.

It would be interesting to see what this interpretation of Theorem 5.7 yields in terms of other properties and structures that have been characterized using c_0. For instance, can we say anything about the existence of spreading models or nice (resp. not very nice) operators on the space?

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