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Euclidean arrangements in Banach spaces

by

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Abstract. We study the way in which the Euclidean subspaces of a Banach space fit together, somewhat in the spirit of the Kashin decomposition. The main tool that we introduce is an estimate regarding the convex hull of a convex body in John's position with a Euclidean ball of a given radius, which leads to a new and simplified proof of the randomized isomorphic Dvoretzky theorem. Our results also include a characterization of spaces with nontrivial cotype in terms of arrangements of Euclidean subspaces.

1. Introduction. A fundamental result in the geometry of Banach spaces is Dvoretzky's theorem (see e.g. [30, 38]), which states that any Banach space X of dimension $n \in \mathbb{N}$ is richly endowed with approximately Euclidean subspaces of dimension $\lfloor c \log n \rfloor$. Besides knowing that there are many Euclidean subspaces, it is not known precisely how these subspaces are arranged within X. In the case $X = \ell_1^n$ it has been shown, going back to the work of Kashin [18], that there exist mutually orthogonal subspaces $E_1, E_2 \subset \mathbb{R}^n$ such that $\mathbb{R}^n = E_1 \oplus E_2$ and for all $i \in \{1, 2\}$ and all $x \in E_i$,

$$c_1|x| \leq \frac{1}{\sqrt{n}}\|x\|_1 \leq c_2|x|$$

where $c_1, c_2 > 0$ are universal constants, $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$. The same decomposition was shown to hold for spaces with universally bounded volume ratio [40, 43] (see Section 3 for more details). Using the results just mentioned, it is easy to show that there exists an orthonormal basis for $\mathbb{R}^n$, say $(e_i)_{i=1}^n$, such that for all $(0.99n)$-sparse vectors $a \in \mathbb{R}^n$,

$$c_1\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \leq \frac{1}{\sqrt{n}}\Big\|\sum_{i=1}^n a_i e_i\Big\|_1 \leq c_2\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}. \tag{1.1}$$

In this paper we seek a collection of Euclidean subspaces of a general finite-dimensional Banach space that fit together like a grid in such a way so that,

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with respect to a particular basis, vectors with various regularity properties act as though they were in a Hilbert space, in the sense of (1.1). We are interested in both isomorphic and almost-isometric type estimates. There are two main types of regularity that we impose. The first is sparsity, and the second is simplicity of support, measured in terms of cyclic length and Kolmogorov complexity.

The most interesting example is ℓ_∞^n which has a $1+\varepsilon$ Kashin-style decomposition but not the richer arrangement of $1+\varepsilon$ Euclidean subspaces as described above. Numerous questions remain unsolved, even for ℓ_∞^n.

2. Main results

2.1. The isomorphic theory. A key innovation of the paper is the following lemma which uses a result of Vershynin [45] on contact points of ∂B_X with the John ellipsoid (see Theorem 3.5 and Corollary 3.6).

LEMMA 2.1. *There exist universal constants $c, c', c_1 > 0$ with $c' > 2c_1^2$ such that the following holds. Consider any Banach space X of dimension $n \in \mathbb{N}$ and let $t \in \mathbb{R}$ with $1 \le t \le c_1\sqrt{n}$. Identify X with $\mathbb{R}^n$ so that B_X is in John's position. Let $K_t = \mathrm{conv}\{tB_2^n, B_X\}$. Let M_t and b_t denote the median and maximum of the Minkowski functional of K_t on S^{n-1}. Then*

$$\frac{M_t}{b_t} \ge ct\sqrt{\frac{1}{n}\log\left(\frac{c'n}{t^2}\right)}.$$

The body $K_{(\rho)} = B_X \cap \rho B_2^n$, related to K_t by duality, has appeared before in the literature. For this we refer the reader to [12, 25] and the references therein. An immediate consequence is a version of the randomized isomorphic Dvoretzky theorem of Litvak, Mankiewicz and Tomczak-Jaegermann [24] (see also the original papers by Milman and Schechtman [31, 32], as well as [14, 15]).

COROLLARY 2.2. *There exist universal constants $c, c_1, c_2, C > 0$ such that the following is true. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that the ellipsoid of maximum volume in $B_X = \{x : \|x\| \le 1\}$ is the standard Euclidean ball B_2^n. Let $1 \le k \le n$ and let $E \in G_{n,k}$ be a random subspace uniformly distributed in $G_{n,k}$. Then with probability at least $1 - C\exp(-c\max\{k, \log n\})$ the following event occurs. For all $x \in E$,*

$$c_1 M^{(k)}|x| \le \|x\| \le c_2\sqrt{\frac{k+\log n}{\log(1+n/k)}}\, M^{(k)}|x|$$

where $M^{(k)}$ is the average value of the Minkowski functional of the set $\mathrm{conv}(B_X \cup tB_2^n)$ on S^{n-1}, with $t = \sqrt{(k+\log n)/\log(1+n/k)}$.

Proof. Assume first that $k < c'n$ for a sufficiently small $c' > 0$. Set $K_t = \operatorname{conv}(B_X \cup tB_2^n)$ and apply Lemma 2.1 followed by Milman's general Dvoretzky theorem (see e.g. Theorem 3.3) with $\varepsilon = 1/2$. We conclude that with probability at least $1 - C\exp(-c\max\{k, \log n\})$ the following event occurs: for all $x \in E$, $c_1 M^{(k)}|x| \le \|x\|_{K_t} \le c_2 M^{(k)}|x|$. Then note that $\|x\|_{K_t} \le \|x\| \le t\|x\|_{K_t}$. If $c'n \le k \le n$ the result follows by John's theorem. ∎

For a vector $a \in \mathbb{R}^n$, let $a^\flat \in \mathbb{R}^n$ denote the indicator function of the support of a, i.e.

$$a_i^\flat = \begin{cases} 0, & a_i = 0, \\ 1, & a_i \neq 0. \end{cases}$$

Let $\|a\|_0 = \|a^\flat\|_1 = |\{i : a_i \neq 0\}|$ denote the *sparsity* of a, let $\|a\|_{\mathrm{Kol}} = C_{\mathrm{Kol}}(a^\flat)$ denote the *Kolmogorov complexity* of $a^\flat$, and let

$$\|a\|_{\mathrm{cyc}} = \min\{k \le n : \exists m \le n, (k \le i < n \Rightarrow a_{(m+i) \bmod n} = 0)\}$$

denote the *cyclic length* of the support of a.

Another direct consequence of Lemma 2.1 (or just as well Corollary 2.2) is as follows.

COROLLARY 2.3. *There exist universal constants $c, c', C > 0$ such that the following is true. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that the ellipsoid of maximum volume in $B_X = \{x : \|x\| \le 1\}$ is the standard Euclidean ball B_2^n. Let $(e_i)_{i=1}^n$ be a random orthonormal basis for $\mathbb{R}^n$ generated by the action of a random orthogonal matrix uniformly distributed in $O(n)$. Then with probability at least $1 - Cn^{-c}$, the following event occurs. For all $a \in \mathbb{R}^n$,*

$$cM_{D(a)}\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n a_i e_i\Big\| \le D(a) M_{D(a)}\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}$$

where $M_{D(a)}$ is the average value of the Minkowski functional of the set $\operatorname{conv}(B_X \cup D(a)B_2^n)$ on S^{n-1}, and the distortion $D(a)$ can be written as

$$c' \min\bigg\{\|a\|_0^{1/2}, \Big(\frac{\|a\|_{\mathrm{cyc}} + \log n}{\log(1 + n\|a\|_{\mathrm{cyc}}^{-1})}\Big)^{1/2}, \Big(\frac{\|a\|_0 + \|a\|_{\mathrm{Kol}} + \log n}{\log(1 + n(\|a\|_0 + \|a\|_{\mathrm{Kol}})^{-1})}\Big)^{1/2}\bigg\}. \tag{2.1}$$

Note also that M_t is nonincreasing in t while tM_t is nondecreasing.

The same dependence on $\|a\|_0$ can be achieved with the use of the Dvoretzky–Rogers factorization [4, 11, 41, 42, 45], but only on a subspace of proportional dimension. Note that in Corollary 2.3, logarithmic sparsity alone is not enough to guarantee bounded Euclidean distortion.

PROBLEM 2.4. *Does there exist a universal constant $C > 0$ and a sequence $(\omega_n)_{n=1}^\infty$ with $\lim_{n\to\infty} \omega_n = \infty$ such that the following is true: For every $n \in \mathbb{N}$ and any real Banach space $(X, \|\cdot\|)$ of dimension n, there is a basis $(e_i)_{i=1}^n$ for X such that for any $a \in \mathbb{R}^n$ with $\|a\|_0 \le \omega_n$,*

$$\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n a_i e_i\Big\| \le C\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} ?$$

Can one take $\omega_n = c\log n$? Can one take $(e_i)_{i=1}^n$ to be orthonormal with respect to the John ellipsoid of B_X?

In Section 2.2 we show that the corresponding $1+\varepsilon$ estimate does not hold in ℓ_∞^n.

Lemma 2.1 can also be used to prove an inequality for the distribution of norms on S^{n-1} that is very similar to a result of Schechtman and Schmuckenschläger [39]. This in turn shows that bodies in John's position with low Dvoretzky dimension have a large Klartag–Vershynin parameter $d_u(K)$, a parameter which is significant partly because of its relation to the outer inclusion in Dvoretzky's theorem [19]. These results are presented in Corollaries 4.3 and 4.4.

Finally, let us note that a forthcoming paper of Chasapis and Giannopoulos [7] further explores consequences of Lemma 2.1, which includes an isomorphic version of the global Dvoretzky theorem of Bourgain, Lindenstrauss and Milman.

2.2. The almost-isometric theory. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$. Using Corollary 2.3 followed by a further application of the randomized Dvoretzky theorem, it follows that we may identify X with $\mathbb{R}^n$ in such a manner that $c_1 B_2^n \subset \mathcal{E} \subset c_2 B_2^n$, where $\mathcal{E}$ is the John ellipsoid of B_X, and we may write X as the internal direct sum of subspaces

$$X = \bigoplus_{i=1}^N H_i$$

where $\dim(H_i) \ge c(\varepsilon)\log n$ and the subspaces $(H_i)_{i=1}^N$ are pairwise orthogonal, and for each $1 \le i \le N$ and each $x \in H_i$,

$$(1-\varepsilon)\widetilde{M}|x| \le \|x\| \le (1+\varepsilon)\widetilde{M}|x|$$

where $\widetilde{M}$ depends on X. However for many spaces such as those with a symmetric basis and those that come with a pre-packaged coordinate system, such as ℓ_∞^n, the Euclidean structure associated to the (exact) John ellipsoid is of particular importance. Answering a question that we posed in an earlier draft of this paper, Konstantin Tikhomirov gave a proof of the following result which we discuss further in Section 5.1. We thank him for allowing us to include it here.

THEOREM 2.5. *There exists a universal constant $c > 0$ with the following property. Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that the ellipsoid of maximum volume in $B_X = \{x : \|x\| \le 1\}$ is the standard Euclidean ball B_2^n. Let $c(\log\log n)^{3/2}/(\log n)^{1/2} < \varepsilon < 1/2$. Then there exists a decomposition $X = \bigoplus_{i=1}^N H_i$ into mutually orthogonal subspaces $(H_i)_{i=1}^N$ with $\dim(H_i) \ge c\varepsilon^2(\log\varepsilon^{-1})^{-1}\log n$ such that for all $1 \le i \le N$ and all $x \in H_i$,*

$$(1-\varepsilon)M^\sharp|x| \le \|x\| \le (1+\varepsilon)M^\sharp|x|$$

where $M^\sharp$ is the median of $\|\cdot\|$ on S^{n-1}.

In Corollary 4.2 we consider the infinite-dimensional case. Theorem 2.5 guarantees the existence of at least one decomposition, and we do not know whether it holds for a typical decomposition. Referring back to Theorem 2.3, we also do not know whether one can achieve a $1+\varepsilon$ estimate for all a such that $\|a\|_{\mathrm{cyc}} \le c(\varepsilon)\omega_n$ for some fixed function ω_n with $\omega_n \to \infty$, such as $\omega_n = \log n$.

DEFINITION 2.6. Let X be an infinite-dimensional Banach space over $\mathbb{R}$. We shall say that X satisfies Definition 2.6 if for all $\varepsilon > 0$ and all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ with the following property. For any finite-dimensional subspace $E \subset X$ with $\dim(E) > N$, there exists a basis $(e_i)_{i=1}^n$ for E such that for any $a \in \mathbb{R}^n$ with $\|a\|_0 \le k$,

$$(1-\varepsilon)\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n a_ie_i\Big\|_X \le (1+\varepsilon)\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}. \tag{2.2}$$

The space c_0 does not have this property (see Lemma 5.3). Going back to the work of Kwapień [21] and Figiel, Lindenstrauss and Milman [9], it is well known that there are intimate connections between the Euclidean structures within a Banach space and the notions of type and cotype. The ℓ_∞^n spaces have, in a sense, the smallest possible collection of Euclidean subspaces. By the Maurey–Pisier–Krivine theorem, these spaces are excluded as subspaces of X precisely when X has nontrivial cotype. This leads naturally to the following result.

THEOREM 2.7. *An infinite-dimensional real Banach space X satisfies Definition 2.6 if and only if it has nontrivial cotype.*

Our proof shows that when X has cotype $q < \infty$ and corresponding cotype constant $\beta \in (0,1]$, such a basis exists provided $0 < \varepsilon < 0.99$ and $k \le c\beta^2\varepsilon^2(\log(en/k))^{-1}n^{2/q}$. For cotype 2 spaces the bound for k can be polished slightly and written as $k \le c\beta^2\varepsilon^2(\log\beta^{-1} + \log\varepsilon^{-1})^{-1}n$.

2.3. Related observations. Our results are closely related to the restricted isometry property of Candès and Tao [6] which plays a fundamental role in compressed sensing. They can be understood as generalized restricted isometry/isomorphism properties for random operators from ℓ_2^n into more general normed spaces. It follows from Theorem 2.7 that nontrivial cotype is a natural condition to assume in the following two results.

PROPOSITION 2.8. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces of dimensions $n, m \in \mathbb{N}$ respectively, with cotype $q_1, q_2 < \infty$ and corresponding cotype constants $\beta_1, \beta_2 \in (0,1]$. Let $0 < \varepsilon < 0.99$ and consider any $k \in \mathbb{N}$ such that*

$$k \leq c\varepsilon^2 (\log(en/k))^{-1} \min\{\beta_1^2 n^{2/q_1}, \beta_2^2 m^{2/q_2}\}.$$

Then there exist bases $(e_i)_{i=1}^n$ and $(f_i)_{i=1}^m$ in X and Y respectively with the following property. Let G be an $m \times n$ random matrix with i.i.d. $N(0,1)$ entries. With probability at least

$$1 - c_1 \exp(-C\beta_2^2 m^{2/q_2} \varepsilon^2)$$

the following event occurs. For all k-sparse vectors $a \in \mathbb{R}^n$,

$$(1-\varepsilon)\Big\|\sum_{j=1}^n a_j e_j\Big\|_X \leq \Big\|\frac{1}{\sqrt{m}}\sum_{i=1}^m \sum_{j=1}^n G_{ij} a_j f_i\Big\|_Y \leq (1+\varepsilon)\Big\|\sum_{j=1}^n a_j e_j\Big\|_X$$

Here, $c, c_1, C > 0$ are universal constants.

Let us say that a vector $x \in X$ is *k-sparse* with respect to a basis $(e_i)_{i=1}^n$ if it can be expressed as a linear combination of no more than k basis vectors. The Johnson–Lindenstrauss lemma [16] does not hold in a general Banach space, even in spaces with nontrivial cotype (see [17] and the references therein). For vectors that are sparse with respect to a particular basis (which can be chosen randomly), the situation is different.

PROPOSITION 2.9. *Let $(X, \|\cdot\|)$ be a real Banach space of dimension $n \in \mathbb{N}$, with cotype $q \in [2, \infty)$ and corresponding cotype constant $\beta \in (0,1]$. Let $0 < \varepsilon < 0.99$ and let $k \in \mathbb{N}$ be such that $k \leq c\beta^2\varepsilon^2(\log(en/k))^{-1} n^{2/q}$. Then there exists a basis $(e_i)_{i=1}^n$ for X with the following property. Let $\Omega \subset X$ be a finite collection of vectors that are each k-sparse with respect to the given basis and let $m = \lfloor C\varepsilon^{-2} \log|\Omega| \rfloor$. Then there exists a linear operator $T : X \to \ell_2^m$ such that for all $x, y \in \Omega$,*

$$(1-\varepsilon)\|x-y\| \leq |Tx - Ty| \leq (1+\varepsilon)\|x-y\|.$$

If in the above proposition one insists on a map $Q : X \to E$, where E is a subspace of X, then we may use Dvoretzky's theorem and modify the bound on m.

3. Background. Most of the background material relevant to the paper can be found in [1, 9, 26, 27, 30, 33, 34, 35]. The letters c, c_1, c_2, c', C etc. denote universal constants that take on different values from one line to the next. They are not arbitrary, but have very specific numerical values that we do not always have control over. The symbols $\mathbb{P}$ and $\mathbb{E}$ denote probability and expected value. For $p \in [1, \infty)$, let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ denote the ℓ_p^n norm of a vector $x \in \mathbb{R}^n$ and let $|\cdot| = \|\cdot\|_2$ be the standard Euclidean norm. For $p = \infty$, $\|x\|_\infty = \max\{|x_i| : 1 \le i \le n\}$. Let $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. The Grassmannian manifold $G_{n,k}$ consists of all k-dimensional linear subspaces of $\mathbb{R}^n$, while the orthogonal group $O(n)$ is the space of all orthogonal $n \times n$ matrices. The spaces S^{n-1}, $G_{n,k}$ and $O(n)$ each have a unique rotation invariant probability measure called Haar measure, which will be denoted in each case as σ_n. Of fundamental importance is Lévy's concentration inequality.

THEOREM 3.1. *Let $f : S^{n-1} \to \mathbb{R}$ be a Lipschitz function and let $M_f = \int_{S^{n-1}} f \, d\sigma_n$. Then for all $t > 0$,*

$$\sigma_n\{\theta \in S^{n-1} : |f(\theta) - M_f| < t \operatorname{Lip}(f)\} \ge 1 - c_1 e^{-c_2 n t^2} \tag{3.1}$$

where $c_1, c_2 > 0$ are universal constants. The same result holds with the mean M_f replaced with anything between (say) the 10th and 90th percentile of f, such as the median, and $\operatorname{Lip}(f)$ *is measured with respect to either the Euclidean metric on S^{n-1} or the geodesic distance.*

A (centrally) *symmetric convex body* $K \subset \mathbb{R}^n$ is a compact, convex set with nonempty interior such that $x \in K$ if and only if $-x \in K$. The associated *Minkowski* and *dual Minkowski functionals* are the norms defined by

$$\|x\|_K = \inf\{t \ge 0 : x \in tK\},$$
$$\|y\|_{K^\circ} = \max\{\langle x, y\rangle : x \in K\},$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product. The body

$$K^\circ = \{y : \|y\|_{K^\circ} \le 1\}$$

is known as the *polar* of K.

The *John ellipsoid* of a convex body K, denoted $\mathcal{E}_K$, is the ellipsoid of maximal volume contained within K. It can be shown via a compactness argument that such an ellipsoid exists. It is also known that $\mathcal{E}_K$ is unique. When $\mathcal{E}_K = B_2^n$, we say that K is *in John's position*, and in this case (assuming K is symmetric)

$$B_2^n \subseteq K \subseteq \sqrt{n}\, B_2^n.$$

Two parameters of particular importance are the mean and the maximum,

$$M(K) = \int_{S^{n-1}} \|\theta\|_K \, d\sigma_n(\theta), \tag{3.2}$$

$$b(K) = \max\{\|\theta\|_K : \theta \in S^{n-1}\}. \tag{3.3}$$

When K is in John's position we have $b \leq 1$, and it can be shown using the Dvoretzky–Rogers lemma that

$$M \geq c\sqrt{\frac{\log n}{n}}. \tag{3.4}$$

Let (Ω, ρ) be a compact metric space and $0 < \varepsilon < 1$. An *ε-net* $\mathcal{N} \subset \Omega$ is a set such that for all $\theta \in \Omega$ there exists $\omega \in \mathcal{N}$ such that $\rho(\theta, \omega) < \varepsilon$ and for all $\omega_1, \omega_2 \in \mathcal{N}$, $\rho(\omega_1, \omega_2) \geq \varepsilon$. Sometimes the latter condition is dropped. Such a set can easily be shown to exist. In the case $\Omega = S^{n-1}$, a volumetric argument yields

$$|\mathcal{N}| \leq \left(\frac{3}{\varepsilon}\right)^n. \tag{3.5}$$

By homogeneity, any $x \in \mathbb{R}^n$ can be expressed as $x = |x|\omega_0 + x'$, where $\omega_0 \in \mathcal{N}$ and $|x'| < \varepsilon|x|$. Iterating this expression yields

$$x/|x| = \omega_0 + \sum_{i=1}^{\infty} \varepsilon_i \omega_i \tag{3.6}$$

where $(\omega_i)_{i=0}^{\infty}$ is a sequence in $\mathcal{N}$ and $0 \leq \varepsilon_i < \varepsilon^i$. Applying the triangle inequality then leads to the following lemma.

LEMMA 3.2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $\delta \in (0, 1/4)$. Let $M > 0$ and let $\mathcal{N}$ be a δ-net in S^{n-1}. Suppose that for all $\omega \in \mathcal{N}$, $(1-\delta)M \leq \|\omega\| \leq (1+\delta)M$. Then for all $x \in \mathbb{R}^n$, $(1-4\delta)M \leq \|x\| \leq (1+4\delta)M$.*

THEOREM 3.3 (The general Dvoretzky theorem). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with parameters M, b as defined by* (3.2) *and* (3.3) *respectively. Let $0 < \varepsilon < 0.99$ and $k \leq c_1\varepsilon^2 M^2 b^{-2} n$, and let $E \in G_{n,k}$ be any fixed subspace. Let T be a random orthogonal matrix uniformly distributed in $O(n)$ and let $E = TF$. Then with probability at least $1 - c_1 \exp(-c_2\varepsilon^2 M^2 b^{-2} n)$ we have, for all $x \in E$, $(1-\varepsilon)M|x| \leq \|x\| \leq (1+\varepsilon)M|x|$.*

Sketch. Let us start by giving Milman's original proof under the slightly stronger assumption that $k \leq c_1\varepsilon^2(\log \varepsilon^{-1})^{-1} M^2 b^{-2} n$. Let $\mathcal{N}$ be an $\varepsilon/4$-net in $S(F) = \{x \in F : |x| = 1\}$. The epsilon net bound (3.5) yields $|\mathcal{N}| \leq (12/\varepsilon)^k$. It follows from the triangle inequality and the definition of b that $\|\cdot\|$ is Lipschitz on $\mathbb{R}^n$ with $\mathrm{Lip}(\|\cdot\|) = b$. For each $\omega \in \mathcal{N}$, $T\omega$ is uniformly distributed in S^{n-1}. The result then follows from Lévy's inequality (3.1) with $t = 4^{-1}\varepsilon M b^{-1}$, the union bound, and Lemma 3.2. To eliminate the factor $(\log \varepsilon^{-1})^{-1}$, one can use a random $n \times n$ matrix Q with i.i.d. standard Gaussian entries, instead of $T \in O(n)$, as well as a more intricate epsilon net argument. The use of a Gaussian matrix allows one to take advantage of the fact that if $u \perp v$ then Qu and Qv are independent. Eliminating the factor $(\log \varepsilon^{-1})^{-1}$ using Gaussian processes/matrices was done by Gordon [13] and Schechtman [36]. Lastly, the matrix $n^{-1/2}Q$ acts as an approximate isometry on F (with respect to the Euclidean norm on F and QF), and so the result for Q can then be transferred to the result for T. ∎

If $(F_i)_{i=1}^N$ is any collection of subspaces of $\mathbb{R}^n$, then we may apply Theorem 3.3 simultaneously to all F_i and modify the corresponding probability (using the union bound). This is usually how we shall apply Theorem 3.3.

If X is any finite-dimensional Banach space, then there exists a basis for X such that with respect to this basis, the unit ball $B_X = \{x : \|x\| \le 1\}$ is in John's position and by (3.4), we can take $k = \lfloor c\varepsilon^2 \log n \rfloor$. The best known dependence on ε for the existence of at least one deterministic subspace E is $c(\varepsilon) = c\varepsilon(\log \varepsilon^{-1})^{-2}$ by Schechtman [37, 38].

The *volume ratio* of a convex body $K \subset \mathbb{R}^n$ is defined as

$$\mathrm{vr}(K) = \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(\mathcal{E}_K)} \right)^{1/n}.$$

This goes back to the paper of Szarek [40] where it was used implicitly to prove Kashin's result, and then explicitly defined in [43] following a suggestion by Pełczyński. The volume ratio theorem states that if K is symmetric, $1 \le k \le n$, and $E \in G_{n,k}$ is a random subspace of dimension k (uniformly distributed with respect to the Haar measure on $G_{n,k}$ corresponding to $\mathcal{E}_K$), then with probability at least $1 - 2^{-n}$,

$$\mathcal{E}_K \cap E \subseteq K \cap E \subseteq (4\pi \mathrm{vr}(K))^{n/(n-k)} (\mathcal{E}_K \cap E).$$

For spaces with universally bounded volume ratio, such as ℓ_1^n where $\mathrm{vr}(B_1^n) \le \sqrt{2\pi/e}$, this gives a version of Dvoretzky's theorem with proportional dimension, as well as the Kashin decomposition mentioned in the introduction. The probability bound $1 - 2^{-n}$ is somewhat arbitrary and can be replaced with $1 - t^n$ for any $t > 0$ if we replace the 4π with ct^{-1}. Setting $t = 1/10$ (say) and using the union bound and the inequality $\binom{n}{k} \le (en/k)^k$ then implies inequality (1.1) for $k = \lceil 0.99n \rceil$.

A Banach space E embeds (linearly) into a space Y *with distortion* $\gamma \ge 1$, denoted $E \hookrightarrow_\gamma Y$, if there exists a linear subspace $F \subseteq Y$ and a linear bijection $T : E \to F$ such that $\|T\| \cdot \|T^{-1}\| = \gamma$. The *Euclidean distortion* of E is defined as

$$d_2(E) = \inf\{\gamma \ge 1 : E \hookrightarrow_\gamma H\}$$

where H is a suitably large Hilbert space. A space X is *finitely representable* in Y if for all finite-dimensional subspaces $E \subset X$ and all $\varepsilon > 0$, $E \hookrightarrow_{1+\varepsilon} Y$. By Dvoretzky's theorem, every Hilbert space is finitely representable in every infinite-dimensional Banach space.

The notions of type and cotype capture the spirit of the L_p spaces in an abstract setting. Let $(\varepsilon_i)_{i=1}^\infty$ denote an i.i.d. sequence of Rademacher random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$. A Banach space X is said to have *type* $p \in [1,2]$ if there exists $\alpha \in [1,\infty)$ such that for all finite sequences $(x_i)_{i=1}^m$ in X,

$$\mathbb{E}\Big\|\sum_{i=1}^m \varepsilon_i x_i\Big\| \le \alpha\Big(\sum_{i=1}^m \|x_i\|^p\Big)^{1/p}.$$

Similarly, X is said to have *cotype* $q \in [2,\infty]$ if there exists $\beta \in (0,1]$ such that for all $(x_i)_{i=1}^m$ in X,

$$\beta\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q} \le \mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|$$

with the appropriate interpretation when $q=\infty$. Any Banach space X has type 1 and cotype ∞, and these are referred to as *trivial* type/cotype. If X has type p and cotype q, then it has type p' and cotype q' for all $p' \in [1,p]$ and $q' \in [q,\infty]$. Type and cotype are inherited by subspaces, and the space L_p $(1 \le p < \infty)$ has type $\min\{p,2\}$ and cotype $\max\{p,2\}$. If E is a finite-dimensional space with cotype $q<\infty$ and corresponding constant β, then with respect to the John ellipsoid of B_E inequality (3.4) can be improved to

$$M \ge c\beta n^{1/q-1/2} \tag{3.7}$$

where $n=\dim(E)$, and the general Dvoretzky theorem guarantees the existence of Euclidean subspaces of dimension $c\varepsilon^2\beta^2 n^{2/q}$. Here we use the notation in [9] where $\beta \in (0,1]$. Some authors refer to β^{-1} as the cotype constant, and others use yet another definition which is equivalent up to a constant C_p.

Let p_X and q_X be the supremum (resp. infimum) over all values of p and q such that X has type p and cotype q. One of the most significant results in the theory of type and cotype is the Maurey–Pisier–Krivine theorem [20, 28], which builds on work by Brunel and Sucheston [5].

THEOREM 3.4. *If X is infinite-dimensional, then ℓ_{p_X} and ℓ_{q_X} are finitely representable in X.*

The following powerful result of Vershynin plays a key role in our work (see [45, Corollary 5.5 and the discussion on p. 269]):

THEOREM 3.5. *There exists a function $\xi : (0,1) \to (1,\infty)$ such that the following is true. Consider any $\varepsilon \in (0,1)$ and let X be any n-dimensional real Banach space that we identify with $\mathbb{R}^n$ so that $B_X = \{x : \|x\| \le 1\}$ is in John's position. Then there exists $m > (1-\varepsilon)n$ and a sequence $(v_i)_{i=1}^m \subset S^{n-1} \cap \partial B_X$ such that for all $a \in \mathbb{R}^m$,*

$$\xi(\varepsilon)^{-1}\Big(\sum_{i=1}^m a_i^2\Big)^{1/2} \le \Big|\sum_{i=1}^m a_i v_i\Big| \le \xi(\varepsilon)\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}.$$

Corollary 3.6. *Let $(X, \|\cdot\|_X)$ be a real Banach space of dimension $n \in \mathbb{N}$ and let $m = \lceil n/2 \rceil$. Then there exists an inner product $\langle\cdot,\cdot\rangle_\sharp$ on X and a sequence $(u_i)_{i=1}^m$ that is orthonormal with respect to $\langle\cdot,\cdot\rangle_\sharp$ such that for all $1 \le i \le m$,*

$$C^{-1} \le \|u_i\|_X \le C,$$
$$C^{-1} \le \|u_i\|_{X^*} \le C,$$

where $\|\cdot\|_{X^*}$ is the dual norm on X under the duality corresponding to $\langle\cdot,\cdot\rangle_\sharp$, i.e. $\|x\|_{X^*} = \sup\{\langle x, y\rangle_\sharp : \|y\|_X \le 1\}$. Furthermore, the ellipsoid of maximum volume in B_X, denoted $\mathcal{E}$, satisfies $c_1\mathcal{E}^\sharp \subseteq \mathcal{E} \subseteq c_2\mathcal{E}^\sharp$, where $\mathcal{E}^\sharp = \{x \in X : \langle x, x\rangle_\sharp \le 1\}$.

Proof. Here we review the theory surrounding Vershynin's result. Identify X with $\mathbb{R}^n$ so that $K = B_X$ is in John's position. By Theorem 3.5, there exists a sequence $(v_i)_{i=1}^m$ of contact points between B_2^n and ∂K such that for any sequence of coefficients $(a_i)_{i=1}^m \in \mathbb{R}^m$,

$$c_1\Big(\sum_{i=1}^m a_i^2\Big)^{1/2} \le \Big|\sum_{i=1}^m a_i v_i\Big| \le c_2\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}.$$

By construction, $\|v_i\|_K = |v_i| = 1$ for all $1 \le i \le m$. These vectors are linearly independent, and can be extended to a basis for $\mathbb{R}^n$, $(v_i)_{i=1}^n$, such that $(v_i)_{i=m+1}^n$ are orthonormal and $\operatorname{span}\{v_i\}_{i=1}^m$ is orthogonal to $\operatorname{span}\{v_i\}_{i=m+1}^n$. Using these properties, for any sequence of coefficients $(a_i)_{i=1}^n \in \mathbb{R}^n$,

$$c_3\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \Big|\sum_{i=1}^n a_i v_i\Big| \le c_4\Big(\sum_{i=1}^n a_i^2\Big)^{1/2},$$

which implies $c \le \|A\|_{2\to 2} \le c'$ and $c \le \|A^{-1}\|_{2\to 2} \le c'$, where A is the $n \times n$ matrix with the vectors $(v_i)_{i=1}^n$ as columns and $\|\cdot\|_{2\to 2}$ denotes the operator norm of a matrix from ℓ_2^n to ℓ_2^n. Let $(u_i)_{i=1}^n$ denote the dual basis of $(v_i)_{i=1}^n$, i.e. the columns of $(A^T)^{-1}$. Since $\|A^T\|_{2\to 2} = \|A\|_{2\to 2}$ and $\|(A^T)^{-1}\|_{2\to 2} = \|A^{-1}\|_{2\to 2}$, we have

$$c_5\Big(\sum_{i=1}^n a_i^2\Big)^{1/2} \le \Big|\sum_{i=1}^n a_i u_i\Big| \le c_6\Big(\sum_{i=1}^n a_i^2\Big)^{1/2}.$$

By the Hahn–Banach theorem, there exists $(w_i)_{i=1}^m$ such that $\|w_i\|_{K^\circ} = 1$ and $\langle v_i, w_i\rangle = 1$. Since $K^\circ \subseteq B_2^n$, $|w_i| \le 1$. Since $|v_i|, |w_i| \le 1$ and $\langle v_i, w_i\rangle = 1$, it follows that $v_i = w_i$. Therefore $\|v_i\|_{K^\circ} = 1$. By definition of $(u_i)_{i=1}^n$, $\langle u_i, v_j\rangle = \delta_{i,j}$, and therefore for each $1 \le i \le m$,

$$|a_i| = \Big|\Big\langle \sum_{j=1}^m a_j u_j, v_i\Big\rangle\Big| \le \Big\|\sum_{j=1}^m a_j u_j\Big\|_K,$$

which implies that for all $a \in \mathbb{R}^m$,

$$\max_{1\le i\le m} |a_i| \le \Big\|\sum_{j=1}^m a_j u_j\Big\|_K \le c_6\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}.$$

Here we have also used the fact that the set K is in John's position, which implies that $\|\cdot\|_K \le |\cdot|$. Define $\langle x,y\rangle_\sharp = \langle A^T x, A^T y\rangle$. Since $A^T u_i = e_i$ for each $1 \le i \le m$, it follows that $(u_i)_{i=1}^m$ are orthonormal with respect to $\langle\cdot,\cdot\rangle_\sharp$. The fact that $c_1\mathcal{E}^\sharp \subseteq B_2^n \subseteq c_2\mathcal{E}^\sharp$ follows from the fact that $\max\{\|A^T\|_{2\to2}, \|(A^T)^{-1}\|_{2\to2}\} \le c$. ∎

The *Kolmogorov complexity* of a finite binary string $b \in \{0,1\}^* = \bigcup_{n=0}^\infty \{0,1\}^n$ is defined as (see for example [23])

$$C_{\mathrm{Kol}}(b) = \min\{\ell(p) : p \in \{0,1\}^*,\ \phi(p) = b\}$$

where ϕ is the universal partial recursive function generated by a specific universal Turing machine, and $\ell(p)$ denotes the length of p. This measures the amount of information contained in the string, which may be much less than its length due to redundancy and the existence of patterns. For example a string of the form

$$b = (0,0,0,\ldots,1,1,1,\ldots,0,0,0,\ldots) \tag{3.8}$$

has Kolmogorov complexity at most $c\log(n+2)$, where $n = \ell(b)$. In order to describe the string, we need to express the fact that a zero never occurs between two 1's, which can be communicated using at most c bits. We must then describe the starting point of the 1's and the ending point, which requires at most $2\log_2(n+1)+c$ bits. The types of strings most relevant to us are those with low complexity, such as those of the form (3.8). This differs from the more common situation where one is interested in strings of high complexity.

4. Isomorphic theory

LEMMA 4.1. *Let $(X_i)_{i=1}^m$ be an i.i.d. $N(0,1)$ sequence and* $(\log m)^{-1/2} \le s \le 1 - c(\log m)^{-1}$. *With probability at least* 0.52,

$$|\{i : s\sqrt{\log m} \le X_i \le 3\sqrt{\log m}\}| \ge c' m^{1-s^2}.$$

Proof. Let Φ denote the cumulative standard normal distribution function. For all $t \ge 1$ (see e.g. [8]),

$$\frac{\phi(t)}{2t} \le 1 - \Phi(t) \le \frac{\phi(t)}{t}$$

where ϕ is the standard normal density. This implies that

$$\mathbb{P}\{s\sqrt{\log m} \le X_i \le 3\sqrt{\log m}\} \ge cm^{-s^2}.$$

Let $Y = |\{i : s\sqrt{\log m} \le X_i \le 3\sqrt{\log m}\}|$. By taking the constant c in the statement of the lemma to be sufficiently large, it follows that $\mathbb{E}Y \geq 1000$ and

$$\sqrt{\mathrm{Var}(Y)} \leq (\mathbb{E}Y)^{1/2} \leq \frac{\mathbb{E}Y}{30}.$$

It then follows from Chebyshev's inequality that $\mathbb{P}\{Y \geq \mathbb{E}Y/10\} \geq 0.52$. ∎

Proof of Lemma 2.1. Without loss of generality we may assume that $n > n_0$ and $t > c_2$, where n_0 and c_2 are sufficiently large universal constants. Let $A = \{x \in X : \|x\| \leq 1\}$ and $m = \lceil n/2 \rceil$. Identify X with $\mathbb{R}^n$ so that the inner product $\langle \cdot, \cdot \rangle_\sharp$ from Corollary 3.6 is the standard Euclidean inner product, and consider the vectors $(u_i)_{i=1}^m$ as in Corollary 3.6. With this coordinate structure, A is not necessarily in John's position. Near the end of the proof we shall use a second coordinate structure. After applying an orthogonal transformation, we may assume that $u_i = e_i$ for all $1 \leq i \leq m$, where $(e_i)_{i=1}^n$ are the standard basis vectors of $\mathbb{R}^n$. Let $A_t = \mathrm{conv}\{A, tB_2^n\}$. Then $\|y\|_{A_t^\circ} = \max\{\|y\|_{A^\circ}, t|y|\}$. Let $\theta \in S^{n-1}$ be a random point uniformly distributed on the sphere. We can simulate $\theta = X/|X|$, where $(X_i)_{i=1}^n$ are i.i.d. $N(0,1)$ variables. Set $s = (1 - 2\log(ct)/\log m)^{1/2}$, in which case $t = cm^{(1-s^2)/2}$. By Lemma 4.1, with probability at least 0.52, $|\Omega| \geq cm^{1-s^2} = ct^2$, where

$$\Omega = \bigl\{1 \leq i \leq m : s\sqrt{\log m} \leq X_i \leq 3\sqrt{\log m}\bigr\}.$$

Let $z = \sum_{i\in\Omega} e_i$. Then $\langle z, X\rangle \geq s(\log m)^{1/2}|\Omega|$. By Corollary 3.6 and the triangle inequality, $\|z\|_{A^\circ} \leq c|\Omega|$ and $|z| = |\Omega|^{1/2}$. By the bound on $|\Omega|$, $c|\Omega| \geq t|\Omega|^{1/2}$, which implies that $\|z\|_{A_t^\circ} \leq c|\Omega|$. We thus have $\|X\|_{A_t} \geq \langle z, X\rangle/\|z\|_{A_t^\circ} \geq cs(\log m)^{1/2}$ (with probability ≥ 0.52). With probability at least 0.99, $|X| \leq c\sqrt{n}$. Therefore, with probability at least 0.51, $\|\theta\|_{A_t} \geq cs\sqrt{(\log n)/n}$. Since this probability bound is strictly greater than 0.5, this leads to a bound on the median of $\|\cdot\|_{A_t}$ on S^{n-1}, i.e. $\mathrm{med}(\|\cdot\|_{A_t}) \geq cs\sqrt{(\log n)/n}$.

Now consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $K = TA$ is in John's position (this is the final position/coordinate structure as in the statement of the lemma). By Corollary 3.6, $c_3|x| \leq |Tx| \leq c_4|x|$ for all $x \in \mathbb{R}^n$. Recalling the definition $K_t = \mathrm{conv}\{tB_2^n, K\}$, which implies that $\|x\|_{K_t^\circ} = \max\{\|x\|_{K^\circ}, t|x|\}$, we see that $c_3\|x\|_{A_t^\circ} \leq \|x\|_{K_t^\circ} \leq c_4\|x\|_{A_t^\circ}$ for all $x \in \mathbb{R}^n$, and therefore $c_3\|x\|_{A_t} \leq \|x\|_{K_t} \leq c_4\|x\|_{A_t}$. This then implies that $M_t \geq cs\sqrt{(\log m)/m}$, and clearly $b_t \leq t^{-1}$. Recalling that $s = (1 - 2\log(ct)/\log m)^{1/2}$, these inequalities imply that

$$\frac{M_t}{b_t} \geq ct\biggl(1 - \frac{2\log(ct)}{\log m}\biggr)^{1/2}\biggl(\frac{\log m}{m}\biggr)^{1/2} ct = \biggl(\frac{\log m - 2\log(ct)}{m}\biggr)^{1/2}$$

$$\geq ct\sqrt{\frac{1}{n}\log\biggl(\frac{c'n}{t^2}\biggr)}. \ ∎$$

Proof of Corollary 2.3. The idea of the proof is as follows: make repeated use of Lemma 2.1 (or just as well Corollary 2.2) and Theorem 3.3 and then use the union bound. The probability of a positive outcome for each group is at least $1-Cn^{-c}$, and there are at most $C\log n$ groups. For the sake of clarity, we choose to provide the details. Upper case letters such as C, C', C_1 denote constants that are sufficiently large, while lower case letters such as c, c', c_1 etc. denote sufficiently small constants in $(0,1)$. Without loss of generality we may take $n > n_0$ for some universal constant $n_0 \in \mathbb{N}$. For each $A \subseteq \{1,\dots,n\}$ let $x_A \in \mathbb{R}^n$ be such that $x_i = 1$ if $i \in A$ and $x_i = 0$ if $x \notin A$, and consider the corresponding coordinate subspace $E_A = \{x \in \mathbb{R}^n : i \notin A \Rightarrow x_i = 0\}$. Define

$$\|A\|_0 := \|x_A\|_0, \quad \|A\|_{\mathrm{cyc}} := \|x_A\|_{\mathrm{cyc}}, \quad \|A\|_{\mathrm{Kol}} := \|x_A\|_{\mathrm{Kol}}.$$

For each $1 \le j \le \log_2 \sqrt{c_1 n}$, set

$$\Lambda(1,j) = \{A \subseteq \{1,\dots,n\} : \|A\|_0^{1/2} = \lfloor c2^j \rfloor\},$$
$$\Lambda(2,j) = \left\{A \subseteq \{1,\dots,n\} : \left(\frac{\|A\|_{\mathrm{cyc}} + \log n}{\log(1+n\|A\|_{\mathrm{cyc}}^{-1})}\right)^{1/2} \le c2^j\right\},$$
$$\Lambda(3,j) = \left\{A \subseteq \{1,\dots,n\} : \left(\frac{\|A\|_0 + \|A\|_{\mathrm{Kol}} + \log n}{\log(1+n(\|A\|_0 + \|A\|_{\mathrm{Kol}})^{-1})}\right)^{1/2} \le c2^j\right\}.$$

Now let $U \in O(n)$ be a uniformly distributed random matrix. Consider the events $\{\Psi(i,j) : 1 \le i \le 3, 1 \le j \le \log_2 \sqrt{c_1 n}\}$ where $\Psi(i,j)$ is the event that for all $A \in \Lambda(i,j)$ and all $a \in E_A$, $\frac{1}{2}M_t|a| \le \|Ua\| \le \frac{3}{2}tM_t|a|$, where $t = C2^j$ and M_t is the median of the Minkowski functional of $K_t = \mathrm{conv}\{B_X, tB_2^n\}$ restricted to S^{n-1}, as in Lemma 2.1. Our task now is to bound the probability of these events, and we do so separately for each value of i.

Case 1: $i = 1$ and $1 \le j \le \log_2 \sqrt{c_1 n}$. We now use Lemma 2.1 with $t = C2^j$ and Theorem 3.3 with $\varepsilon = 1/2$, $k = \lfloor c2^j \rfloor^2 < 4c_1 n$,

$$N = |\Lambda(1,j)| = \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

and $(F_i)_{i=1}^N$ any enumeration of $\{E_A : A \in \Lambda(1,j)\}$. This implies that with probability at least

$$1 - C\exp\left(-ct^2 \log\left(\frac{c'n}{t^2}\right)\right) \ge 1 - Cn^{-c}$$

the following event occurs: for all $A \in \Lambda(1,j)$ and all $a \in E_A$, $\frac{1}{2}M_t|a| \le \|Ua\|_{K_t} \le \frac{3}{2}M_t|a|$ and therefore $\frac{1}{2}M_t|a| \le \|Ua\| \le \frac{3}{2}tM_t|a|$. i.e. $\mathbb{P}(\Psi(1,j)) \ge 1 - Cn^{-c}$.

Case 2: $i = 2$ and $1 \le j \le \log_2 \sqrt{c_1 n}$. We now follow the same procedure as in Case 1. Set $t = C2^j$, and consider the function $\psi(s) = s^2 \log(c'n/s^2)$. It follows from the inequality defining $\Lambda(2,j)$ and the fact that ψ is increasing on $C \le s \le c\sqrt{n}$, that ψ evaluated at

$$s = c^{-1}\left(\frac{\|A\|_{\mathrm{cyc}} + \log n}{\log(1+n\|A\|_{\mathrm{cyc}}^{-1})}\right)^{1/2}$$

is bounded above by $\psi(t)$, for any particular $A \in \Lambda(2,j)$. Simplifying the resulting inequality yields $c^{-1}\|A\|_{\mathrm{cyc}} \le t^2\log(c'n/t^2)$. Here one also uses the fact that $\|A\|_{\mathrm{cyc}} < cn$, which follows directly from the definition of $\Lambda(2,j)$. Then define $k = \lceil ct^2\log(c'n/t^2)\rceil$. Note that every $A \in \Lambda(2,j)$ is contained in a set of the form

$$A'_{m,k} = \{(m+i) \bmod n : 0 \le i \le k-1\}.$$

Thus we may take $N = n$ and apply Lemma 2.1 and Theorem 3.3 with $\varepsilon = 1/2$ and $F_i = A'_{i,k}$. Our conclusion is that $\mathbb{P}(\Psi(2,j)) \ge 1 - Cn^{-c}$.

CASE 3: $i = 3$ and $1 \le j \le \log_2\sqrt{c_1 n}$. This is similar to Case 2. Set $t = C2^j$. Repeating the above argument, we see that $\|A\|_0 + \|A\|_{\mathrm{Kol}} \le ct^2\log(c'n/t^2)$. We may then set $k = \lceil ct^2\log(c'n/t^2)\rceil$ and $\varepsilon = 1/2$. Since any binary string b can be written as $b = \phi(p)$ for some other binary string p (here ϕ is the universal partial recursive function involved in the definition of Kolmogorov complexity, see Section 3), the number of strings b with $C_{\mathrm{Kol}}(b) \le k$ is at most $|\{p \in \{0,1\}^* : \ell(p) \le k\}| \le 2^{k+1}$. Therefore $N = |\Lambda(3,j)| \le 2^{k+1}$. It then follows after a third use of Lemma 2.1 and Theorem 3.3 that $\mathbb{P}(\Psi(3,j)) \ge 1 - Cn^{-c}$.

By the union bound

$$P(\Psi) \ge 1 - Cn^{-c}\log(n) \ge 1 - Cn^{-c}$$

where Ψ denotes the intersection of all $\Psi(i,j)$ as $1 \le i \le 3$ and $1 \le j \le \log_2\sqrt{c_1 n}$. This completes the probabilistic argument. For the remainder of the proof we assume that Ψ occurs and show that the conclusion of Theorem 2.3 holds. Consider any $a \in \mathbb{R}^n$, and the associated quantity $D(a)$ as defined by (2.1). If the minimum in (2.1) is attained at $\|a\|_0^{1/2}$ and $\|a\|_0^{1/2} \le c'n$ then there exists $1 \le j \le \log_2\sqrt{c_1 n}$ such that $\|a\|_0^{1/2} \le \lfloor c2^j\rfloor$ and $a \in E_A$ for some $A \in \Lambda(1,j)$. Since $\Psi(1,j) \supseteq \Psi$, we have $\frac{1}{2}M_t|a| \le \|Ua\| \le \frac{3}{2}tM_t|a|$ with $t = 2^j \le C'\|a\|_0^{1/2}$ and the conclusion of the theorem holds. Otherwise, if $\|a\|_0^{1/2} > c'n$ then by John's theorem $n^{-1/2}|a| \le \|Ua\| \le |a|$, and the conclusion still holds. Similar arguments hold when the minimum is attained at either of the other two terms involving $\|a\|_{\mathrm{cyc}}$ and $\|a\|_{\mathrm{Kol}}$. ■

COROLLARY 4.2. *Let X be an infinite-dimensional Banach space over $\mathbb{R}$ with a Schauder basis $(e_i)_{i=1}^\infty$. For any $N \in \mathbb{N}$ and $\varepsilon > 0$, X admits an FDD (finite-dimensional decomposition) $(E_n)_{n=1}^\infty$ where* $\dim(E_n) \ge N$ *and $d_2(E_n) \le (1+\varepsilon)$.*

Proof. For all $n \in \mathbb{N}$ define $U_n = \mathrm{span}\{e_j : (n-1)\exp(cN) < j \leq n\exp(cN)\}$. Then apply Corollary 2.3 followed by Theorem 3.3 to U_n to obtain $U_n = V_1^{(n)} \oplus V_2^{(n)} \oplus \cdots \oplus V_{N(n)}^{(n)}$, where $d_{\mathrm{BM}}(V_i^{(n)}, \ell_2^{k_i}) \leq 1+\varepsilon$, with $k_i = \dim(V_i^{(n)})$. We now claim that

$$X = V_1^{(1)} \oplus V_2^{(1)} \oplus \cdots \oplus V_{N(1)}^{(1)} \oplus V_1^{(2)} \oplus V_2^{(2)} \oplus \cdots \oplus V_{N(2)}^{(2)} \oplus \cdots .$$

The main subtlety here is convergence, however it follows using John's theorem that the norms of the partial sum projections of U_n onto $\bigoplus_{i=1}^{k} V_i^{(n)}$ for $1 \leq k \leq N(n)$ are all bounded above by e^{cN}. ∎

It was shown by Schechtman and Schmuckenschläger [39] that if $K \subset \mathbb{R}^n$ is a convex body in John's position, then for all $t \geq 0$,

$$\mathbb{P}\{\|G\|_K \leq t\} \leq \mathbb{P}\{\|G\|_\infty \leq t\} \tag{4.1}$$

where G is a standard normal random vector in $\mathbb{R}^n$. Using Lemma 2.1 we may recover a very similar type of estimate.

Corollary 4.3. *There exist universal constants $C, c, c_1, c_2 > 0$ such that the following holds. Let $K \subset \mathbb{R}^n$ be a symmetric convex body in John's position. Then for all $1 \leq t \leq c'\sqrt{\log n}$,*

$$\sigma_n\{\theta \in S^{n-1} : \|\theta\|_K \leq t/\sqrt{n}\} < C\exp(-c_1 n \exp(-c_2 t^2))$$

where σ_n is normalized Haar measure on S^{n-1}.

Proof. We first assume that $C' \leq t \leq c\sqrt{\log n}$. Set $s = c\sqrt{n}\exp(-ct^2)$ and consider $K_s = \mathrm{conv}\{K, sB_2^n\}$. By Lemma 2.1,

$$M_s \geq c\sqrt{\frac{1}{n}\log\left(\frac{c'n}{s^2}\right)} = \frac{2t}{\sqrt{n}}$$

and $\mathrm{Lip}(\|\cdot\|_{K_s}) \leq s^{-1}$. By Lévy's inequality,

$$\begin{aligned}\sigma_n\left\{\theta \in S^{n-1} : \frac{1}{2}M_s \leq \|\theta\|_{K_s} \leq \frac{3}{2}M_s\right\} &\geq 1 - C\exp(-cns^2M_s^2)\\ &\geq 1 - C\exp(-cnt^2\exp(-ct^2)),\end{aligned}$$

and the desired bound follows because $\|\theta\|_K \geq \|\theta\|_{K_s}$. If $1 \leq t < C'$, then the result follows by readjusting the constants in the bound. ∎

We refer to [19, 22] and Theorem 3.1 in [10] for related small ball estimates.

For a convex body $K \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(K)$, the parameter $d_u(K)$ is defined for each $u > 1$ as

$$d_u(K) = \min\left\{n, -\log\sigma_n\left\{\theta \in S^{n-1} : \|\theta\|_K \leq \frac{1}{u}M\right\}\right\}$$

where M is the mean of $\|\cdot\|_K$ on S^{n-1}. It was shown by Klartag and Vershynin [19] that the outer inclusion in the randomized Dvoretzky theorem holds with uniformly bounded distortion for all dimensions $1 \le l \le c(u)d_u(K)$. It is known that $d_u(K) \ge ck(K)$, for $u \ge 2$ say, where $k(K) = nM^2$ (here $b = 1$), and that for $1 \le p \le \infty$, $d_{C_p}(B_p^n) \ge c_p n$ where $c_p, C_p > 0$ depend only on p (and can be taken independent of p for $1 \le p \le 2$).

Corollary 4.4. *For all $\varepsilon \in (0, 1/2)$ and all $T > 0$ there exists $u_0 \le CT/\sqrt{\varepsilon}$ such that for all $n \ge \exp(C\varepsilon^{-1})$ and any symmetric convex body $K \subset \mathbb{R}^n$ in John's position with*

$$M(K) \le T\sqrt{\frac{\log n}{n}}$$

we have

$$d_{u_0}(K) \ge cn^{1-\varepsilon}.$$

Proof. Set $t = c\sqrt{\varepsilon \log n}$ and $u_0 = M\sqrt{n}t^{-1} \le CT/\sqrt{\varepsilon}$. The result then follows from Corollary 4.3 or equation (4.1). ∎

5. Almost-isometric theory

5.1. Almost-isometric decompositions in John's position. For the entirety of this subsection, let $(X, \|\cdot\|)$ denote a real normed space of dimension $n \in \mathbb{N}$ that we identify with $\mathbb{R}^n$ so that B_X is in John's position. For any subspace $E \subset X$ let $b(E)$ denote the Lipschitz constant of $\|\cdot\|$ restricted to E, and $M^\sharp(E)$ the median of $\|\cdot\|$ restricted to $S^{n-1} \cap E$. Let $M^\sharp = M^\sharp(X)$. All random objects that we consider (points, subspaces and orthogonal matrices) are distributed according to Haar measure on the appropriate space. Lemmas 5.1 and 5.2, as well as the proof of Theorem 2.5, are taken (with permission) from [44], with minor modifications.

Lemma 5.1. *There exists $c > 0$ such that the following is true. Let $k \ge (M^\sharp)^2 n$ and $F \in G_{n,k}$. Let $U \in O(n)$ be a random orthogonal matrix and let $E = UF$. Then with probability at least $1 - 2^{-k}$, $b(E) \le c\sqrt{k/n}$.*

Proof. Let $\mathcal{N}$ be a 1/2-net in $S^{n-1} \cap F$ with $|\mathcal{N}| \le 6^k$. The result follows by applying Lévy's inequality (3.1) with $t = c_3\sqrt{k/n}$ and applying the series representation (3.6), the union bound, and the triangle inequality. ∎

Lemma 5.2. *There exists a universal constant $c_1 > 0$ such that the following is true. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with the unit ball in John's position, let $\theta \in S^{n-1}$ be a random point and $E \in G_{n,k}$ a random subspace (for some $k < n$). Let $\varepsilon > 0$ be such that*

$$\mathbb{P}\{\big|\|\theta\| - M^\sharp\big| \ge \varepsilon M^\sharp\} \le 1/4.$$

Then

$$\mathbb{P}\{|M^\sharp(E) - M^\sharp| \le \varepsilon M^\sharp\} \ge 1 - 2\exp(-c_1 k).$$

Proof. Let $(v_i)_{i=1}^k$ be i.i.d. random points on S^{n-1}. Let

$$\alpha = \mathbb{P}\{M^\sharp(E) \le (1-\varepsilon)M^\sharp\},$$
$$\mathcal{M} = \{H \in G_{n,k} : M^\sharp(H) \le (1-\varepsilon)M^\sharp\}.$$

By definition of $\mathcal{M}$, for all $1 \le i \le k$ and any $H \in \mathcal{M}$ we have the 'conditional' probability $\mathbb{P}\{\|v_i\| \le (1-\varepsilon)M^\sharp : v_i \in H\} \ge 1/2$, and therefore

$$\mathbb{P}\big\{|\{i : \|v_i\| \le (1-\varepsilon)M^\sharp\}| \ge k/2\big\} \ge \alpha/2.$$

Of course the event $\{v_i \in H\}$ has measure zero and this does not fit into the classical definition of conditional probability. However, it can be justified using a construction involving Fubini's theorem on $O(n) \times O(k)$. On the other hand, from the condition imposed on ε, $\mathbb{P}\{\|v_i\| \le (1-\varepsilon)M^\sharp\} \le 1/4$ and

$$\mathbb{E}|\{i : \|v_i\| \le (1-\varepsilon)M^\sharp\}| \le k/4.$$

By Hoeffding's inequality,

$$\mathbb{P}\big\{|\{i : \|v_i\| \le (1-\varepsilon)M^\sharp\}| \ge k/2\big\} \le \exp(-c_1 k).$$

The bound on $\mathbb{P}\{M^\sharp(E) \ge (1+\varepsilon)M^\sharp\}$ follows similar lines. ■

Proof of Theorem 2.5. If $M^\sharp \ge c_1\varepsilon^{-1}((\log n)/n)^{1/2}$, the statement follows from Theorem 3.3 and we may assume without loss of generality that $M^\sharp < c_1\varepsilon^{-1}((\log n)/n)^{1/2}$. Let

$$N = \lfloor n^{1-\varepsilon^2(\log\varepsilon^{-1})^{-2}} \rfloor$$

and let $H_1 \oplus \cdots \oplus H_N$ be a decomposition of $\mathbb{R}^n$ into mutually orthogonal subspaces of dimension either k or $k+1$, with $k \approx n^{\varepsilon^2(\log\varepsilon^{-1})^{-2}}$, and let $U \in O(n)$ be a random orthogonal matrix. From Lemmas 5.1 and 5.2, it follows that with high probability, for all i, $M^\sharp(UH_i)/b(UH_i) \ge c_4\varepsilon^{-1}(\log\varepsilon^{-1})\sqrt{(\log k)/k}$. It now follows by Theorem 3.3 that each UH_i can be decomposed yet again into approximately Euclidean subspaces of dimension $c\varepsilon^2(\log\varepsilon^{-1})^{-1}\log n$. ■

5.2. Grid structures

LEMMA 5.3. *The space c_0 does not satisfy Definition* 2.6.

Proof. Assume for the sake of a contradiction that c_0 satisfies Definition 2.6. Let $n \ge 3$, $k = 2$ and consider any $0 < \varepsilon < (\sqrt{2}-1)^4$. This ensures that both of the following inequalities hold:

$$(1+\varepsilon+2\sqrt{\varepsilon}) < \sqrt{2}(1-\varepsilon), \tag{5.1}$$

$$(1+\varepsilon)^2 < 2(1-\varepsilon)^2. \tag{5.2}$$

Let $(f_i)_{i=1}^n$ be a basis for ℓ_∞^n such that for any $a \in \mathbb{R}^n$ with $\|a\|_0 \le 2$, (2.2) holds. Let T be the $n \times n$ matrix with the vectors $(f_i)_{i=1}^n$ as columns. For all 2-sparse vectors $x \in \mathbb{R}^n$, (2.2) can be written as

$$(1-\varepsilon)|x| \le \|Tx\|_\infty \le (1+\varepsilon)|x|. \tag{5.3}$$

It follows by setting $x = e_j$ (the jth standard basis vector of $\mathbb{R}^n$) that for all $1 \le j \le n$,

$$1 - \varepsilon \le \max_{1\le i\le n} |T_{i,j}| \le 1 + \varepsilon.$$

In particular, there exists $\nu = \nu(j)$ such that $|T_{\nu,j}| \ge 1 - \varepsilon$. Since this holds for each column, there are at least n entries of the matrix such that $|T_{i,j}| \ge 1-\varepsilon$. By duality, for all $1 \le i \le n$ and any $j, l \in \{1,\dots,n\}$,

$$T_{i,j}^2 + T_{i,l}^2 \le (1+\varepsilon)^2.$$

By (5.2), there can be at most one such entry per row, and we conclude that there is exactly one in every row. Hence $\nu \in S_n$ is a permutation of the set $\{1,\dots,n\}$. For all $1 \le i \le n$, if $j \ne \nu^{-1}(i)$ then $T_{i,j}^2 = T_{i,j}^2 + T_{i,\nu^{-1}(i)}^2 - T_{i,\nu^{-1}(i)}^2 \le (1+\varepsilon)^2 - (1-\varepsilon)^2 = 4\varepsilon$ and $|T_{i,j}| \le 2\sqrt{\varepsilon}$. The matrix T is therefore a small perturbation of a permutation matrix. If we now take $x = e_1 + e_2$, then $|x| = \sqrt{2}$ and for all $1 \le i \le n$,

$$|T_{i,1} + T_{i,2}| \le 1 + \varepsilon + 2\sqrt{\varepsilon} < \sqrt{2}(1-\varepsilon)$$

by (5.1), which implies that $\|Tx\|_\infty < \sqrt{2}(1-\varepsilon)$. This contradicts (5.3), and the result follows. ■

Proof of Theorem 2.7. If X fails to have nontrivial cotype, then it follows from the Maurey–Pisier–Krivine theorem that ℓ_∞ (equivalently c_0) is finitely representable in X, and therefore X fails to satisfy Definition 2.6 by Lemma 5.3. On the other hand, if X has cotype q for some $q < \infty$, and cotype constant $\beta \in (0,1]$, then it follows by Theorem 3.3 that X satisfies Definition 2.6. Here we also use the fact that $\binom{n}{k} \le (en/k)^k$, and the bound $M \ge c\beta n^{1/q-1/2}$ from (3.7). The corresponding probability bound is positive if $k \le c\beta^2\varepsilon^2(\log(en/k))^{-1}n^{2/q}$. In the case $q = 2$, this bound can be polished. A simple calculation shows that if $0 < s < 1/3$ and $T \ge \max\{e, 3s^{-1}\log s^{-1}\}$, then $T^{-1}\log T \le s$. Applying this with $T = n/k$ we see that $k \le c\beta^2\varepsilon^2(\log\beta^{-1} + \log\varepsilon^{-1})^{-1}n$ is sufficient. ■

6. Remaining proofs. We state the following result for completeness and because we could not find exactly what we wanted in the literature. See [2] for a similar statement that applies to a wider class of random matrices, but has slightly weaker dependence on ε.

LEMMA 6.1. *Consider any $m, n, k \in \mathbb{N}$ and $0 < \varepsilon < 0.99$ such that $k \le c\varepsilon^2(\log(en/k))^{-1}m$. Let U be a random $m \times n$ matrix with i.i.d. $N(0,1)$ entries. With probability at least $1 - c_1\exp(-c_2 m\varepsilon^2)$, the inequality*

$$(1-\varepsilon)|x| \le m^{-1/2}|Ux| \le (1+\varepsilon)|x|$$

holds simultaneously for all k-sparse vectors $x \in \mathbb{R}^n$.

Proof. Let $E \in G_{n,k}$ be any k-dimensional coordinate subspace of $\mathbb{R}^n$. By Schechtman's version of the general Dvoretzky theorem ([36, Theorem 2], or the proof of Theorem 3.3 here), the required bound holds for all $x \in E$ with probability at least $1 - c_1 \exp(-c_2 m\varepsilon^2)$. There are at most $\binom{n}{k} \leq (en/k)^k$ such coordinate subspaces, and the result follows from the union bound. ∎

Proof of Proposition 2.8. We know from Theorem 2.7 that there exists a basis in X, say $(e_i)_{i=1}^n$ such that for all k-sparse vectors $a \in \mathbb{R}^n$,

$$(1 - \varepsilon/6)|a| \leq \Big\| \sum_{i=1}^n a_i e_i \Big\|_X \leq (1 + \varepsilon/6)|a|$$

Identify X with $\mathbb{R}^n$ using this basis. Identify Y with $\mathbb{R}^m$ in such a way that B_Y is in John's position, and then readjust the coordinate structure by scalar multiplication so that the mean of $\|\cdot\|_Y$ in S^{m-1} obeys $M(\|\cdot\|_Y) = 1$. Let G be a random $m \times n$ matrix with i.i.d. $N(0,1)$ entries. For each k-dimensional coordinate subspace $E \subset X$, GE has dimension k with probability 1 and is uniformly distributed in $G_{m,k}$. Therefore, using Theorem 3.3 and (3.7), with probability at least $1 - c_1 \exp(-c\beta_2^2 m^{2/q_2} \varepsilon^2)$, for all $y \in GE$, $(1 - \varepsilon/6)|y| \leq \|y\|_Y \leq (1 + \varepsilon/6)|y|$. By applying the union bound, with probability at least $1 - c_1 n^k \exp(-c\beta_2^2 m^{2/q_2} \varepsilon^2)$, for all k-sparse vectors $x \in X$ we have $(1 - \varepsilon/6)|Gx| \leq \|Gx\|_Y \leq (1 + \varepsilon/6)|Gx|$. The result now follows from Lemma 6.1 and the inequality $1-\varepsilon \leq (1-\varepsilon/6)(1+\varepsilon/6)^{-2} \leq (1-\varepsilon/6)^{-2}(1+\varepsilon/6) \leq 1 + \varepsilon$. ∎

Proof of Proposition 2.9. The difference between any two k-sparse vectors is $2k$-sparse. The result now follows from Theorem 2.7 and the comment following the statement of the theorem (by readjusting the constants involved) as well as the usual form of the Johnson–Lindenstrauss lemma. ∎

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References

[1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. 233, Springer, New York, 2006.
[2] R. Baraniuk, M. Davenport, R. DeVore and M. Wakin, *A simple proof of the restricted isometry property for random matrices*, Constr. Approx. 28 (2008), 253–263.
[3] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, Amer. Math. Soc. Colloq. Publ. 48, Amer. Math. Soc., Providence, RI, 2000.

[4] J. Bourgain and S. Szarek, *The Banach–Mazur distance to the cube and the Dvoretzky–Rogers factorization*, Israel J. Math. 62 (1988), 169–180.

[5] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory 7 (1974), 294–299.

[6] E. J. Candès and T. Tao, *Decoding by linear programming*, IEEE Trans. Inform. Theory 51 (2005), 4203–4215.

[7] G. Chasapis and A. Giannopoulos, *Euclidean regularization in John's position*, preprint.

[8] R. Dudley, *Uniform Central Limit Theorems*, Cambridge Stud. Adv. Math. 63, Cambridge Univ. Press, 1999.

[9] T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. 139 (1977), 53–94.

[10] O. Friedland and O. Guédon, *Random embedding of ℓ_p^n into ℓ_r^N*, Math. Ann. 350 (2011), 953–972.

[11] A. Giannopoulos, *A proportional Dvoretzky–Rogers factorization result*, Proc. Amer. Math. Soc. 124 (1996), 233–241.

[12] A. Giannopoulos, V. D. Milman and A. Tsolomitis, *Asymptotic formulas for the diameter of sections of symmetric convex bodies*, J. Funct. Anal. 223 (2005), 86–108.

[13] Y. Gordon, *Some inequalities for Gaussian processes and applications*, Israel J. Math. 50 (1985), 265–289.

[14] Y. Gordon, O. Guédon and M. Meyer, *An isomorphic Dvoretzky's theorem for convex bodies*, Studia Math. 127 (1998), 191–200.

[15] O. Guédon, *Gaussian version of a theorem of Milman and Schechtman*, Positivity 1 (1997), 1–5.

[16] W. B. Johnson and J. Lindenstrauss, *Extensions of Lipschitz mappings into Hilbert space*, in: Conference in Modern Analysis and Probability (New Haven, CT, 1982), Contemp. Math. 26, Amer. Math. Soc., Providence, RI, 1984, 189–206.

[17] W. B. Johnson and A. Naor, *The Johnson–Lindenstrauss lemma almost characterizes Hilbert space, but not quite*, Discrete Comput. Geom. 43 (2010), 542–553.

[18] B. Kašin, *The widths of certain finite-dimensional sets and classes of smooth functions*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), 334–351 (in Russian).

[19] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky's theorem*, Israel J. Math. 157 (2007), 193–207.

[20] J. L. Krivine, *Sous-espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. (2) 104 (1976), 1–29.

[21] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. 44 (1972), 583–595.

[22] R. Latała and K. Oleszkiewicz, *Small ball probability estimates in terms of width*, Studia Math. 169 (2005), 305–314.

[23] M. Li and P. Vitányi, *An Introduction to Kolmogorov Complexity and its Applications*, Texts Monogr. Computer Sci., Springer, 1993.

[24] A. E. Litvak, P. Mankiewicz and N. Tomczak-Jaegermann, *Randomized isomorphic Dvoretzky theorem*, C. R. Math. Acad. Sci. Paris 335 (2002), 345–350.

[25] A. E. Litvak and N. Tomczak-Jaegermann, *Random aspects of high dimensional convex bodies*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1745, Springer, Berlin, 2000, 169–190.

[26] J. Matoušek, *Lectures on Discrete Geometry*, Grad. Texts in Math. 212, Springer, New York, 2002, 169–190.

[27] B. Maurey, *Type, cotype and K-convexity*, in: Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, 1299–1332.

[28] B. Maurey et G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.
[29] V. D. Milman, *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies*, Funktsional. Anal. i Prilozhen. 5 (1971), no. 4, 28–37 (in Russian); English transl.: Functional Anal. Appl. 5 (1971), 288–295.
[30] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces*, Lecture Notes in Math. 1200, Springer, Berlin, 1986.
[31] V. D. Milman and G. Schechtman, *An "isomorphic" version of Dvoretzky's theorem*, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), 541–544.
[32] V. D. Milman and G. Schechtman, *An "isomorphic" version of Dvoretzky's theorem II*, in: Convex Geometric Analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ. 34, Cambridge Univ. Press, Cambridge, 1999, 159–164.
[33] V. D. Milman and M. Sharir, *A new proof of the Maurey–Pisier theorem*, Israel J. Math. 33 (1979), 73–87.
[34] G. Pisier, *Probabilistic methods in the geometry of Banach spaces*, in: Probability and Analysis (Varenna, 1985), Lecture Notes in Math. 1206, Springer, Berlin, 1986, 167–241.
[35] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge Tracts in Math. 94, Cambridge Univ. Press, 1989.
[36] G. Schechtman, *A remark concerning the dependence on ε in Dvoretzky's theorem*, in: Geometric Aspects of Functional Analysis (1987–88), Lecture Notes in Math. 1376, Springer, Berlin, 1989, 274–277.
[37] G. Schechtman, *Two observations regarding embedding subsets of Euclidean space in normed spaces*, Adv. Math. 200 (2006), 125–135.
[38] G. Schechtman, *Euclidean sections of convex bodies*, in: Asymptotic Geometric Analysis, Fields Inst. Comm. 68, Springer, New York, 2013, 271–288.
[39] G. Schechtman and M. Schmuckenschläger, *A concentration inequality for harmonic measures on the sphere*, in: Geometric Aspects of Functional Analysis (1992–1994), Oper. Theory Adv. Appl. 77, Birkhäuser, Basel, 1995, 255–273.
[40] S. Szarek, *On Kashin's almost Euclidean orthogonal decomposition of l_n^1*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), 691–694.
[41] S. Szarek, *Spaces with large distance to ℓ_∞^n and random matrices*, Amer. J. Math. 112 (1990), 899–942.
[42] S. Szarek and M. Talagrand, *An "isomorphic" version of the Sauer–Shelah lemma and the Banach–Mazur distance to the cube*, in: Geometric Aspects of Functional Analysis (1987–88), Lecture Notes in Math. 1376, Springer, Berlin, 1989, 105–112.
[43] S. Szarek and N. Tomczak-Jaegermann, *On nearly Euclidean decompositions for some classes of Banach spaces*, Compos. Math. 40 (1980), 367–385.
[44] K. Tikhomirov, private communication.
[45] R. Vershynin, *John's decompositions: selecting a large part*, Israel J. Math. 122 (2001), 253–277.

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