STUDIA MATHEMATICA 227 (1) (2015)

The Kadec–Pełczyński–Rosenthal subsequence splitting lemma for JBW*-triple preduals

by

Antonio M. Peralta (Granada and Riyadh)
and Hermann Pfitzner (Orléans)

Abstract. Any bounded sequence in an L^1-space admits a subsequence which can be written as the sum of a sequence of pairwise disjoint elements and a sequence which forms a uniformly integrable or equiintegrable (equivalently, a relatively weakly compact) set. This is known as the Kadec–Pełczyński–Rosenthal subsequence splitting lemma and has been generalized to preduals of von Neuman algebras and of JBW*-algebras. In this note we generalize it to JBW*-triple preduals.

1. Introduction. Up to a subsequence any bounded sequence in an L^1-space splits into (i.e. can be written as) the sum of two sequences of opposite nature: one which is pairwise disjointly supported, and another one which converges weakly or, equivalently, is uniformly integrable. The paper of Kadec–Pełczyński [33] contains a forerunner of this subsequence splitting lemma, its explicit formulation appears in [7, p. 68] (with a reference to Rosenthal's [47]), whereas the authors of [2, p. 250] call it folklore and refer to [12]. Note in passing that the splitting lemma also holds for L^p-spaces with $0 < p < \infty$ [43, 45], but in this note we concentrate on $p = 1$. For some generalizations and applications see [51, 43, 25, 14, 44, 45, 32].

In this note we generalize the splitting lemma to preduals of JBW*-triples as stated in our main result (Thm. 6.1). The main result gives a positive answer to [41, Question 3] and a proof to [21, Conjecture 4.4]. On the way from the classical result for L^1-spaces to JBW*-triple preduals we find the following stages: Randrianantoanina [43] has shown the splitting lemma for von Neumann preduals. In [21], Fernández-Polo, Ramírez and the first author of this note adopted Randrianantoanina's approach in order to prove the splitting lemma for preduals of JBW*-algebras. In the present

2010 *Mathematics Subject Classification*: Primary 17C65, 46L70, 46B08; Secondary 46B04, 46L51.
Key words and phrases: Kadec–Pełczyński–Rosenthal subsequence splitting lemma, JBW*-triples, weak compactness, uniform integrability, L-embedded Banach spaces.

DOI: 10.4064/sm227-1-5

note we follow Raynaud and Xu [45] who, shortly after Randrianantoanina, recovered his result by means of ultraproduct techniques.

Although this note is intended to be self-contained, it can be considered, in some sense, a continuation of [39] in that it uses a main result of [39] (see the proof of Thm. 6.1) which allows us to obtain a disjointly supported sequence from one being only almost isometric to ℓ_1.

As it is possible to define a topology on arbitrary L-embedded Banach spaces X which on bounded sets equals the usual measure topology when X is $L^1[0,1]$ [41], it makes sense to conjecture a splitting lemma for L-embedded spaces; see [41, §6] for a precise wording. However, examples show that in general, L-embedded spaces fail such a splitting lemma [41, Ex. 6.2]. So JBW*-triple preduals seem to be the biggest class of L-embedded Banach spaces known to admit a splitting property for bounded sequences. It remains an open problem to find (reasonable) conditions on L-embedded spaces to ensure the possibility of splitting.

2. Notation. Basic notions and properties not explained here (or alluded to too succinctly) can be found for Banach spaces in [13, 20, 31] and for JBW*-triples in [11, 10], but also in the introductory sections of [39]. Throughout this article we will use the following notation. The unit ball of a Banach space X is written B_X, and the dual X^*. Given an ultrafilter $\mathcal{U}$ on an index set I, and a family $(X_i)_{i\in I}$ of Banach spaces, we denote by $(X_i)_{\mathcal{U}}$ the corresponding ultraproduct of the X_i, and if $X_i = X$ for all i, we write $(X)_{\mathcal{U}}$ (or simply $X_{\mathcal{U}}$) for the ultrapower of X. We refer to [28] for basic facts and definitions concerning ultraproducts. Elements of $(X_i)_{\mathcal{U}}$ are written $\widetilde{x} = [x_i]_{\mathcal{U}}$, in which case (x_i) is called a *representing family* or a *representative* of $\widetilde{x}$. We have $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_i\|$ independently of the representative. We recall that there is a canonical isometric embedding $\widehat{\ } : X \hookrightarrow (X)_{\mathcal{U}}$, $x \mapsto [x]_{\mathcal{U}}$, and shall write $\widehat{X}$ and $\widehat{x}$ for the image of X and x, respectively, under this embedding. A normalized sequence (x_k) in a Banach space is said to *span* ℓ_1 *asymptotically* if there exists a sequence (δ_n) such that $0 \le \delta_n \to 0$ and

$$\sum_{k\ge 1} |\alpha_k| \ge \Big\|\sum_{k\ge 1} \alpha_k x_k\Big\| \ge \sum_{k\ge 1} (1-\delta_k)|\alpha_k|, \qquad \forall \alpha_k \in \mathbb{C}.$$

Moreover, throughout this article, W will denote a JBW*-triple with predual W_* and triple product $\{\cdot,\cdot,\cdot\}$. The Peirce projections associated with a tripotent e are denoted by $P_k(e) : W \to W$, $k = 0,1,2$, the ranges of $P_k(e)$ by $W_k(e)$, whence we have the Peirce decomposition $W = W_2(e) \oplus W_1(e) \oplus W_0(e)$ [11, p. 32]. The *Peirce rules* are

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = \{0\}$$

and

$$\{E_i(u), E_j(u), E_k(u)\} \subseteq E_{i-j+k}(u),$$

where $E_{i-j+k}(u) = \{0\}$ whenever $i - j + k \notin \{0, 1, 2\}$ ([22] or [11, Thm. 1.2.44]).

When $X = W_*$ is the predual of the JBW*-triple W, the conventions explained above hold accordingly, for example we write $\widetilde{\phi} = [\phi_i]_{\mathcal{U}} \in (W_*)_{\mathcal{U}}$ and $\widehat{W_*} \subset (W_*)_{\mathcal{U}}$.

The orthogonality of two elements $a, b \in W$ is written $a \perp b$, which by definition means $\{a, b, W\} = 0$ (see [9, Lem. 1] for equivalent characterizations).

Two elements $\varphi, \psi \in W_*$ are called *orthogonal*, in symbols $\varphi \perp \psi$, if $s(\varphi) \perp s(\psi)$ where $s(\varphi)$ is the support tripotent of φ, uniquely determined by the fact that $\varphi|_{W_2(s(\varphi))}$ is a faithful normal positive functional on the JBW*-algebra $W_2(s(\varphi))$ such that $\varphi = \varphi P_2(s(\varphi))$ [22, Prop. 2]. For any tripotent $e \in W$ such that $\varphi(e) = \|\varphi\|$, in particular for $s(\varphi)$, we have $\varphi = \varphi P_2(e)$ [22, Prop. 1]. Recall that $\varphi \perp \psi$ if and only if they are *L-orthogonal*, that is, $\|\alpha\varphi + \beta\psi\| = |\alpha|\|\varphi\| + |\beta|\|\psi\|$ for all scalars α, β (cf. [24, 19]; see [39] for quantified versions).

According to the notation in [45], we shall say that a functional $\widetilde{\varphi} = [\varphi_i]_{\mathcal{U}} \in (W_*)_{\mathcal{U}}$ is *disjoint from* $W_* \equiv \widehat{W_*}$ whenever $\widetilde{\varphi} \perp \widehat{\phi}$ for every $\phi \in W_*$. We recall the *Jordan identity*

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}, \tag{2.1}$$

which by definition of triple systems is valid for all a, b, x, y, z in a JB*-triple E. We also recall from [23, Cor. 3] that

$$\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|. \tag{2.2}$$

It follows from the so-called *Gelfand–Naimark axiom* for JB*-triples ($\|\{a, a, a\}\| = \|a\|^3$ for all $a \in E$) that the quadratic operator $Q(a) : E \to E$, $x \mapsto \{a, x, a\}$, has norm $\|a\|^2$. We finally recall that $P_2(e) = Q(e)^2$ for every tripotent $e \in E$ [11, p. 32].

3. Preliminary results

3.1. Banach spaces. The following way of constructing asymptotically isometric ℓ_1-copies is reminiscent of a construction of Godefroy ([27, IV.2.5] or [42, Thm. 2]).

LEMMA 3.1. *Let X be a Banach space, and let $\mathcal{U}$ be an ultrafilter on an index set I. We denote $\widetilde{X} = (X)_{\mathcal{U}}$ and write $\widehat{X}$ for the image of X under the canonical embedding* $\widehat{\ } : X \hookrightarrow (X)_{\mathcal{U}}$, $\widehat{x} = [x]_{\mathcal{U}}$. *Suppose that a bounded* *family (x_i) in X is such that $[x_i]_{\mathcal{U}}$ is non-zero and is L-orthogonal to $\widehat{X}$ in the sense that*

$$\|\widehat{y} + [x_i]_{\mathcal{U}}\| = \|\widehat{y}\| + \|[x_i]_{\mathcal{U}}\|, \qquad \forall y \in X. \tag{3.1}$$

Then there is a sequence $(x_{i_n})_{n\in\mathbb{N}}$ such that $(x_{i_k}/\|x_{i_k}\|)$ spans ℓ_1 asymptotically.

Proof. By hypothesis we have

$$\lim_{\mathcal{U}} \|y + \alpha x_i\| = \|\widehat{y}\| + |\alpha|\, \|[x_i]_{\mathcal{U}}\|, \qquad \forall \alpha \in \mathbb{C},\ y \in X. \tag{3.2}$$

Let (δ_n) be a sequence of strictly positive numbers converging to 0. Set $\eta_1 = \frac{1}{3}\delta_1$ and $\eta_{n+1} = \frac{1}{3}\min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction on $n \in \mathbb{N}$ we will construct $i_n \in I$ such that

$$\sum_{k=1}^{n} (1-\delta_k)|\alpha_k| + \eta_n \sum_{k=1}^{n} |\alpha_k| \le \Big\| \sum_{k=1}^{n} \alpha_k \frac{x_{i_k}}{\|x_{i_k}\|} \Big\| \tag{3.3}$$

for all $n \in \mathbb{N}$ and $\alpha_k \in \mathbb{C}$.

Suppose without loss of generality that all x_i are of norm one. For the first induction step we choose any $i_1 \in I$. For the induction step $n \mapsto n+1$ we suppose that $x_{i_1}, \dots, x_{i_n}$ are constructed so that (3.3) holds. Fix $\alpha = (\alpha_k)_{k=1}^{n+1}$ in the unit sphere of ℓ_1^{n+1} such that $\alpha_{n+1} \neq 0$. Then (3.2) yields

$$\begin{aligned}
\lim_{\mathcal{U}} \Big\| \sum_{k=1}^{n} \alpha_k x_{i_k} + \alpha_{n+1} x_i \Big\| &= \Big\| \sum_{k=1}^{n} \alpha_k \widehat{x}_{i_k} \Big\| + |\alpha_{n+1}|\, \|[x_i]_{\mathcal{U}}\| \\
&\overset{(3.3)}{\ge} \sum_{k=1}^{n} (1-\delta_k)|\alpha_k| + \eta_n \sum_{k=1}^{n} |\alpha_k| + |\alpha_{n+1}| \\
&= \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k| + \eta_n \sum_{k=1}^{n+1} |\alpha_k| - (\eta_n - \delta_{n+1})|\alpha_{n+1}| \\
&\ge \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k| + \min(\eta_n, \delta_{n+1}) > \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k| + 2\eta_{n+1},
\end{aligned}$$

because $\|\alpha\| = 1$ and $|\alpha_{n+1}| \le 1$. Thus, there exists $U \in \mathcal{U}$ such that

$$\Big\| \sum_{k=1}^{n} \alpha_k x_{i_k} + \alpha_{n+1} x_i \Big\| > \sum_{k=1}^{n} (1-\delta_k)|\alpha_k| + 2\eta_{n+1}, \qquad \forall i \in U.$$

Choose a finite $\eta_{n+1}/2$-net $(\alpha^l)_{l=1}^{L_{n+1}}$, with $\alpha_{n+1}^l \neq 0$ for $l \le L$, in the unit sphere of ℓ_1^{n+1} in the sense that for each α in that unit sphere there is $l \leq L_{n+1}$ such that $\|\alpha - \alpha^l\| = \sum_{k=1}^{n+1} |\alpha_k - \alpha_k^l| < \eta_{n+1}/2$. Then we may repeat the reasoning above finitely many times for $l = 1, \ldots, L_{n+1}$ to get $x_{i_{n+1}}$ such that

$$\Big\|\sum_{k=1}^{n+1} \alpha_k^l x_{i_k}\Big\| > \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k^l| + 2\eta_{n+1}, \qquad \forall l \leq L_{n+1}.$$

For each α in the unit sphere of ℓ_1^{n+1} choose $l \leq L_{n+1}$ with $\|\alpha - \alpha^l\| < \eta_{n+1}$. Then

$$\begin{aligned}\Big\|\sum_{k=1}^{n+1} \alpha_k x_{i_k}\Big\| &\geq \Big\|\sum_{k=1}^{n+1} \alpha_k^l x_{i_k}\Big\| - \Big\|\sum_{k=1}^{n+1} (\alpha_k - \alpha_k^l) x_{i_k}\Big\| \\ &\geq \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k^l| + 2\eta_{n+1} - \|\alpha - \alpha^l\| \\ &\geq \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k| - \frac{\eta_{n+1}}{2} + 2\eta_{n+1} - \frac{\eta_{n+1}}{2} \\ &= \sum_{k=1}^{n+1} (1-\delta_k)|\alpha_k| + \eta_{n+1} \sum_{k=1}^{n+1} |\alpha_k|.\end{aligned}$$

This extends to all $\alpha \in \ell_1^{n+1}$, and thus ends the induction and the proof. ∎

An ultrafilter $\mathcal{U}$ on a set I is called *countably incomplete* if it contains a sequence (U_n) such that $\bigcap_n U_n = \emptyset$. Ultrafilters on $\mathbb{N}$ are countably incomplete. The following lemma is essentially contained in [45, end of the proof of Thm. 4.6]. For the sake of completeness we give a detailed proof.

LEMMA 3.2. *Let $\mathcal{U}$ be a countably incomplete ultrafilter on a set I, and let X be a Banach space. Consider a sequence $(\widetilde{x}^{(n)})$ and an element $\widetilde{x}$ in the ultrapower $X_{\mathcal{U}}$ such that $\|\widetilde{x}^{(n)} - \widetilde{x}\| \to 0$ and for each $n \in \mathbb{N}$, $\widetilde{x}^{(n)}$ admits a representative $\widetilde{x}^{(n)} = [x_i^{(n)}]_{\mathcal{U}}$ with $\{x_i^{(n)} : i \in I\}$ relatively weakly compact in X. Then $\widetilde{x}$ also admits a representative $\widetilde{x} = [x_i]_{\mathcal{U}}$ with $\{x_i : i \in I\}$ relatively weakly compact in X.*

Proof. We use the notation of the hypothesis and may further assume that $\|\widetilde{x}^{(n)} - \widetilde{x}\| < 1/n$. Let $\widetilde{x} = [x'_i]_{\mathcal{U}}$. Let (U_n) in $\mathcal{U}$ be such that $\bigcap_n U_n = \emptyset$. We may further assume that $U_1 \supset U_2 \supset \cdots$ and $\|x_i^{(n)} - x'_i\| < 1/n$ for all $i \in U_n$.

Set $x_i = 0$ for $i \notin U_1$ and $x_i = x_i^{(n_i)}$ for $i \in U_1$, where n_i is defined by $i \in U_{n_i} \setminus U_{n_i+1}$. By construction, $\|x_i - x'_i\| < 1/n$ for $i \in U_n$. Hence $[x_i]_{\mathcal{U}} = [x'_i]_{\mathcal{U}} = \widetilde{x}$.

Fix $n \geq 1$ and $i \in U_1$. If $n > n_i$ then

$$\min_{j \leq n} \|x_i - x_i^{(j)}\| \leq \|x_i - x_i^{(n_i)}\| = 0,$$

and if $n \leq n_i$ then

$$\|x_i - x_i^{(n)}\| = \|x_i^{(n_i)} - x_i^{(n)}\| \leq \|x_i^{(n_i)} - x_i'\| + \|x_i' - x_i^{(n)}\| < \frac{1}{n_i} + \frac{1}{n} \leq \frac{2}{n}.$$

From both cases we see that $\min_{j \leq n} \|x_i - x_i^{(j)}\| < 2/n$ for all $i \in U_1$ and $n \geq 1$. This means that given n there is a family (y_i) in the relatively weakly compact union $\{0\} \cup \bigcup_{j=1}^{n} \{x_i^{(j)} : i \in I\}$ which is at most $2/n$ away from (x_i).

Now let $x^{**} \in X^{**}$ be a weak*-limit of the x_i along an ultrafilter $\mathcal{V}$ on I. Denote by α the distance from x^{**} to X and suppose $\alpha > 0$. Take a natural number n such that $2/n < \alpha/2$, $\|y_i - x_i\| \leq 2/n$ for every i, and set $y = \text{weak-lim}_{\mathcal{V}} y_i$. Let $x^* \in B_{X^*}$ be such that $|(x^{**}-y)(x^*)| > \|x^{**}-y\| - \alpha/2$. The contradiction

$$\alpha \leq \|x^{**} - y\| < \lim_{\mathcal{V}} |x^*(x_i - y_i)| + \frac{\alpha}{2} \leq \lim_{\mathcal{V}} \|x_i - y_i\| + \frac{\alpha}{2} \leq \frac{2}{n} + \frac{\alpha}{2} < \alpha$$

shows that $\alpha = 0$. Hence $x^{**} \in X$, and $\{x_i : i \in I\}$ is relatively weakly compact in X. ∎

3.2. JBW*-triples. Using the first half of [28, Cor. 7.6], Becerra and Martín [6, Prop. 5.5] have shown the stability of the class of JBW*-triple preduals under ultraproducts. By using also the second half, the following improvement can be obtained.

THEOREM 3.3. *Let $(W_i)_{i \in I}$ be a family of JBW*-triples, $\mathcal{U}$ an ultrafilter on I, and let $\mathcal{W} = X^*$, where $X = ((W_i)_*)_{\mathcal{U}}$. Then $\mathcal{W}$ is a JBW*-triple and the canonical embedding $\mathcal{J} : (W_i)_{\mathcal{U}} \to \mathcal{W}$ (defined by $\mathcal{J}([x_i]_{\mathcal{U}})([\varphi_i]_{\mathcal{U}}) = \lim_{\mathcal{U}} \varphi_i(x_i)$) is an isometric triple homomorphism with weak*-dense image.*

Proof. Let $E = (W_i)_{\mathcal{U}}$. As a consequence of [28, Cor. 7.6], there are an ultrafilter $\mathcal{B}$ on an index set I', a contractive projection P on $(E)_{\mathcal{B}}$, and a surjective linear isometry $T : \mathcal{W} \to V$ where $V = P((E)_{\mathcal{B}})$.

Let $\widetilde{E} = (E)_{\mathcal{B}}$ and let $j_E : E \to \widetilde{E}$ be the canonical embedding of E into its ultrapower. Still according to [28, Cor. 7.6], the restriction of T to $\mathcal{J}(E)$ is E's canonical embedding into $(E)_{\mathcal{B}}$, that is, $T(\mathcal{J}(x)) = j_E(x)$ for all $x \in E$. In particular, P acts as the identity on $j_E(E)$. By the contractive projection theorem (cf. [50], [35], [11, Thm. 3.3.1]), $V = P(\widetilde{E})$ is a JB*-triple via $\{P(a), P(b), P(c)\}_V = P(\{a,b,c\}_{\widetilde{E}})$. Since T is a surjective linear isometry, the product $\{a,b,c\}_{\mathcal{W}} = T^{-1}\{T(a), T(b), T(c)\}_V$ defines a JB*-triple structure on $\mathcal{W}$. The mapping $T : (\mathcal{W}, \{\cdot,\cdot,\cdot\}_{\mathcal{W}}) \to (V, \{\cdot,\cdot,\cdot\}_V)$ is a triple isomorphism by construction.

Let $x, y, z \in E$. Then

$$\begin{aligned}\{\mathcal{J}(x), \mathcal{J}(y), \mathcal{J}(z)\}_{\mathcal{W}} &= T^{-1}(\{T\mathcal{J}(x), T\mathcal{J}(y), T\mathcal{J}(z)\}_V) \\ &= T^{-1}(P\{T\mathcal{J}(x), T\mathcal{J}(y), T\mathcal{J}(z)\}_{\widetilde{E}}) \\ &= T^{-1}(P\{j_E(x), j_E(y), j_E(z)\}_{\widetilde{E}}) \\ &= T^{-1}(P\{[x]_{\mathcal{B}}, [y]_{\mathcal{B}}, [z]_{\mathcal{B}}\}_{\widetilde{E}}) \\ &= T^{-1}(P[\{x, y, z\}_E]_{\mathcal{B}}) = T^{-1}(Pj_E\{x, y, z\}_E) \\ &= T^{-1}(j_E(\{x, y, z\}_E)) = \mathcal{J}(\{x, y, z\}_E),\end{aligned}$$

which shows that $\mathcal{J}$ preserves the triple product. By [28, Prop. 7.3] (or [49, Sec. 11, Cor. p. 78]), the image of $\mathcal{J}$ is weak*-dense in $\mathcal{W}$. ∎

We isolate here a technical result which will be needed later.

LEMMA 3.4. *Let W be a JBW*-triple, let $z \in W$, $\phi \in W_*$ and denote by $s(\phi)$ the support tripotent of ϕ. If $z \perp s(\phi)$, then $\phi\{x, y, z\} = 0$ for all $x, y \in W$.*

Proof. We write $s = s(\phi)$ for short. Since $z \perp s$, and hence $z \in W_0(s)$, it follows from the Peirce rules that $\{x, y, z\} = a + b$, where

$$\begin{aligned}a &= \{P_1(s)(x), P_0(s)(y), z\} + \{P_2(s)(x), P_1(s)(y), z\} \subseteq W_1(s), \\ b &= \{P_0(s)(x), P_0(s)(y), z\} + \{P_1(s)(x), P_1(s)(y), z\} \subseteq W_0(s).\end{aligned}$$

Therefore, $\phi\{x, y, z\} = \phi(a + b) = \phi P_2(s(\phi))(a + b) = 0$. ∎

4. Using the strong*-topology. In [4, Prop. 1.2], Barton and Friedman showed that for a JBW*-triple W, the mapping $x \mapsto \|x\|_\varphi := (\varphi\{x, x, s(\varphi)\})^{1/2}$, where $s(\varphi)$ is the support tripotent of $\varphi \in W_*$, defines a pre-Hilbertian seminorm on W. Moreover, $\varphi\{x, x, s(\varphi)\} = \varphi\{x, x, z\}$ whenever $\varphi(z) = \|\varphi\| = \|z\| = 1$.

It is known that the identity

$$\|x\|_\varphi^2 = \|P_1(e)(x)\|_\varphi^2 + \|P_2(e)(x)\|_\varphi^2 \tag{4.1}$$

holds for all $x \in W$, $\varphi \in W_*$, and tripotents e such that $\|\varphi\| = 1 = \varphi(e)$. Indeed, although the proof of (4.1) in [36, Lem. 4.2] does not cover the general case, it is very close. For the general case, let $x = x_0 + x_1 + x_2$ be the Peirce decomposition associated with e (i.e. $x_k = P_k(e)(x)$, $k = 0, 1, 2$). By the Peirce rules,

$$\|x\|_\varphi^2 = \varphi\{x, x, e\} = \varphi\{x_1, x_1, e\} + \varphi\{x_2, x_2, e\} + \varphi\{x_0, x_1, e\} + \varphi\{x_1, x_2, e\},$$

hence (4.1) because $|\varphi\{x_0, x_1, e\}| \le \underline{\varphi\{x_0, x_0, e\}}\varphi\{x_1, x_1, e\} = 0$ ([4, Prop. 1.2] and $x_0 \perp e$) and $\varphi\{x_1, x_2, e\} = \overline{\varphi\{x_2, x_1, e\}} = 0$ (Peirce rules and [4, Prop. 1.2]).

In [5] the strong*-topology $s^*(W, W_*)$ on a JBW*-triple W is defined as the locally convex topology generated by the family $\{\|\cdot\|_\varphi : \varphi \in W_*, \|\varphi\| = 1\}$. On a von Neumann algebra the strong* topology in the von Neumann sense [48, 1.8.7] and the strong*-topology in the triple sense coincide [5, pp. 258–259].

The following proposition resembles [21, Cor. 2.6]. It says that a bounded net (a_λ) is strong*-null if and only if the net $\{a_\lambda, x, y\}$ is weak*-null uniformly in $x, y \in B_W$.

Proposition 4.1. *Let (a_λ) be a bounded net in a JBW*-triple W.*

(a) *The net (a_λ) is strong*-null if and only if for each $\varphi \in W_*$,*

$$\sup\{|\varphi\{a_\lambda, x, y\}| : x, y \in B_W\} \xrightarrow{\lambda} 0. \tag{4.2}$$

(b) *If (a_λ) is strong*-null then, for each $b \in W$ and each $\varphi \in W_*$,*

$$\sup\{|\varphi\{b, a_\lambda, y\}| : y \in B_W\} \xrightarrow{\lambda} 0. \tag{4.3}$$

Proof. Without loss of generality, we suppose that $(a_\lambda) \subset B_W$. First we notice that the "if" part of (a) follows from

$$\|a_\lambda\|_\varphi^2 = \varphi\{a_\lambda, a_\lambda, s(\varphi)\} \le \sup\{|\varphi\{a_\lambda, x, y\}| : x, y \in B_W\}.$$

For the "only if" part of (a) and for (b) we first consider the case when W is a von Neumann algebra considered as a JBW*-triple via $\{a, b, c\} = (ab^*c + cb^*a)/2$. In this case it is enough to consider a positive $\varphi \in W_*$. We may assume $\|\varphi\| = 1$. By the Cauchy–Schwarz inequality, $|\varphi(a_\lambda x^* y)|^2 \le \varphi(a_\lambda a_\lambda^*)\varphi(y^* x x^* y) \le \varphi(a_\lambda a_\lambda^*)$ and similarly $|\varphi(y x^* a_\lambda)|^2 \le \varphi(a_\lambda^* a_\lambda)$. Thus

$$2|\varphi\{a_\lambda, x, y\}| = |\varphi(a_\lambda x^* y) + \varphi(y x^* a_\lambda)| \le (\varphi(a_\lambda^* a_\lambda))^{1/2} + (\varphi(a_\lambda a_\lambda^*))^{1/2},$$

and similarly, for $b \in W$,

$$2|\varphi\{b, a_\lambda, y\}| \le (\varphi_b(a_\lambda^* a_\lambda))^{1/2} + (\varphi_{b^*}(a_\lambda a_\lambda^*))^{1/2}$$

where φ_b and φ_{b^*} are positive normal functionals on W defined by $c \mapsto \varphi(bcb^*)$ and $c \mapsto \varphi(b^* cb)$, respectively. This shows (4.2) and (4.3) for von Neumann algebras W.

To pass to general JBW*-triples W we first make three observations.

Observation 1. The property expressed in the proposition is stable under ℓ_∞ sums. More precisely, let $(W_j)_{j\in J}$ be a family of JBW*-triples such that each W_j satisfies the proposition accordingly. Set $W = \bigoplus_{j\in J}^{\ell_\infty} W_j$, which is a JBW*-triple in a canonical way ([34, p. 523] or [11, Ex. 3.1.4]). Then W satisfies the proposition, too. Indeed, let $(a_\lambda)_\lambda = ((a_{\lambda,j})_{j\in J})_\lambda$ be a strong*-null net in B_W. Given $\varphi = (\varphi_j) \in W_* = \bigoplus_{j\in J}^{\ell_1} W_{j,*}$ we have $\varphi\{a_\lambda, x, y\} = \sum_J \varphi_j\{a_{\lambda,j}, x_j, y_j\}$. For any $\varepsilon > 0$ there is a finite subset $F \subset J$ such that (by (2.2))

$$\sum_{j\in J\setminus F} |\varphi_j\{a_{\lambda,j}, x_j, y_j\}| \le \sum_{j\in J\setminus F} \|\varphi_j\| < \varepsilon/2$$

uniformly in $x = (x_j), y = (y_j) \in B_W$. Since $(a_{\lambda,j})_\lambda$ is strong*-null in W_j for each j we have $\sum_{j\in F} \varphi_j\{a_{\lambda,j}, x_j, y_j\} \xrightarrow{\lambda} 0$ uniformly in $x_j, y_j \in B_{W_j}$. This proves (4.2). The argument for (4.3) is similar.

Observation 2. By [3, Cor. 9] (see also [23, Cor. 1, 2]), every JBW*-triple W can be identified with (i.e. is JBW*-triple isometrically isomorphic to) a weak*-closed JB*-subtriple of a JBW*-algebra M. (Recall that a JBW*-algebra with product $a \circ b$ is a JBW*-triple with the triple product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*, \tag{4.4}$$

cf. [11, Lem. 3.1.6]). In turn, every JBW*-algebra M can be (uniquely) decomposed as a direct ℓ_∞-sum $M = M_1 \oplus_\infty M_2$ where M_1 is a weak*-closed subtriple of a von Neumann algebra and M_2 is a purely exceptional JBW*-algebra (cf. [26, Thm. 7.2.7]). Moreover, M_2 embeds as a JBW*-subalgebra into an ℓ_∞-sum of finite-dimensional exceptional JBW*-algebras ([26, Lem. 7.2.2 and Thm. 7.2.7]).

Observation 3. For each JBW*-subtriple F of a JBW*-triple W, the strong*-topology of F coincides with the restriction to F of the strong*-topology of W, that is, $s^*(F, F_*) = s^*(W, W_*)|_F$ (cf. [8, Cor.]). Hence the property expressed in the proposition passes from JBW*-triples to weak*-closed subtriples.

For an arbitrary JBW*-triple W, the proposition can now be reduced, via the previous three observations, to the von Neumann case, which has been proved above, and to the fact that finite-dimensional JBW*-triples satisfy the proposition trivially. ∎

Analogously to [43, 45] and to [21], we define uniform integrability in JBW*-triple preduals:

Definition 4.2. Let W be a JBW*-triple. A bounded subset K of W_* is said to be *uniformly integrable* if

$$\lim_{n\to\infty} \sup\{\|\varphi Q(x_n)\| : \varphi \in K\} = 0$$

for each strong*-null sequence (x_n) in W.

This definition turns out to be equivalent to relative weak compactness and is therefore equivalent to the corresponding definitions of [43, Def. 2.2], [45, Def. 4.1] (to be read only for the case $p = 1$) and [21, Def. 2.1]. As in [21], this will be a consequence of some characterizations of relative weak compactness in JBW*-triple preduals taken from [37].

For the reader's convenience we first recall some more results concerning the strong*-topology. Let u, v be two tripotents in a JBW*-triple W. We write $u \le v$ if $v - u \perp u$, which is equivalent to $\{v, u, v\} = u$ [11, 1.2.43]. The Peirce space $W_2(v)$ becomes a unital JB*-algebra with product $a \circ b = \{a, v, b\}$ and involution $a^* = \{v, a, v\}$; further, from this product the original triple product can be recovered by (4.4) [11, p. 20]. Now it is not difficult to see that u is a symmetric projection in $W_2(v)$. On a JBW*-algebra M a strong*-topology in the algebraic sense is defined by the family of seminorms of the form $x \mapsto \|x\|_\phi = \phi\{x, x, 1\}^{1/2} = (\phi(x^* \circ x))^{1/2}$, where $\phi \in M_*$ is positive and of norm one [26, 4.1.3]. Rodríguez-Palacios [46, Prop. 3] has shown that this topology coincides with $s^*(M, M_*)$ when M is considered as a JBW*-triple.

Let now (q_n) be a decreasing weak*-null sequence of tripotents in W. Then (q_n) is a weak*-null sequence of projections in $M = W_2(q_1)$. For any positive $\phi \in (W_2(q_1))_*$ we have $\|q_n\|_\phi^2 = \phi(q_n^* \circ q_n) = \phi(q_n) \to 0$, which shows that (q_n) is strong*-null in the algebraic and in the triple sense in $W_2(q_1)$, hence it is also strong*-null in W (cf. [8, Cor.]). To sum up, a decreasing weak*-null sequence of tripotents in W is also strong*-null.

Similarly, we can show that a sequence (e_n) of pairwise orthogonal tripotents in W is strong*-null in W. It is known that (e_n) is summable with respect to the weak*-topology of W. Moreover, the element $e := \sigma(W, W_*)-\sum_n e_n$ is a tripotent in W and $e_n \le e$ for every $n \in \mathbb{N}$, that is, the sequence (e_n) lies in the JBW*-algebra $W_2(e)$ (cf. [30, Cor. 3.13]) and we have $e_n \circ e_n^* = e_n$ for all n. Further, $e_n \to 0$ with respect to the weak*-topology (of W and) of $W_2(e)$. As in the preceding paragraph, we deduce from $\|e_n\|_\phi^2 = \phi(e_n^* \circ e_n) = \phi(e_n) \to 0$ that (e_n) is strong*-null in $W_2(e)$ and finally in W.

We can now show the connections between uniform integrability and relative weak compactness. The main aspect of the following proposition is the equivalence of (i) and (ii); other equivalences are standard or, like (vii), at least implicitly known but perhaps not stated in the literature.

Proposition 4.3. *Let K be a bounded subset in the predual of a JBW*-triple W. The following statements are equivalent:*

(i) *K is relatively weakly compact.*
(ii) *K is uniformly integrable.*
(iii) *For each strong*-null sequence (e_n) of tripotents we have*

$$\lim_{n\to\infty} \sup\{\|\varphi P_2(e_n)\| : \varphi \in K\} = 0. \tag{4.5}$$

(iv) *For each decreasing strong*-null sequence (e_n) of tripotents we have (4.5).*

(v) *For each sequence* (e_n) *of pairwise orthogonal tripotents we have* (4.5).

(vi) *For each decreasing weak***-null (equivalently decreasing strong***-null) sequence* (e_n) *of tripotents we have*

$$\lim_{n\to\infty} \sup\{|\varphi(e_n)| : \varphi \in K\} = 0. \tag{4.6}$$

(vii) *For each sequence* (e_n) *of pairwise orthogonal tripotents we have* (4.6).

Proof. We use the notation $\|x\|^2_{\varphi_1,\varphi_2} = \|x\|^2_{\varphi_1} + \|x\|^2_{\varphi_2}$. From [37, Thm. 1.1, Cor. 1.4] we infer that (i) is equivalent to (vi) and also to the following statement.

(1) There exist norm-one elements $\psi_1, \psi_2 \in W_*$ with the following property: Given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in W$ with $\|x\| \le 1$ and $\|x\|_{\psi_1,\psi_2} < \delta$, we have $|\varphi(x)| < \varepsilon$ for each $\varphi \in K$.

We have (vi)⇒(i)⇒(1) and show (1)⇒(ii): Let (x_n) be strong*-null in W, in fact in B_W, and take $\psi \in \{\psi_1, \psi_2\}$ where ψ_1, ψ_2 are from (1). Given $\varepsilon > 0$ choose $\delta > 0$ according to (1). Let $y \in B_W$, and set $z = Q(x_n)(y)$. From the Jordan identity (2.1) we get

$$\begin{aligned}\{z, z, s(\psi)\} &= \{\{x_n, y, x_n\}, z, s(\psi)\}\\ &= \{x_n, y, \{x_n, z, s(\psi)\}\} + \{x_n, \{y, x_n, z\}, s(\psi)\} - \{x_n, z, \{x_n, y, s(\psi)\}\}.\end{aligned}$$

Hence

$$\|Q(x_n)(y)\|^2_\psi \le 3 \sup\{|\psi\{x_n, a, b\}| : a, b \in B_W\} \xrightarrow{n} 0$$

uniformly in $y \in B_W$ by Proposition 4.1(a). Thus, there is n_0 such that $\|Q(x_n)(y)\|_{\psi_1,\psi_2} < \delta$ for all $n \ge n_0$ and all $y \in B_W$. Now $\|\varphi Q(x_n)\| = \sup_{y\in B_W} |\varphi(Q(x_n)(y))| \le \varepsilon$ by (1), which shows (ii).

The implication (ii)⇒(iii) follows from $\|\varphi P_2(e_n)\| = \|\varphi Q(e_n)^2\| \le \|\varphi Q(e_n)\|$. The implication (iii)⇒(iv) is trivial, and so is (iii)⇒(v) if we take into account that, as seen above, a sequence of pairwise orthogonal tripotents is strong*-null.

(iv)⇒(vi): We have commented that a decreasing weak*-null sequence of tripotents is strong*-null. Thus the desired implication follows from $|\varphi(e_n)| = |\varphi(P_2(e_n)(e_n))| \le \|\varphi P_2(e_n)\|$.

From the same inequality we also deduce (v)⇒(vii).

(vii)⇒(i): In order to show that K is relatively weakly compact, it is enough to show that the restriction $K_{|\mathcal{C}}$ is so for each maximal abelian subtriple $\mathcal{C}$ of W (cf. [37, Thm. 1.1]). But such $\mathcal{C}$'s are isometric to von Neumann algebras (see, for example, [29, Cor. 6.4]), thus the desired implication follows from Akemann's criterion (see, for example, [1], [45, 4.14(ii)]). ■

We will use the following definitions of functionals on W. For $a,b,x\in W$ and $\varphi\in W_*$, the maps $\{a,b,\varphi\}$, $\{\varphi,b,a\}$ and $\{a,\varphi,b\}$ defined by

$$\{a,b,\varphi\}(x)=\{\varphi,b,a\}(x):=\varphi\{b,a,x\} \tag{4.7}$$

and

$$\{a,\varphi,b\}(x):=\overline{\varphi\{a,x,b\}}$$

are well-defined elements of W_* (by the separate weak*-continuity of the triple product). Further, $\{a,b,\varphi\}$ and $\{\varphi,b,a\}$ are linear in b and φ and conjugate linear in a, whereas $\{a,\varphi,b\}$ is conjugate linear in a,b,φ. Although these properties are more than enough for what we need, it is worth pointing out that (4.7) defines natural actions of W on W_* and allows one to consider W_* as a Banach triple module over W. The notion of Banach triple module has been introduced in the recent paper [40] by Russo and the first author of this note. A bit more concretely in our context, compare, for example, $\{\mathcal{W},\mathcal{W},\widehat{W_*}\}$ in Proposition 5.3 with $\mathcal{A}L_1(\mathcal{a})+L_1(\mathcal{a})\mathcal{A}$ in the proof of [45, Thm. 4.6b].

Corollary 4.4. *Let ϕ be a normal functional in the predual of a JBW*-triple W. Let $(a_i)_{i\in I}$, $(b_i)_{i\in I}$ be two bounded families of elements in W. Then the set $\{\{a_i,b_i,\phi\}:i\in I\}$ is relatively weakly compact in W_*.*

Proof. We may assume that $a_i,b_i\in B_W$ for every $i\in I$. Let (e_n) be a decreasing strong*-null sequence of tripotents in W. Proposition 4.1(a) implies that

$$\sup\{|\{a_i,b_i,\phi\}(e_n)|:i\in I\}\le\sup\{|\phi\{b,a,e_n\}|:a,b\in B_W\}\overset{n}{\to}0.$$

The desired statement follows from [37, Thm. 1.1, Cor. 1.4] (cited here as (vi)⇒(i) in Theorem 4.3). ■

Proposition 4.5. *Let E be a weak*-dense JB*-subtriple of a JBW*-triple W. Then for all $\phi\in W_*$ and $y,z\in W$, the functional $\{z,y,\phi\}\in W_*$ is in the norm-closure of the set $\{\{a,b,\phi\}:a,b\in\rho B_E\}$, where $\rho=\max\{\|y\|,\|z\|\}$.*

Proof. We can assume that $\max\{\|y\|,\|z\|\}=1$. By the Kaplansky density theorem [5, Cor. 3.3] it follows that

$$B_W=\overline{B_E}^{s^*(W,W_*)}.$$

Let (y_λ) and (z_μ) be two nets in B_E converging in the strong*-topology of W to y and z, respectively. By Proposition 4.1, we have

$$\|\{z_\mu,y-y_\lambda,\phi\}\|\le\sup\{|\phi\{y-y_\lambda,z,x\}|:z,x\in B_W\}\overset{\lambda}{\to}0$$

uniformly in μ, and

$$\|\{z-z_\mu,y,\phi\}\|\le\sup\{|\phi\{y,z-z_\mu,x\}|:x\in B_W\}\overset{\mu}{\to}0.$$

Finally, the identity $\{z, y, \phi\} - \{z_\mu, y_\lambda, \phi\} = \{z - z_\mu, y, \phi\} + \{z_\mu, y - y_\lambda, \phi\}$ gives the desired statement. ∎

5. Using structural projections. A linear subspace J of a JBW*-triple W is an *inner ideal* in W if $\{J, W, J\} \subseteq J$. Clearly, inner ideals are subtriples. Edwards and Rüttimann [17, Lem. 2.3] established the following characterization: A weak*-closed subtriple J of W is an inner ideal of W if and only if

$$J = \bigcup_{e \in \mathrm{Trip}(J)} W_2(e) \tag{5.1}$$

where $\mathrm{Trip}(J)$ is the set of tripotents contained in J. Note in passing that in von Neumann algebras (viewed as JBW*-triples) left and right ideals and sets of the form aWb $(a, b, \in W)$ are inner ideals, whereas weak*-closed inner ideals are of the form pWq with projections $p, q \in W$ [16, Thm. 3.16].

Examples of inner ideals can be given as follows. Let $M \subset W$. Then $M^\perp$, the (*orthogonal*) *annihilator* of M, defined by

$$M^\perp := \{y \in W : y \perp x,\ \forall x \in M\},$$

is a weak*-closed (by the separate weak*-continuity of the triple product) inner ideal of W (cf. [18, Lem. 3.2]).

A linear projection P on W is said to be *structural* when

$$\{P(a), b, P(c)\} = P\{a, P(b), c\}, \qquad \forall a, b, c \in W.$$

Such a projection is contractive and weak*-continuous and its pre-adjoint $P_* : W_* \to W_*$ has range

$$P_*(W_*) = P(W)_\sharp := \{\varphi \in W_* : \|\varphi\| = \|\varphi|_{P(W)}\|\},$$

where, of course, $\|\varphi|_{P(W)}\| = \sup_{\|P(x)\| \le 1} |\varphi(P(x))|$ (see [15, Thm. 5.3]). Note in passing that structural projections on a von Neumann algebra M are of the form $x \mapsto pxq$ where p, q are centrally equivalent projections in M [15, Thm. 6.1].

This circle of ideas culminates in the result of [15, Thm. 5.4], where Edwards, McCrimmon and Rüttimann proved that every weak*-closed inner ideal J in a JBW*-triple W is the range of a unique structural projection P on W. It is also known (cf. [15, Lem. 5.2]) that

$$P_*(W_*) = J_* = \bigcup_{e \in \mathrm{Trip}(J)} W_{*,2}(e).$$

Given a subset $Z \subset W_*$ we henceforth write

$$Z^\perp := \{\varphi \in W_* : \varphi \perp \phi,\ \forall \phi \in Z\}.$$

LEMMA 5.1. *Let Z be a subset in the predual of a* JBW*-*triple* W. *Let* $S(Z) := \{s(\phi) : \phi \in Z\}$ *in* W *and write* $J = S(Z)^{\perp} \subseteq W$. *Suppose* $P : W \to W$ *is the unique structural projection on* W *whose image is the weak*-closed inner ideal* J. *Then* $P_*(W_*) = Z^{\perp}$.

Proof. Let φ be a functional in $P_*(W_*) = P(W)_\sharp$. Then there exists a tripotent $e \in P(W) = J$ such that $\varphi(e) = \|\varphi\|$, and hence $\varphi = \varphi P_2(e)$. It follows that $s(\varphi) \in W_2(e)$ (cf. [22, proof of Prop. 2]), and thus $s(\varphi) \in J = S(Z)^{\perp}$ by (5.1). We deduce that $s(\varphi) \perp s(\phi)$ for every $\phi \in Z$, or equivalently, $\varphi \in Z^{\perp}$. This shows that $J_* \subseteq Z^{\perp}$.

Take now $\varphi \in Z^{\perp}$. In this case, $s(\varphi) \perp s(\phi)$ for every $\phi \in Z$. Therefore, $s(\varphi) \in S(Z)^{\perp} = J$, and hence $\varphi \in P(W)_\sharp = P_*(W_*) = J_*$. ■

We now describe the situation in which the theory above will be used. Let W be a JBW*-triple and let $\mathcal{U}$ be an ultrafilter on a set I. Henceforth, we write $\mathcal{W} = ((W_*)_{\mathcal{U}})^*$, $S(\widehat{W_*}) = \{s(\widehat{\phi}) \in \mathcal{W} : \phi \in W_*\}$ and

$$\mathcal{J} = (S(\widehat{W_*}))^{\perp} = \{y \in \mathcal{W} : y \perp s(\widehat{\phi}),\ \forall s(\widehat{\phi}) \in S(\widehat{W_*})\}.$$

Then $\mathcal{J}$ is a weak*-closed inner ideal in $\mathcal{W}$. Further, we denote by $\mathcal{P}_{\mathcal{U}} : \mathcal{W} \to \mathcal{W}$ the unique structural projection on $\mathcal{W}$ whose image is $\mathcal{J}$.

The following corollary is immediate from Lemma 5.1.

COROLLARY 5.2. *In the situation just described, a functional* $\widetilde{\varphi} = [\varphi_i]_{\mathcal{U}}$ *in* $(W_*)_{\mathcal{U}} = \mathcal{W}_*$ *is disjoint from* $W_* \equiv \widehat{W_*}$ *if and only if* $(\mathcal{P}_{\mathcal{U}})_*(\widetilde{\varphi}) = \widetilde{\varphi}$. ■

PROPOSITION 5.3. *In the situation described before Corollary* 5.2 *we have*

$$\ker((\mathcal{P}_{\mathcal{U}})_*) = \overline{\mathrm{span}}^{\|\cdot\|}\{\mathcal{W}, \mathcal{W}, \widehat{W_*}\} = \overline{\mathrm{span}}^{\|\cdot\|}\{W_{\mathcal{U}}, W_{\mathcal{U}}, \widehat{W_*}\}. \tag{5.2}$$

Proof. Take $x, y \in \mathcal{W}$, and $\widehat{\phi} \in \widehat{W_*}$. Since each element of $\mathcal{J}$ is orthogonal to the support tripotent $s(\widehat{\phi})$ of $\widehat{\phi}$, we have $\widehat{\phi}\{x, y, \mathcal{J}\} = 0$ by Lemma 3.4, that is, $\{y, x, \widehat{\phi}\}(\mathcal{J}) = 0$. Equivalently,

$$(\mathcal{P}_{\mathcal{U}})_*\{y, x, \widehat{\phi}\} = \{y, x, \widehat{\phi}\}\mathcal{P}_{\mathcal{U}} = 0.$$

This shows that $\ker((\mathcal{P}_{\mathcal{U}})_*) \supseteq \overline{\mathrm{span}}^{\|\cdot\|}\{\mathcal{W}, \mathcal{W}, \widehat{W_*}\}$.

In order to show that equality holds suppose that $z \in \mathcal{W}$ vanishes on $\{\mathcal{W}, \mathcal{W}, \widehat{W_*}\}$. Then $0 = \{y, x, \widehat{\phi}\}(z) = \widehat{\phi}\{x, y, z\}$ for all $x, y \in \mathcal{W}$ and all $\widehat{\phi} \in \widehat{W_*}$. Taking $x = s(\widehat{\phi}) \in \mathcal{W}$ and $y = z$ we get

$$\|z\|_{\widehat{\phi}}^2 = \widehat{\phi}\{z, z, s(\widehat{\phi})\} = 0.$$

By (4.1),

$$\begin{aligned} 0 = \|z\|_{\widehat{\phi}}^2 &= \|P_2(s(\widehat{\phi}))(z)\|_{\widehat{\phi}}^2 + \|P_1(s(\widehat{\phi}))(z)\|_{\widehat{\phi}}^2 \\ &= \widehat{\phi}\{P_2(s(\widehat{\phi}))(z), P_2(s(\widehat{\phi}))(z), s(\widehat{\phi})\} + \widehat{\phi}\{P_1(s(\widehat{\phi}))(z), P_1(s(\widehat{\phi}))(z), s(\widehat{\phi})\}. \end{aligned}$$

By [22, Lem. 1.5] (see also [38]), the triples $\{P_2(s(\widehat{\phi}))(z), P_2(s(\widehat{\phi}))(z), s(\widehat{\phi})\}$ and $\{P_1(s(\widehat{\phi}))(z), P_1(s(\widehat{\phi}))(z), s(\widehat{\phi})\}$ are positive elements in the JBW*-algebra $\mathcal{W}_2(s(\widehat{\phi}))$, and it follows from the faithfulness of $\widehat{\phi}$ on $\mathcal{W}_2(s(\widehat{\phi}))$ that both are zero. Another application of [22, Lem. 1.5] (see also [38]) shows that $P_2(s(\widehat{\phi}))(z) = P_1(s(\widehat{\phi}))(z) = 0$. We have shown that $z \in \mathcal{W}_0(s(\widehat{\phi}))$, equivalently, $z \perp s(\widehat{\phi})$ for every $\widehat{\phi} \in \widehat{W_*}$, that is, $z \in \mathcal{J} = \mathcal{P}_{\mathcal{U}}(\mathcal{W})$. Hence z vanishes on the annihilator of $\mathcal{P}_{\mathcal{U}}(\mathcal{W})$ in $\mathcal{W}_*$, that is, on $\ker((\mathcal{P}_{\mathcal{U}})_*)$. By the Hahn–Banach theorem we get the first equality of (5.2).

If we keep in mind that $W_{\mathcal{U}}$ is a weak*-dense JB*-subtriple of $\mathcal{W}$ (cf. Theorem 3.3), then the second equality of (5.2) is a consequence of Proposition 4.5. ∎

The main result of this section shows how, in the case of a countably incomplete ultrafilter $\mathcal{U}$, the projection $(\mathcal{P}_{\mathcal{U}})_*$ determines when an element $\widetilde{\varphi} \in (W_*)_{\mathcal{U}}$ admits a representative which, as a set, is relatively weakly compact in W_*.

Theorem 5.4. *Consider the situation described before Corollary* 5.2. *Suppose the ultrafilter $\mathcal{U}$ is countably incomplete. Then $(\mathcal{P}_{\mathcal{U}})_*(\widetilde{\varphi}) = 0$ if and only if we can write $\widetilde{\varphi} = [\varphi_i]_{\mathcal{U}}$ for some relatively weakly compact set $\{\varphi_i : i \in I\}$ in W_*.*

Proof. “Only if”: Every $\widetilde{\varphi} \in \{W_{\mathcal{U}}, W_{\mathcal{U}}, \widehat{W_*}\}$ can be written in the form

$$\widetilde{\varphi} = \{[a_i]_{\mathcal{U}}, [b_i]_{\mathcal{U}}, \widehat{\phi}\} = [\{a_i, b_i, \phi\}]_{\mathcal{U}}$$

where $[a_i]_{\mathcal{U}}, [b_i]_{\mathcal{U}} \in W_{\mathcal{U}}$ and $\phi \in W_*$. Corollary 4.4 proves that the set $\{\{a_i, b_i, \phi\} : i \in I\}$ is relatively weakly compact in W_*. Thus, every $\widetilde{\varphi} \in \{W_{\mathcal{U}}, W_{\mathcal{U}}, \widehat{W_*}\}$, and in fact every $\widetilde{\varphi} \in \mathrm{span}\{W_{\mathcal{U}}, W_{\mathcal{U}}, \widehat{W_*}\}$, admits a representative $\widetilde{\varphi} = [\varphi_i]_{\mathcal{U}}$ where $\{\varphi_i : i \in I\}$ is relatively weakly compact in W_*. By Lemma 3.2, the same statement still holds for all $\widetilde{\varphi} \in \overline{\mathrm{span}}^{\|\cdot\|}\{W_{\mathcal{U}}, W_{\mathcal{U}}, \widehat{W_*}\}$, and hence for all $\widetilde{\varphi} \in \ker((\mathcal{P}_{\mathcal{U}})_*)$ by Proposition 5.3.

“If”: Suppose that $\widetilde{\varphi} = [\varphi_i]_{\mathcal{U}}$ where $\{\varphi_i : i \in I\}$ is relatively weakly compact in W_*. Write

$$\widetilde{\varphi} = \widetilde{\psi} + \widetilde{\xi}$$

where $\widetilde{\psi}$ is in $(\mathcal{P}_{\mathcal{U}})_*$'s range and $\widetilde{\xi} \in \ker((\mathcal{P}_{\mathcal{U}})_*)$. By the “only if” implication, we can find a representative $\widetilde{\xi} = [\xi_i]_{\mathcal{U}}$ for some relatively weakly compact set $\{\xi_i : i \in I\}$ in W_*. If we set $\psi_i = \varphi_i - \xi_i$, it follows from the above that the ψ_i's form a relatively weakly compact representative of $\widetilde{\psi}$. Since $\widetilde{\psi}$ is in the image of $(\mathcal{P}_{\mathcal{U}})_*$, we infer from Corollary 5.2 that $\widetilde{\psi}$ is disjoint from $(\widehat{W_*})^{\perp}$, hence $\|\widetilde{\psi} + \widehat{\phi}\| = \|\widetilde{\psi}\| + \|\widehat{\phi}\|$ for all $\phi \in W_*$. If we had $\widetilde{\psi} = [\psi_i]_{\mathcal{U}} \neq 0$ then by Lemma 3.1 the set $\{\psi_i : i \in I\}$ would contain an ℓ_1-sequence, which, however, is not possible for a relatively weakly compact set. Hence $(\mathcal{P}_{\mathcal{U}})_*(\widetilde{\varphi}) = \widetilde{\psi} = 0$. ∎

6. Main result. Finally, we are in a position to prove the main result, a generalization of the Kadec–Pełczyński–Rosenthal subsequence splitting lemma to preduals of JBW*-triples. As already mentioned in the introduction, JBW*-triple preduals seem to constitute the largest known class of L-embedded Banach spaces fulfilling a splitting property for bounded sequences.

THEOREM 6.1. *Let W be a JBW*-triple, and let (φ_n) be a bounded sequence in W_*. Then there is a subsequence (φ_{n_k}) which can be written $\varphi_{n_k} = \psi_k + \xi_k$ where the ψ_k's are pairwise orthogonal and (ξ_k) converges weakly to some $\xi \in W_*$.*

Proof. We apply Theorem 5.4 with $I = \mathbb{N}$ and $\mathcal{U}$ a free ultrafilter over $\mathbb{N}$. Consider $\widetilde{\varphi} = [\varphi_n]_{\mathcal{U}}$ and $\widetilde{\tau} = \widetilde{\varphi} - (\mathcal{P}_{\mathcal{U}})_*(\widetilde{\varphi})$ in $(W_*)_{\mathcal{U}}$. Then $(\mathcal{P}_{\mathcal{U}})_*(\widetilde{\tau}) = 0$. By Theorem 5.4 we can write $\widetilde{\tau} = [\tau_n]_{\mathcal{U}}$ where the set $\{\tau_n : n \in \mathbb{N}\}$ is relatively weakly compact in W_*.

Set $\omega_n = \varphi_n - \tau_n$ and $\widetilde{\omega} = [\omega_n]_{\mathcal{U}}$. Then $\widetilde{\omega} = (\mathcal{P}_{\mathcal{U}})_*(\widetilde{\varphi})$ and $\widetilde{\omega} \perp \widehat{W_*}$ (cf. Corollary 5.2). If $\widetilde{\omega} = 0$, then $\lim_{\mathcal{U}} \|\varphi_n - \tau_n\| = 0$, and hence

$$\lim_{k\to\infty} \|\varphi_{n_k} - \tau_{n_k}\| = 0$$

for appropriate subsequences; we can further assume, by the theorem of Eberlein–Šmulyan, that (τ_{n_k}) converges weakly to some ξ. Setting $\psi_k = 0$ and $\xi_k = (\varphi_{n_k} - \tau_{n_k}) + \tau_{n_k}$, we get the conclusion in the case $\widetilde{\omega} = 0$.

If $\widetilde{\omega} \neq 0$, then by Lemma 3.1 there is a seminormalized (= bounded and uniformly away from 0) subsequence (ω_{n_l}) such that $(\omega_{n_l}/\|\omega_{n_l}\|)$ spans ℓ_1 asymptotically, hence almost isometrically. It follows from [39, Thm. 4.1] that there are a further subsequence of (ω_{n_l}) (which we still denote by (ω_{n_l})) and a sequence (ψ_l') of pairwise orthogonal norm-one functionals in W_* such that $\|\omega_{n_l}/\|\omega_{n_l}\| - \psi_l'\| \to 0$. Moreover, there is a subsequence $(\tau_{n_{l_k}})$ which converges weakly to some ξ (Eberlein–Šmulyan theorem). It remains to set

$$\psi_{l_k} = \|\omega_{n_{l_k}}\|\psi'_{n_{l_k}} \quad \text{and} \quad \xi_{l_k} = \tau_{n_{l_k}} + (\omega_{n_{l_k}} - \psi_{l_k}),$$

and to replace l_k by k. ∎

Acknowledgements. We thank Y. Raynaud for helpful conversations.

This work was essentially done while the second author visited the mathematical department of the University of Granada in June 2014; he expresses his thanks to the department and particularly to Antonio Peralta for a generous and pleasant organisation of the visit.

The first author was partially supported by the Spanish Ministry of Economy and Competitiveness project no. MTM2014-58984-P, Junta de Andalucía grant FQM375, and the Deanship of Scientific Research at King Saud University (Saudi Arabia) research group no. RG-1435-020.

References

[1] C. A. Akemann, *The general Stone–Weierstrass problem*, J. Funct. Anal. 4 (1969), 277–294.

[2] M. A. Akcoglu and L. Sucheston, *On ergodic theory and truncated limits in Banach lattices*, in: Measure Theory (Oberwolfach, 1983), Lecture Notes in Math. 1089, Springer, 1984, 241–262.

[3] T. J. Barton, T. Dang and G. Horn, *Normal representations of Banach Jordan triple systems*, Proc. Amer. Math. Soc. 102 (1988), 551–555.

[4] T. Barton and Y. Friedman, *Grothendieck's inequality for JB^*-triples and applications*, J. London Math. Soc. II. Ser. 36 (1987), 513–523.

[5] T. Barton and Y. Friedman, *Bounded derivations of JB^*-triples*, Quart. J. Math. Oxford Ser. (2) 41 (1990), 255–268.

[6] J. Becerra Guerrero and M. Martín, *The Daugavet property of C^*-algebras, JB^*-triples, and of their isometric preduals*, J. Funct. Anal. 224 (2005), 316–337.

[7] J. Bourgain and H. P. Rosenthal, *Martingales valued in certain subspaces of L^1*, Israel J. Math. 37 (1980), 54–75.

[8] L. J. Bunce, *Norm preserving extensions in JBW^*-triple preduals*, Quart. J. Math. 52 (2001), 133–136.

[9] M. J. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno and A. M. Peralta, *Orthogonality preservers in C^*-algebras, JB^*-algebras and JB^*-triples*, J. Math. Anal. Appl. 348 (2008), 220–233.

[10] M. Cabrera and A. Rodríguez Palacios, *Non-Associative Normed Algebras: Volume 1, The Vidav–Palmer and Gelfand–Naimark Theorems*, Cambridge Univ. Press, 2014.

[11] Ch. -H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge Univ. Press, 2012.

[12] D. Dacunha-Castelle et M. Schreiber, *Techniques probabilistes pour l'étude de problèmes d'isomorphismes entre espaces de Banach*, Ann. Inst. H. Poincaré Sect. B 10 (1974), 229–277.

[13] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, New York, 1984.

[14] P. G. Dodds, T. K. Dodds and F. A. Sukochev, *Banach–Saks properties in symmetric spaces of measurable operators*, Studia Math. 178 (2007), 125–166.

[15] C. M. Edwards, K. McCrimmon and G. T. Rüttimann, *The range of a structural projection*, J. Funct. Anal. 139 (1996), 196–224.

[16] C. M. Edwards and G. T. Rüttimann, *Inner ideals in W^*-algebras*, Michigan Math. J. 36 (1989), 147–159.

[17] C. M. Edwards and G. T. Rüttimann, *A characterization of inner ideals in JB^*-triples*, Proc. Amer. Math. Soc. 116 (1992), 1049–1057.

[18] C. M. Edwards and G. T. Rüttimann, *Peirce inner ideals in Jordan*-triples*, J. Algebra 180 (1996), 41–66.

[19] C. M. Edwards and G. T. Rüttimann, *Orthogonal faces of the unit ball in a Banach space*, Atti Sem. Mat. Fis. Univ. Modena 49 (2001), 473–493.

[20] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, Berlin, 2011.

[21] F. J. Fernández-Polo, A. M. Peralta and M. I. Ramirez, *A Kadec–Pełczyński dichotomy-type theorem for preduals of JBW^*-algebras*, Israel J. Math., to appear.

[22] Y. Friedman and B. Russo, *Structure of the predual of a JBW^*-triple*, J. Reine Angew. Math. 356 (1985), 67–89.

[23] Y. Friedman and B. Russo, *The Gelfand–Naimark theorem for JB^*-triples*, Duke Math. J. 53 (1986), 139–148.

[24] Y. Friedman and B. Russo, *Conditional expectation and bicontractive projections on Jordan C^*-algebras and their generalizations*, Math. Z. 194 (1987), 227–236.

[25] M. González and A. Martínez-Abejón, *Ultrapowers of $L_1(\mu)$ and the subsequence splitting principle*, Israel J. Math. 122 (2001), 189–206.

[26] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monogr. Stud. Math. 21, Pitman, Boston, 1984.

[27] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[28] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[29] T. Ho, A. M. Peralta and B. Russo, *Ternary weakly amenable C^*-algebras and JB^*-triples*, Quart. J. Math. 64 (2013), 1109–1139.

[30] G. Horn, *Characterization of the predual and ideal structure of a JBW^*-triple*, Math. Scand. 61 (1987), 117–133.

[31] W. B. Johnson and J. Lindenstrauss, *Handbook of the Geometry of Banach Spaces, Volumes 1 and 2*, Elsevier, Amsterdam, 2001, 2003.

[32] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. 19 (1979), no. 217.

[33] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), 161–176.

[34] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 183 (1983), 503–529.

[35] W. Kaup, *Contractive projections on Jordan C^*-algebras and generalizations*, Math. Scand. 54 (1984), 95–100.

[36] A. M. Peralta, *Little Grothendieck's theorem for real JB^*-triples*, Math. Z. 237 (2001), 531–545.

[37] A. M. Peralta, *Some remarks on weak compactness in the dual space of a JB^*-triple*, Tohoku Math. J. (2) 58 (2006), 149–159.

[38] A. M. Peralta, *Positive definite hermitian mappings associated with tripotent elements*, Expo. Math. 33 (2015), 252–258.

[39] A. M. Peralta and H. Pfitzner, *Perturbation of ℓ_1-copies in preduals of JBW^*-triples*, preprint, 2014.

[40] A. M. Peralta and B. Russo, *Automatic continuity of derivations on C^*-algebras and JB^*-triples*, J. Algebra 399 (2014), 960–977.

[41] H. Pfitzner, *L-embedded Banach spaces and measure topology*, Israel J. Math. 205 (2015), 421–451.

[42] H. Pfitzner, *A note on asymptotically isometric copies of ℓ^1 and c_0*, Proc. Amer. Math. Soc. 129 (2001), 1367–1373.

[43] N. Randrianantoanina, *Kadec–Pełczyński decomposition for Haagerup L^p-spaces*, Math. Proc. Cambridge Philos. Soc. 132 (2002), 137–154.

[44] N. Randrianantoanina, *Embeddings of ℓ_p into non-commutative spaces*, J. Austral. Math. Soc. 74 (2003), 331–350.

[45] Y. Raynaud and Q. Xu, *On subspaces of non-commutative L_p-spaces*, J. Funct. Anal. 203 (2003), 149–196.

[46] Á. Rodríguez-Palacios, *On the strong* topology of a JBW^*-triple*, Quart. J. Math. Oxford Ser. (2) 42 (1991), 99–103.

[47] H. P. Rosenthal, *Sous-espaces de L^1*, Cours de troisième cycle, Univ. Paris VI, unpublished, 1979.

[48] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer, Berlin, 1971.
[49] B. Sims, *"Ultra"-techniques in Banach space theory*, Queen's Papers Pure Appl. Math. 60, Queen's Univ. Kingston, 1982.
[50] L. L. Stachó, *A projection principle concerning biholomorphic automorphisms*, Acta Sci. Math. (Szeged) 44 (1982), 99–124.
[51] L. W. Weis, *Banach lattices with the subsequence splitting property*, Proc. Amer. Math. Soc. 105 (1989), 87–96.

Antonio M. Peralta
Departamento de Análisis Matemático
Universidad de Granada
Facultad de Ciencias
18071 Granada, Spain

Hermann Pfitzner
Université d'Orléans
BP 6759
F-45067 Orléans Cedex 2, France
E-mail: pfitzner@labomath.univ-orleans.fr

Current address:
Department of Mathematics
College of Science
King Saud University
P.O. Box 2455-5
Riyadh 11451, Kingdom of Saudi Arabia
E-mail: aperalta@ugr.es

Received October 30, 2014
Revised version May 19, 2015 (8119)