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On property (β) of Rolewicz in Köthe–Bochner sequence spaces

by

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Abstract. We study property (β) in Köthe–Bochner sequence spaces $E(X)$, where E is any Köthe sequence space and X is an arbitrary Banach space. The question of whether or not this geometric property lifts from X and E to $E(X)$ is examined. We prove that if $\dim X = \infty$, then $E(X)$ has property (β) if and only if X has property (β) and E is orthogonally uniformly convex. It is also showed that if $\dim X < \infty$, then $E(X)$ has property (β) if and only if E has property (β). Our results essentially extend and improve those from [14] and [15].

1. Introduction. Köthe–Bochner spaces $E(X)$ of vector-valued functions are generalizations of the Lebesgue–Bochner and Orlicz–Bochner spaces. They have been investigated by many authors (see for example [2], [10], [14], [15], [18], [21], [22] and [28]). A survey of geometry in Köthe–Bochner spaces can be found in [29]. One of the fundamental problems in these spaces is the question whether or not a geometric property lifts from X and E to $E(X)$. Although the answer is often the same in the case of function and sequence Köthe–Bochner spaces, the really peculiar situation is when the respective criteria are different. Property (H) turns out to be one of such properties. This property is also known as the Radon–Riesz or Kadec–Klee property (KK) ([11]). Property (KK) in Köthe–Bochner spaces was studied in [1] and [22]. It is known that it lifts from X to $E(X)$ when E is a Köthe sequence space, but it need not lift if E is a Köthe function space ([1], [22] and [28]). The same situation is for the uniform Kadec–Klee property (UKK) and nearly uniform convexity (NUC) ([18]). Both are stronger notions than (KK) and were introduced by Huff in [11]. He proved that a Banach space is nearly uniformly convex if and only if it has the uniform Kadec–Klee property and is reflexive. Moreover, it is known that if X is a separable Banach space without the Schur property and E is a Köthe sequence space, then $E(X)$ has the uniform Kadec–Klee property iff X does

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and E is uniformly monotone. Furthermore, if X has the Schur property and E is uniformly monotone, then $E(X)$ has the uniform Kadec–Klee property ([18]). It has also been proved that if X is an infinite-dimensional Banach space, then $E(X)$ is nearly uniformly convex iff both E and X have the same property and E is uniformly monotone (proved independently in [18] and [27]). Furthermore, if X is finite-dimensional, then $E(X)$ is nearly uniformly convex iff E is nearly uniformly convex ([18] and [27]). However, the uniform Kadec–Klee property and nearly uniform convexity do not lift from X and E to $E(X)$ if E is a Köthe function space (see Remark 3 below).

In this paper we study property (β) in Köthe–Bochner sequence spaces $E(X)$, where E is any Köthe sequence space and X is an arbitrary Banach space. Property (β) was introduced by Rolewicz in [33]. He proved the implications (UC) $\Rightarrow$ (β) $\Rightarrow$ (NUC), where (UC) denotes uniform convexity. Moreover, the class of spaces with an equivalent norm with property (β) coincides neither with that of superreflexive spaces ([24] and [31]) nor with the class of nearly uniformly convexifiable spaces ([23]). It is known that property (β) coincides with reflexivity in Orlicz sequence spaces, and property (β) and uniform convexity are equivalent in Orlicz–Lorentz function spaces ([5] and [16]). This property was also studied in Calderón–Lozanovskiĭ spaces ([16]). One of the reasons that property (β) is important is that if a Banach space X has property (β), then both X and X^* have the fixed point property (FPP). For X, this follows from the implications (β) $\Rightarrow$ (NUC) and (NUC) $\Rightarrow$ (FPP) ([6] and [33]). Moreover, if $X \in (\beta)$, then X^* has normal structure ([26]) and hence the weak fixed point property (WFPP) (see Kirk [13]). Since (WFPP) and (FPP) coincide in reflexive spaces and property (β) implies reflexivity, it follows that X^* has the fixed point property.

We will show that if $\dim X = \infty$, then $E(X)$ has property (β) if and only if X has property (β) and E is orthogonally uniformly convex. It is also noted that if $\dim X < \infty$, then $E(X)$ has property (β) if and only if E has property (β). It is worth mentioning that in the function case the situation is different. Then property (β) does not lift from X and E to $E(X)$ (see Remark 3 below).

The orthogonal uniform convexity $(\mathrm{UC}^{\perp})$ was introduced in [16]. It is known that the implications

$$(\mathrm{UC}) \Rightarrow (\beta) \Rightarrow (\mathrm{UC}^{\perp}) \tag{1}$$

hold in any Köthe function space and the second implication cannot be reversed in general ([16], [17] and [33]). Moreover,

$$(\mathrm{UC}) \Rightarrow (\mathrm{UC}^{\perp}) \Rightarrow (\beta) \tag{2}$$

in any Köthe sequence space and the converse of any of these implications is not true in general ([17]). However, ($\mathrm{UC}^{\perp}$) $\Leftrightarrow$ (β) in any symmetric Köthe sequence space ([20]).

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of natural, real and non-negative real numbers, respectively. Let $(\mathbb{N}, 2^{\mathbb{N}}, m)$ be the counting measure space and $l_0 = l_0(m)$ be the linear space of all real sequences.

Let $E = (E, \leq, \|\cdot\|_E)$ be a *Banach sequence lattice* over the measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$, that is, E is a Banach space which is a subspace of l_0 endowed with the natural coordinatewise semi-order relation, and E satisfies the conditions:

(i) if $x \in E,\ y \in l_0,\ |y| \leq |x|$, i.e. $|y(i)| \leq |x(i)|$ for every $i \in \mathbb{N}$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists a sequence x in E that is positive on the whole $\mathbb{N}$ (see [30]).

Banach sequence lattices are often called *Köthe sequence spaces.*

A Banach lattice E is said to be *strictly monotone* ($E \in$ (SM)) if for every $0 \leq y \leq x$ with $y \neq x$ we have $\|y\|_E < \|x\|_E$. We say that a Banach lattice E is *uniformly monotone* ($E \in$ (UM)) if for every $q \in (0,1)$ there exists $p \in (0,1)$ such that for all $0 \leq y \leq x$ satisfying $\|x\|_E = 1$ and $\|y\|_E \geq q$ we have $\|x - y\|_E \leq 1 - p$ (see [9]). Then the *modulus* $p(\cdot)$ *of uniform monotonicity* of E is defined as follows:

$$p(q) = \inf\{1 - \|x - y\|_E : \|x\|_E = 1,\ \|y\|_E \geq q,\ 0 \leq y \leq x\}.$$

A Banach lattice E is called *order continuous* ($E \in$ (OC)) if for every $x \in E$ and every sequence $(x_m) \in E$ such that $0 \leq x_m \leq |x|$ and $x_m \to 0$ we have $\|x_m\|_E \to 0$ (see [30]).

For any subset A of X, we denote by $\mathrm{conv}(A)$ the convex hull of A. Let $(X, \|\cdot\|_X)$ be a real Banach space, and $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively.

A Banach space $(X, \|\cdot\|_X)$ is said to be *uniformly convex* ($X \in$ (UC)) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in S(X)$ the inequality $\|x - y\|_X \geq \varepsilon$ implies $\|x + y\|_X \leq 2(1 - \delta)$ (see [3]).

We say that for a given $\varepsilon > 0$ a sequence $(x_n) \subset X$ is *ε-separated* if

$$\mathrm{sep}\,(x_n) = \inf\{\|x_n - x_m\|_X : n \neq m\} > \varepsilon.$$

Although the original definition of property (β) uses the Kuratowski measure of noncompactness (see [33]), the following equivalent condition proved by Kutzarova in [25] is more convenient for our considerations.

THEOREM 1. *A Banach space X has property (β) iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in B(X)$ and each sequence (x_n) in*

$B(X)$ with $\operatorname{sep}(x_n) \geq \varepsilon$ *there is an index k for which*

$$\|x + x_k\|_X \leq 2(1-\delta). \tag{3}$$

A Banach space is said to be *nearly uniformly convex* ($X \in$ (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subset B(X)$ with $\operatorname{sep}(x_n) \geq \varepsilon$ we have $\operatorname{conv}((x_n)) \cap (1-\delta)B(X) \neq \emptyset$.

A Banach space X is said to have the *uniform Kadec–Klee property* ($X \in$ (UKK)) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that $\|x\|_X \leq 1-\delta$ whenever $(x_n) \subset B(X)$, $x_n \xrightarrow{w} x$ and $\operatorname{sep}(x_n) \geq \varepsilon$.

Recall that a Banach space X has the *Schur property* (written $X \in$ (SP)) if every weakly null sequence is norm null. Every Schur space is (UKK) and the converse is not true ([11]).

Now, let us define the type of spaces to be considered in this paper. For a real Banach space $(X, \|\cdot\|_X)$, denote by $\mathcal{M}(\mathbb{N}, X)$, or just $\mathcal{M}(X)$, the space of all sequences $x = (x(i))_{i=1}^{\infty}$ such that $x(i) \in X$ for all $i \in \mathbb{N}$. Define

$$\widetilde{x}(\cdot) = \|x(\cdot)\|_X \quad \text{for } x \in \mathcal{M}(X), \qquad E(X) = \{x \in \mathcal{M}(X) : \widetilde{x} \in E\}.$$

Then $E(X)$ equipped with the norm $\|x\| = \|\widetilde{x}\|_E$ becomes a Banach space and it is called a *Köthe–Bochner sequence space.*

2. Auxiliary lemmas. Write $r \wedge s = \min\{r,s\}$ and $r \vee s = \max\{r,s\}$ for $r,s \in \mathbb{R}$. For every $x \in X \setminus \{0\}$ let $\widehat{x} = x/\|x\|$.

Lemma 1 (Lemma 1.1 in [10]). *If $x, y \in X \setminus \{0\}$, then*

$$\|x+y\|_X \leq \big|\|x\|_X - \|y\|_X\big| + \{\|x\|_X \wedge \|y\|_X\}\|\widehat{x} + \widehat{y}\|_X.$$

Lemma 2 (Lemma 1 in [22]). *Let X be a separable Banach space and E be an order continuous Köthe sequence space. If $f_n, f \in E(X)$ and $f_n \xrightarrow{w} f$ in $E(X)$, then $f_n(i) \xrightarrow{w} f(i)$ in X for every $i \in \mathbb{N}$.*

Lemma 3 (Theorem 1 in [15]). *A Banach space X has property (β) if and only if for every $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that for each $x \in X \setminus \{0\}$ and each sequence (x_n) in $X \setminus \{0\}$ with* $\operatorname{sep}(x_n/\|x_n\|_X) \geq \varepsilon_0$ *there is an index k for which*

$$\left\|\frac{x+x_k}{2}\right\|_X \leq \frac{1}{2}(\|x\|_X + \|x_k\|_X)\left(1 - \frac{2\delta_0\{\|x\|_X \wedge \|x_k\|_X\}}{\|x\|_X + \|x_k\|_X}\right).$$

The following property was introduced in [16]:

Definition 1. We say that a Köthe space $(E, \|\cdot\|_E)$ is *orthogonally uniformly convex* ($E \in (\mathrm{UC}^{\perp})$) if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for any couple $x, y \in B(E)$ the inequality $\|x\chi_{A_{xy}}\|_E \vee \|y\chi_{A_{xy}}\|_E \geq \varepsilon$ implies $\|(x+y)/2\|_E \leq 1-\delta$, where

$$A_{xy} = \operatorname{supp} x \div \operatorname{supp} y, \qquad A \div B = (A \setminus B) \cup (B \setminus A).$$

Obviously if $E \in (\mathrm{UC})$, then $E \in (\mathrm{UC}^{\perp})$. It is known that any uniformly convex Banach function lattice is uniformly monotone ([9]). Moreover

LEMMA 4 (Lemma 3 in [16]). *Let E be any Köthe space. If $E \in (\mathrm{UC}^{\perp})$, then $E \in (\mathrm{UM})$.*

The converse implication is not true as examples of L_1 and l_1 show.

3. Results. It is known that if $\dim X = \infty$ and $E(X) \in (\mathrm{NUC})$, then $E \in (\mathrm{UM})$ ([18] and [27]). If we consider property (β) in an analogous situation, we get the following

THEOREM 2. *Let E be a Köthe sequence space and X be an infinite-dimensional Banach space. Then $E \in (\mathrm{UC}^{\perp})$ whenever $E(X) \in (\beta)$.*

Proof. Since X is isometrically embedded in $E(X)$ and property (β) is inherited by subspaces, X has property (β). Hence X is reflexive. Assume that $E \notin (\mathrm{UC}^{\perp})$. Then there exists $\varepsilon > 0$ and sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in $B(E)$ with

$$\|x_n \chi_{A_{x_n y_n}}\|_E \vee \|y_n \chi_{A_{x_n y_n}}\|_E \ge \varepsilon, \qquad \|x_n + y_n\|_E > 2(1 - 1/n). \tag{4}$$

Define $A_n = A_{x_n y_n}$. Divide A_n into two disjoint subsets $A_n^1 = \operatorname{supp} x_n \setminus \operatorname{supp} y_n$ and $A_n^2 = \operatorname{supp} y_n \setminus \operatorname{supp} x_n$. We divide the proof into two parts.

1. Suppose that $\|x_n \chi_{A_n^1}\|_E \ge \varepsilon$. Fix $n \in \mathbb{N}$. Since X is a reflexive infinite-dimensional Banach space, it fails to have the Schur property. Consequently, there exists a sequence $(u_k)_{k=1}^{\infty}$ in $S(X)$ such that $u_k \xrightarrow{w} 0$. Define $v_k^n(i) = u_k x_n(i)$ for $i \in A_n^1$. Then $v_k^n(i) \xrightarrow{w} 0$ as $k \to \infty$ for all $i \in A_n^1$ and $\|v_k^n(i)\|_X = |x_n(i)|$ for all $i \in A_n^1$ and $k \in \mathbb{N}$. Then, applying the Hahn–Banach theorem, it is easy to prove that for every $i \in \mathbb{N}$ there exists a subsequence $(w_k^n(i))_{k=1}^{\infty}$ of $(v_k^n(i))_{k=1}^{\infty}$ such that $\operatorname{sep}(w_k^n(i))_X \ge |x_n(i)|/2$. Using the well known diagonal method, we conclude that for every $n \in \mathbb{N}$ there exists $(w_k^n)_{k=1}^{\infty}$ in $E(X)$ such that

$$\operatorname{sep}(w_k^n(i))_X \ge |x_n(i)|/2 \quad \text{for every } i \in A_n^1. \tag{5}$$

Let $z \in S(X)$. For $n, k \in \mathbb{N}$ define

$$f_k^n = z x_n \chi_{\mathbb{N} \setminus A_n^1} + w_k^n \chi_{A_n^1}, \qquad f^n = z y_n.$$

Then $f_k^n, f^n \in B(E(X))$ for all $n, k \in \mathbb{N}$. Moreover, by (5), we get

$$\|f_i^n - f_j^n\| = \big\| \|w_i^n - w_j^n\|_X \, \chi_{A_n^1} \big\|_E \ge \big\| (|x_n(\cdot)|/2) \chi_{A_n^1} \big\|_E \ge \varepsilon/2$$

for all $n, i, j \in \mathbb{N}$, $i \ne j$. Then $\operatorname{sep}((f_k^n)_{k=1}^{\infty})_{E(X)} \ge \varepsilon/2$ for every $n \in \mathbb{N}$. On the other hand, by (4), it follows that

$$\begin{aligned} \|f^n + f_k^n\| &= \big\| \|z(x_n + y_n)(\cdot)\|_X \, \chi_{\mathbb{N} \setminus A_n^1} + \|w_k^n(\cdot)\|_X \, \chi_{A_n^1} \big\|_E \\ &= \|x_n + y_n\|_E > 2(1 - 1/n) \end{aligned}$$

for all $n, k \in \mathbb{N}$. By Theorem 1 we conclude that $E(X) \notin (\beta)$.

2. If $\|y_n \chi_{A_n^2}\|_E \geq \varepsilon$, then the proof is analogous.

REMARK 1. The claim of Theorem 2 is a little surprising. Notice that if $E(X) \in (\beta)$, then obviously $E \in (\beta)$. However, orthogonal uniform convexity is essentially stronger than property (β) in Köthe sequence spaces ([17], [20]).

REMARK 2. The assertion of Theorem 2 does not hold for $X = \mathbb{R}$, because there exists a Köthe sequence space E with property (β) which is not uniformly monotone ([17] or [24]).

THEOREM 3. *Let E be a Köthe sequence space and X be an infinite-dimensional Banach space. Then $E(X)$ has property (β) if and only if X has property (β) and E is orthogonally uniformly convex.*

Proof. Necessity. Since X is isometrically embedded in $E(X)$, we have $X \in (\beta)$. By Theorem 2, E is orthogonally uniformly convex.

Sufficiency. Let $\varepsilon > 0$. In view of Lemma 3, property (β) can be equivalently considered on the unit sphere in place of the unit ball. Take $x, x_n \in S(E(X)), n = 1, 2, \ldots$, such that $\operatorname{sep}(x_n)_{E(X)} \geq \varepsilon$. Let

$$0 < \lambda < \varepsilon/128, \qquad 1 - \lambda/2 < u < 1. \tag{6}$$

For $n \neq m$ define

$$A_{n,m} = \left\{ i \in \mathbb{N} : \frac{\|x_n(i)\|_X \wedge \|x_m(i)\|_X}{\|x_n(i)\|_X \vee \|x_m(i)\|_X} < u \right\}, \qquad B_{n,m} = \mathbb{N} \setminus A_{n,m}.$$

First we prove that passing to a subsequence if necessary, we may assume that either

$$\begin{aligned} &\|(x_n - x_m)\chi_{A_{n,m}}\| \geq \varepsilon/64 \qquad \text{for all } n \neq m, \text{ or} \\ &\|(x_n - x_m)\chi_{A_{n,m}}\| < \varepsilon/64 \qquad \text{for all } n \neq m. \end{aligned} \tag{7}$$

1. Consider the element x_1 and the sequence $(x_n)_{n=2}^\infty$. There exists a subsequence $(x_n^{(1)})_{n=1}^\infty$ of $(x_n)_{n=2}^\infty$ such that either

$$\begin{aligned} &\|(x_1 - x_n^{(1)})\chi_{A_{1,n}}\| \geq \varepsilon/64 \qquad \text{for every } n \in \mathbb{N}, \text{ or} \\ &\|(x_1 - x_n^{(1)})\chi_{A_{1,n}}\| < \varepsilon/64 \qquad \text{for every } n \in \mathbb{N}. \end{aligned}$$

The sets $A_{1,n}$ correspond to the pairs $(x_1, x_n^{(1)})$. Set $w_1^{(1)} = x_1$ and $w_{n+1}^{(1)} = x_n^{(1)}$ for $n \in \mathbb{N}$.

2. Take the element $x_1^{(1)}$ and the sequence $(x_n^{(1)})_{n=2}^\infty$. There exists a subsequence $(x_n^{(2)})_{n=1}^\infty$ of $(x_n^{(1)})_{n=2}^\infty$ such that either

$$\begin{aligned} &\|(x_1^{(1)} - x_n^{(2)})\chi_{A_{1,n}}\| \geq \varepsilon/64 \qquad \text{for every } n \in \mathbb{N}, \text{ or} \\ &\|(x_1^{(1)} - x_n^{(2)})\chi_{A_{1,n}}\| < \varepsilon/64 \qquad \text{for every } n \in \mathbb{N}. \end{aligned}$$

Set $w_1^{(2)} = x_1^{(1)}$ and $w_{n+1}^{(2)} = x_n^{(2)}$ for $n \in \mathbb{N}$. Continuing in this way, we conclude that there exists a sequence $(j_k)_{k=1}^{\infty}$ of natural numbers and a sequence of subsequences $(w_n^{(j_k)})_{n=1}^{\infty}, k = 1, 2, \ldots$, such that

$$(w_n^{(j_1)})_{n=1}^{\infty} \supset (w_n^{(j_2)})_{n=1}^{\infty} \supset \ldots$$

and either

$$\|(w_1^{(j_k)} - w_n^{(j_k)})\chi_{A_{1,n}}\| \geq \varepsilon/64 \quad \text{for all } k, n \in \mathbb{N},\ n \geq 2, \text{ or}$$
$$\|(w_1^{(j_k)} - w_n^{(j_k)})\chi_{A_{1,n}}\| < \varepsilon/64 \quad \text{for all } k, n \in \mathbb{N},\ n \geq 2.$$

Define $y_n = w_1^{(j_n)}$ for $n \in \mathbb{N}$. The subsequence (y_n) of (x_n) satisfies the required condition (7).

We divide the remainder of the proof into two parts.

I. Suppose that

$$\|(x_n - x_m)\chi_{A_{n,m}}\| \geq \varepsilon/64 \quad \text{for all } n \neq m. \tag{8}$$

We claim that for all $v, z \in X$ satisfying

$$\|z\|_X \wedge \|v\|_X < u(\|z\|_X \vee \|v\|_X)$$

we have

$$\|z - v\|_X \leq \big|\|z\|_X - \|v\|_X\big|\bigg(1 + \frac{2u}{1-u}\bigg), \tag{9}$$

where $u \in (0,1)$ is defined in (6). If $\|z\|_X \geq \|v\|_X$, then $\|z\|_X - \|v\|_X > (1/u - 1)\|v\|_X$. Hence

$$\begin{aligned}\|z - v\|_X &\leq \|z\|_X + \|v\|_X = \|z\|_X - \|v\|_X + 2\|v\|_X \\ &\leq \|z\|_X - \|v\|_X + 2u\,\frac{\|z\|_X - \|v\|_X}{1-u} = (\|z\|_X - \|v\|_X)\bigg(1 + \frac{2u}{1-u}\bigg).\end{aligned}$$

If $\|z\|_X < \|v\|_X$, the proof is analogous. Applying (8), (9) and the definition of $A_{n,m}$, we get

$$\begin{aligned}\frac{\varepsilon}{64} &\leq \|(x_n - x_m)\chi_{A_{n,m}}\| = \big\|\|(x_n - x_m)(\cdot)\|_X\,\chi_{A_{n,m}}\big\|_E \\ &\leq \frac{1+u}{1-u}\,\big\|(\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X)\chi_{A_{n,m}}\big\|_E \\ &\leq \frac{1+u}{1-u}\,\big\|\|x_n(\cdot)\|_X - \|x_m(\cdot)\|_X\big\|_E\end{aligned}$$

for all $n \neq m$. Set

$$y(\cdot) = \|x(\cdot)\|_X, \qquad y_n(\cdot) = \|x_n(\cdot)\|_X.$$

Then $\|y\|_E = \|y_n\|_E = 1$ and $\operatorname{sep}(y_n)_E \geq (1-u)\varepsilon/64(1+u)$. Since $E \in (\mathrm{UC}^{\perp})$, by (2) we get $E \in (\beta)$. Take

$$\delta_1 = \delta\bigg(\frac{(1-u)\varepsilon}{64(1+u)}\bigg)$$

from Theorem 1. Then there exists $k \in \mathbb{N}$ such that $\|y + y_k\|_E \leq 2(1 - \delta_1)$. Finally,

$$\|x + x_k\| = \big\|\|(x + x_k)(\cdot)\|_X\big\|_E \leq \|y + y_k\|_E \leq 2(1 - \delta_1).$$

II. Assume that

$$\|(x_n - x_m)\chi_{A_{n,m}}\| < \varepsilon/64 \quad \text{for all } n \neq m. \tag{10}$$

Then

$$\|(x_n - x_m)\chi_{B_{n,m}}\| \geq 63\varepsilon/64 \quad \text{for all } n \neq m.$$

By Lemma 4 we conclude that $E \in$ (UM). Moreover, (UM) $\Rightarrow$ (OC) in any Banach function lattice. It is known that a Köthe sequence space is order continuous iff it is absolutely continuous, i.e. for every $x \in E$ we have $\lim_{n\to\infty} \|x - x^{(n)}\|_E = 0$, where $x^{(n)} = (x(1), \ldots, x(n), 0, 0, \ldots)$ (see [4]). Hence we may assume that $0 < \operatorname{card} B_{n,m} < \infty$ for every $n \neq m$ and

$$\|(x_n - x_m)\chi_{B_{n,m}}\| \geq \varepsilon/4. \tag{11}$$

We will prove that, passing to subsets of $B_{n,m}$ for all $n \neq m$ if necessary, and denoting them again by $B_{n,m}$, we get

$$\|(x_n - x_m)\chi_{B_{n,m}}\| \geq \varepsilon/16 \tag{12}$$

and

$$0 < \operatorname{card} B < \infty, \tag{13}$$

where

$$B_n = \bigcup_{m=n+1}^{\infty} B_{n,m}, \qquad B = \bigcup_{n=1}^{\infty} B_n.$$

Suppose that (13) does not hold with B being the sum of the original sets $B_{n,m}$. We consider two cases:

(a) Assume that $\operatorname{card} B_n = \infty$ for some $n \in \mathbb{N}$. Define $D_k = \bigcup_{m=n+1}^{k} B_{n,m}$ for $k = n+1, n+2, \ldots$ Then $\|x_n(\cdot)\|_X \geq \|x_n(\cdot)\|_X\, \chi_{B_n \setminus D_k} \to 0$ as $k \to \infty$. Since E is order continuous, $\big\|\|x_n(\cdot)\|_X\, \chi_{B_n \setminus D_k}\big\|_E \to 0$ as $k \to \infty$. Take $k_0 \in \mathbb{N}$ such that

$$\big\|\|x_n(\cdot)\|_X\, \chi_{B_n \setminus D_{k_0}}\big\|_E < u\varepsilon/16. \tag{14}$$

Define $B^1_{n,m} = B_{n,m} \cap D_{k_0}$ for $m > n$ and $B^1_n = \bigcup_{m=n+1}^{\infty} B^1_{n,m}$. Then $\operatorname{card} B^1_n < \infty$ because $\operatorname{card} D_{k_0} < \infty$. Moreover, we will show that

$$\|(x_n - x_m)\chi_{B^1_{n,m}}\| \geq \varepsilon/8 \quad \text{for every } m = n+1, n+2, \ldots \tag{15}$$

In view of (11), inequality (15) is clear for $m = n+1, \ldots, k_0$, because then $B^1_{n,m} = B_{n,m}$. Suppose that (15) does not hold for some $m > k_0$. Then, by (11), (14) and the definition of $B_{n,m}$, we get

$$\begin{aligned}\varepsilon/4 &\le \|(x_n-x_m)\chi_{B_{n,m}}\| \le \|(x_n-x_m)\chi_{B^1_{n,m}}\| + \|(x_n-x_m)\chi_{B_{n,m}\setminus D_{k_0}}\| \\ &\le \varepsilon/8 + \|x_n\chi_{B_{n,m}\setminus D_{k_0}}\| + \frac{1}{u}\,\|x_n\chi_{B_{n,m}\setminus D_{k_0}}\| \\ &\le \varepsilon/8 + \|x_n\chi_{B_n\setminus D_{k_0}}\| + \frac{1}{u}\,\|x_n\chi_{B_n\setminus D_{k_0}}\| < \varepsilon/4,\end{aligned}$$

which is a contradiction.

(b) By case (a) and inequality (15) we may assume that $\operatorname{card} B_n < \infty$ for every $n \in \mathbb{N}$ and

$$\|(x_n-x_m)\chi_{B_{n,m}}\| \ge \varepsilon/8 \quad \text{for all } n \ne m. \tag{16}$$

Suppose $\operatorname{card} B = \infty$ with B being the union of the sets $B_{n,m}$ constructed in case (a). Set $D_k = \bigcup_{n=1}^k B_n$ for $k = 1, 2, \ldots$ Then $0 \leftarrow \|x_1(\cdot)\|_X\,\chi_{B\setminus D_k} \le \|x_1(\cdot)\|_X$ as $k \to \infty$. Since E is order continuous, $\big\|\|x_1(\cdot)\|_X\,\chi_{B\setminus D_k}\big\|_E \to 0$ as $k \to \infty$. Take $k_0 \in \mathbb{N}$ such that

$$\big\|\|x_1(\cdot)\|_X\,\chi_{B\setminus D_{k_0}}\big\|_E < u\varepsilon/128. \tag{17}$$

Define $C_{n,m} = B_{n,m} \cap D_{k_0}$ for $n \ne m$, $C_n = \bigcup_{m=n+1}^\infty C_{n,m}$ for $n \in \mathbb{N}$, and $C = \bigcup_{n=1}^\infty C_n$. Then $\operatorname{card} C < \infty$ since $\operatorname{card} D_{k_0} < \infty$. Moreover, we will prove that

$$\|(x_n-x_m)\chi_{C_{n,m}}\| \ge \varepsilon/16 \quad \text{for all } m \ne n. \tag{18}$$

By (16), inequality (18) is clear for every $n = 1, \ldots, k_0$ and $m > n$, because then $C_{n,m} = B_{n,m}$. Suppose that (18) does not hold for some $n > k_0$ and $m > n$. Divide the set $B \setminus D_{k_0}$ into the following subsets:

$$\begin{aligned}F_1 &= \{i \in B\setminus D_{k_0} : \|x_1(i)\|_X \wedge \|x_n(i)\|_X \ge u(\|x_1(i)\|_X \vee \|x_n(i)\|_X)\},\\ F_2 &= \{i \in B\setminus D_{k_0} : \|x_1(i)\|_X \wedge \|x_n(i)\|_X < u(\|x_1(i)\|_X \vee \|x_n(i)\|_X)\},\\ F_3 &= \{i \in B\setminus D_{k_0} : \|x_1(i)\|_X \wedge \|x_m(i)\|_X \ge u(\|x_1(i)\|_X \vee \|x_m(i)\|_X)\},\\ F_4 &= \{i \in B\setminus D_{k_0} : \|x_1(i)\|_X \wedge \|x_m(i)\|_X < u(\|x_1(i)\|_X \vee \|x_m(i)\|_X)\}.\end{aligned}$$

Notice that $F_2 \subset A_{1,n}$ and $F_4 \subset A_{1,m}$. Then, by (10), (16) and (17), we get

$$\begin{aligned}\varepsilon/8 &\le \|(x_n-x_m)\chi_{B_{n,m}}\| \le \|(x_n-x_m)\chi_{C_{n,m}}\| + \|(x_n-x_m)\chi_{B_{n,m}\setminus D_{k_0}}\| \\ &< \varepsilon/16 + \|(x_n-x_1)\chi_{B\setminus D_{k_0}}\| + \|(x_1-x_m)\chi_{B\setminus D_{k_0}}\| \\ &\le \varepsilon/16 + \|(x_n-x_1)\chi_{F_1}\| + \|(x_n-x_1)\chi_{F_2}\| \\ &\quad + \|(x_1-x_m)\chi_{F_3}\| + \|(x_1-x_m)\chi_{F_4}\| \\ &\le \varepsilon/16 + (1+1/u)\|x_1\chi_{F_1}\| + \|(x_n-x_1)\chi_{A_{1,n}}\| \\ &\quad + (1+1/u)\|x_1\chi_{F_3}\| + \|(x_1-x_m)\chi_{A_{1,m}}\| < \varepsilon/8,\end{aligned}$$

which is a contradiction. Hence we conclude that conditions (12) and (13) hold with $B_{n,m}, B_n$ and B replaced by $C_{n,m}, C_n$ and C, respectively.

We claim that for all $n \neq m$ there exists $i \in B$ such that

$$\begin{aligned} &\|x_n(i) - x_m(i)\|_X \ge \lambda(\|x_n(i)\|_X \vee \|x_m(i)\|_X), \\ &\|x_n(i)\|_X \wedge \|x_m(i)\|_X \ge u(\|x_n(i)\|_X \vee \|x_m(i)\|_X), \end{aligned} \tag{19}$$

where λ, u are defined in (6). Indeed, if not, then there exist $n, m \in \mathbb{N}$ with $n \neq m$ such that for any $i \in B$, either

$$\begin{aligned} &\|x_n(i) - x_m(i)\|_X < \lambda(\|x_n(i)\|_X \vee \|x_m(i)\|_X), \quad \text{or} \\ &\|x_n(i)\|_X \wedge \|x_m(i)\|_X < u(\|x_n(i)\|_X \vee \|x_m(i)\|_X). \end{aligned} \tag{20}$$

In view of inequalities (6), (10), (12) and (20), we get

$$\varepsilon/16 \le \|(x_n - x_m)\chi_{B_{n,m}}\| \le \|(x_n - x_m)\chi_B\| \le 2\lambda + \|(x_n - x_m)\chi_{A_{n,m}}\| < \varepsilon/32.$$

This contradiction proves the claim. We will prove that:

(+) there exists a subset $B_0 \subset B$ and a subsequence $(z_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(z_n(i), z_m(i))$ satisfies conditions (19) for all $n \neq m$, $i \in B_0$ and $(z_n(i), z_m(i))$ satisfies conditions (20) for all $n \neq m$ and $i \in B \backslash B_0$.

Denote by F_B the family of all non-empty subsets of B. Since $\operatorname{card} B < \infty$, we have $\operatorname{card} F_B < \infty$.

1. Consider the element x_1 and the sequence $(x_n)_{n=2}^\infty$. Since $\operatorname{card} B < \infty$, applying condition (19) we conclude that there exists a subsequence $(x_n^{(1)})_{n=1}^\infty$ of $(x_n)_{n=2}^\infty$ and a set $B_1 \in F_B$ such that

$$\begin{aligned} &(x_1(i), x_n^{(1)}(i)) \text{ satisfies (19) for all } n \in \mathbb{N} \text{ and } i \in B_1, \\ &(x_1(i), x_n^{(1)}(i)) \text{ satisfies (20) for all } n \in \mathbb{N} \text{ and } i \in B \setminus B_1. \end{aligned}$$

Define $y_1^{(1)} = x_1$ and $y_{n+1}^{(1)} = x_n^{(1)}$ for $n \in \mathbb{N}$.

2. Consider the element $x_1^{(1)}$ and the sequence $(x_n^{(1)})_{n=2}^\infty$. There exists a subsequence $(x_n^{(2)})_{n=1}^\infty$ of $(x_n^{(1)})_{n=2}^\infty$ and a set $B_2 \in F_B$ such that

$$\begin{aligned} &(x_1^{(1)}(i), x_n^{(2)}(i)) \text{ satisfies (19) for all } n \in \mathbb{N} \text{ and } i \in B_2, \\ &(x_1^{(1)}(i), x_n^{(2)}(i)) \text{ satisfies (20) for all } n \in \mathbb{N} \text{ and } i \in B \setminus B_2. \end{aligned}$$

Set $y_1^{(2)} = x_1^{(1)}$ and $y_{n+1}^{(2)} = x_n^{(2)}$ for $n \in \mathbb{N}$. Since $\operatorname{card} F_B < \infty$, proceeding analogously we conclude that there exists a set $B_0 \in F_B$, a sequence $(j_k)_{k=1}^\infty$ of natural numbers and a sequence of subsequences $(y_n^{(j_k)})_{n=1}^\infty$, $k = 1, 2, \ldots$, such that

$$(y_n^{(j_1)})_{n=1}^\infty \supset (y_n^{(j_2)})_{n=1}^\infty \supset \ldots$$

and for every $k \in \mathbb{N}$,

$$\begin{aligned} &(y_1^{(j_k)}(i), y_n^{(j_k)}(i)) \text{ satisfies (19) for all } n \in \mathbb{N},\ n \ge 2, \text{ and } i \in B_0, \\ &(y_1^{(j_k)}(i), y_n^{(j_k)}(i)) \text{ satisfies (20) for all } n \in \mathbb{N},\ n \ge 2 \text{ and } i \in B \setminus B_0. \end{aligned}$$

Define $z_n = y_1^{(j_n)}$ for $n \in \mathbb{N}$. In this way we have constructed the sequence $(z_n)_{n=1}^{\infty}$ satisfying condition (+). Denote this subsequence of (x_n) again by (x_n). We will prove that

$$\|x_n \chi_{B_0}\| \geq \varepsilon/64 \tag{21}$$

for all $n \in \mathbb{N}$ except at most one element. Suppose to the contrary that $\|x_n \chi_{B_0}\| < \varepsilon/64$ for $n \in \{n_1, n_2\}$, $n_1 \neq n_2$. By (+) we obtain $B \setminus B_0 = D_1 \cup D_2$, where

$$D_1 = \{i \in B \setminus B_0 : \|x_{n_1}(i) - x_{n_2}(i)\|_X < \lambda(\|x_{n_1}(i)\|_X \vee \|x_{n_2}(i)\|_X)\},$$
$$D_2 = \{i \in B \setminus B_0 : \|x_{n_1}(i)\|_X \wedge \|x_{n_2}(i)\|_X < u(\|x_{n_1}(i)\|_X \vee \|x_{n_2}(i)\|_X)\}.$$

Notice that $D_2 \subset A_{n_1,n_2}$. Hence, by (12), we get

$$\begin{aligned}\varepsilon/16 &\leq \|(x_{n_1} - x_{n_2})\chi_{B_{n_1,n_2}}\| \leq \|(x_{n_1} - x_{n_2})\chi_B\| \\ &\leq \|(x_{n_1} - x_{n_2})\chi_{B_0}\| + \|(x_{n_1} - x_{n_2})\chi_{B \setminus B_0}\| \\ &< \|x_{n_1}\chi_{B_0}\| + \|x_{n_2}\chi_{B_0}\| + 2\lambda + \|(x_{n_1} - x_{n_2})\chi_{A_{n_1,n_2}}\|.\end{aligned}$$

Then, by (6), we get $\|(x_{n_1} - x_{n_2})\chi_{A_{n_1,n_2}}\| \geq \varepsilon/64$, which contradicts (10). This proves inequality (21). Hereafter we assume that (21) is satisfied for every $n \in \mathbb{N}$.

For $x \in X \setminus \{0\}$ set $\widehat{x} = x/\|x\|_X$. We claim that for any $w, z \in B(X)$ satisfying $\|w\|_X \wedge \|z\|_X \geq u(\|w\|_X \vee \|z\|_X)$ and $\|w - z\|_X \geq \lambda$, we have

$$\|\widehat{w} - \widehat{z}\|_X \geq \lambda/2. \tag{22}$$

Indeed, by Lemma 1, we get

$$\begin{aligned}\lambda \leq \|w - z\|_X &\leq \big|\|w\|_X - \|z\|_X\big| + (\|w\|_X \wedge \|z\|_X)(\|\widehat{w} - \widehat{z}\|_X) \\ &\leq 1 - u + \|\widehat{w} - \widehat{z}\|_X,\end{aligned}$$

which proves the claim in view of (6). Fix $i \in B_0$. Let

$$w_{nm}(i) = \frac{x_n(i)}{\|x_n(i)\|_X \vee \|x_m(i)\|_X}, \qquad z_{nm}(i) = \frac{x_m(i)}{\|x_n(i)\|_X \vee \|x_m(i)\|_X}.$$

Then $w_{nm}(i), z_{nm}(i) \in B(X)$. Moreover, condition (+) yields

$$\begin{aligned}\|w_{nm}(i)\|_X \wedge \|z_{nm}(i)\|_X &\geq u(\|w_{nm}(i)\|_X \vee \|z_{nm}(i)\|_X), \\ \|w_{nm} - z_{nm}\|_X &\geq \lambda.\end{aligned}$$

Since $\widehat{w_{nm}(i)} = \widehat{x_n(i)}$ and $\widehat{z_{nm}(i)} = \widehat{x_m(i)}$, applying (22) with w, z being w_{nm}, z_{nm}, respectively, we get

$$\|\widehat{x_n(i)} - \widehat{x_m(i)}\|_X \geq \lambda/2 \tag{23}$$

for all $n \neq m$ and $i \in B_0$.

Let $\delta_E^{\perp}(\cdot)$ be the function $\delta(\cdot)$ from Definition 1. Define the constants

$$\delta_2 = \delta_E^{\perp}(\varepsilon/64), \quad \delta_3 = \delta_E^{\perp}(\varepsilon/128), \quad 0 < \alpha < \min\{\delta_2/8, \delta_3/6, \varepsilon/128\}. \tag{24}$$

In the remaining part of the proof we will consider a few cases.

II.1. Assume that

$$\|x\chi_{B_0}\| < 8\alpha. \tag{25}$$

Let $z_1 = \|x(\cdot)\|_X \, \chi_{\mathbb{N}\setminus B_0}$ and $z_2 = \|x_n(\cdot)\|_X$ for some $n \in \mathbb{N}$. Then $z_1, z_2 \in B(E)$. Define $G = \operatorname{supp} z_1 \div \operatorname{supp} z_2$. By (21), we get $\|z_2\chi_G\|_E \ge \varepsilon/64$. Since $E \in (\mathrm{UC}^{\perp})$, we conclude that $\|z_1 + z_2\|_E \le 2(1-\delta_2)$, where δ_2 is defined in (24). Consequently, by (24) and (25), we get

$$\begin{aligned}\|x + x_n\| &\le \big\| \|x(\cdot)\|_X \, \chi_{B_0} + \|x(\cdot)\|_X \, \chi_{\mathbb{N}\setminus B_0} + \|x_n(\cdot)\|_X \big\|_E \\ &\le 8\alpha + \|z_1 + z_2\|_E \le 8\alpha + 2 - 2\delta_2 \le 2(1-\delta_2/2).\end{aligned}$$

II.2. Suppose that

$$\|x\chi_{B_0}\| \ge 8\alpha. \tag{26}$$

Divide the set B_0 into two disjoint subsets

$$C = \{i \in B_0 : \|x(i)\|_X \wedge \|x_1(i)\|_X \ge \alpha(\|x(i)\|_X \vee \|x_1(i)\|_X)\},$$

$$D = \{i \in B_0 : \|x(i)\|_X \wedge \|x_1(i)\|_X < \alpha(\|x(i)\|_X \vee \|x_1(i)\|_X)\}.$$

II.2.1. Assume that

$$\|x\chi_C\| \ge 4\alpha.$$

Take $\delta_0 = \delta_0(\lambda/2)$ from Lemma 3. For every $i \in C$ consider the element $x(i) \in X$ and the sequence $(x_n(i))_{n=1}^{\infty}$ in X. By the definition of C and B_0, we have $x(i)$, $x_n(i) \ne 0$ for every $n \in \mathbb{N}$. By (23) we get $\operatorname{sep}(x_n(i)/\|x_n(i)\|_X) \ge \lambda/2$ for every $i \in C$. Hence, by Lemma 3, there exists $k_0 = k_0(i) \in \mathbb{N}$ such that

$$\begin{aligned}&\left\| \frac{x(i) + x_{k_0}(i)}{2} \right\|_X \\ &\qquad \le \frac{\|x(i)\|_X + \|x_{k_0}(i)\|_X}{2} \left(1 - \frac{2\delta_0\{\|x(i)\|_X \wedge \|x_{k_0}(i)\|_X\}}{\|x(i)\|_X + \|x_{k_0}(i)\|_X} \right).\end{aligned} \tag{27}$$

For every $i \in C$ and any subsequence $(u_n(i))_{n=1}^{\infty}$ of $(x_n(i))_{n=1}^{\infty}$, define

$$N(i, (u_n(i))) = \{k \in \mathbb{N} : x(i), u_k(i) \text{ satisfy (27)}\}.$$

Let $i_1 \in C$. Property (β) of X implies that $\operatorname{card} N(i_1, (x_n(i_1))) = \infty$. Thus, we can find a subsequence $(x_{n_k}(i_1))_{k=1}^{\infty}$ such that $x(i_1), x_{n_k}(i_1)$ satisfy (27) for every $k \in \mathbb{N}$. Take $i_2 \in C$ and consider the sequence $(x_{n_k}(i_2))_{k=1}^{\infty}$. Similarly, $\operatorname{card} N(i_2, (x_{n_k}(i_2))) = \infty$. Consequently, there exists a subsequence $(x_{n_{k_j}}(i_2))_{j=1}^{\infty}$ such that $x(i_2), x_{n_{k_j}}(i_2)$ satisfy (27) for every $j \in \mathbb{N}$. After a finite number of steps we obtain a subsequence $(x_m)_{m=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x(i), x_m(i)$ satisfy (27) for all $i \in C$ and $m \in \mathbb{N}$. Since, by condition (+), $\|x_m(i)\|_X \wedge \|x_1(i)\|_X \ge u(\|x_m(i)\|_X \vee \|x_1(i)\|_X)$ for all $i \in B_0$ and $m \in \mathbb{N}$, we have

$$\|x(i)\|_X \wedge \|x_m(i)\|_X \ge \alpha u(\|x(i)\|_X \vee \|x_m(i)\|_X) \quad \text{for all } m \in \mathbb{N} \text{ and } i \in C.$$

Consequently, by (27),

$$\left\|\frac{x(i)+x_m(i)}{2}\right\|_X \le \frac{1}{2}\,(1-\eta)(\|x(i)\|_X + \|x_m(i)\|_X)$$

for all $m \in \mathbb{N}$ and $i \in C$, where $\eta = 2\delta_0\alpha u/(1+\alpha u)$. Hence

$$\left\|\frac{x+x_m}{2}(\cdot)\right\|_X \le \frac{\|x(\cdot)\|_X + \|x_m(\cdot)\|_X}{2} - \frac{\eta}{2}\,(\|x(\cdot)\|_X + \|x_m(\cdot)\|_X)\chi_C$$

for every $m \in \mathbb{N}$. Denote by $p(\cdot)$ the modulus of uniform monotonicity of E. Then, by the uniform monotonicity of E, we get $\|(x+x_m)/2\| \le 1 - p_1$ for every $m \in \mathbb{N}$, where $p_1 = p(2\alpha\eta)$.

II.2.2. Suppose that

$$\|x\chi_C\| < 4\alpha. \tag{28}$$

Then, by (26), $\|x\chi_D\| \ge 4\alpha$. Let

$$D^1 = \{i \in D : \|x(i)\|_X = \|x_1(i)\|_X \vee \|x(i)\|_X\}, \qquad D^2 = D \setminus D^1.$$

If $\|x\chi_{D^2}\| \ge 2\alpha$, then $\|x_1\chi_{D^2}\| > 2$, a contradiction. Hence $\|x\chi_{D^2}\| < 2\alpha$. Consequently, by (28),

$$\|x\chi_{D^2\cup C}\| < 6\alpha. \tag{29}$$

On the other hand, $\|x_1\chi_{D^1}\| < \alpha$. Hence, by (21) and (24), we get

$$\|x_1\chi_{D^2\cup C}\| \ge \varepsilon/128. \tag{30}$$

Let $z_1 = \|x(\cdot)\|_X\,\chi_{\mathbb{N}\setminus(D^2\cup C)}$ and $z_2 = \|x_1(\cdot)\|_X$. Then $z_1, z_2 \in B(E)$. Define $G = \operatorname{supp} z_1 \div \operatorname{supp} z_2$. Hence, by (30), $\|z_2\chi_G\|_E \ge \varepsilon/128$. Since $E \in (\mathrm{UC}^{\perp})$, we conclude that $\|z_1+z_2\|_E \le 2(1-\delta_3)$, where δ_3 is defined in (24). Consequently, by (24) and (29), we get

$$\begin{aligned}\|x+x_1\| &\le \big\|\|x(\cdot)\|_X\,\chi_{D^2\cup C} + \|x(\cdot)\|_X\,\chi_{\mathbb{N}\setminus(D^2\cup C)} + \|x_1(\cdot)\|_X\big\|_E \\ &\le 6\alpha + \|z_1+z_2\|_E \le 6\alpha + 2 - 2\delta_3 \le 2(1-\delta_3/2).\end{aligned}$$

Combining all the above cases we conclude that $\|x+x_k\| \le 2(1-\omega)$ for some $k \in \mathbb{N}$, where $\omega = \min\{\delta_1, \delta_2/2, \delta_3/3, p_1\}$. This finishes the proof.

THEOREM 4. *Let E be a Köthe sequence space and X be a finite-dimensional Banach space. Then $E(X)$ has property (β) if and only if E has property (β).*

Proof. Necessity. This is clear, since E is isometrically embedded in $E(X)$ and property (β) is inherited by subspaces.

Proof. Sufficiency. Suppose that $E \in (\beta)$. Hence $E \in (\mathrm{OC})$ and E is reflexive. Clearly, X is also reflexive. From Theorem 5.3 of [7] it follows that $(E(X))^* = E'(X^*)$, where X^* is the dual of X and E' is the Köthe dual of E, i.e.

$$E' = \Big\{ y \in l_0 : \|y\|_{E'} = \sup_{\|x\|_E \le 1} \sum_{i=1}^{\infty} x(i)y(i) < \infty \Big\}.$$

Furthermore, $E \in (\mathrm{OC})$ iff $E' = E^*$ (see [30]). Consequently, $E(X)$ is reflexive.

Let $\varepsilon > 0$. Take $f \in B(E(X))$. Let (f_n) be a sequence in $B(E(X))$ such that $\operatorname{sep}(f_n)_{E(X)} \ge \varepsilon$. Since $E(X)$ is reflexive, passing to a subsequence if necessary we may assume that $f_n \xrightarrow{w} g$ in $E(X)$. In view of Lemma 2, $f_n(i) \xrightarrow{w} g(i)$ in X for every $i \in \mathbb{N}$. Note that every finite-dimensional space X has the Schur property. Hence $f_n(i) \to g(i)$ strongly in X for every $i \in \mathbb{N}$. Therefore

$$\big\| \|f_n(\cdot) - g(\cdot)\|_X \, \chi_I \big\|_E \to 0 \tag{31}$$

for every $I \subset \mathbb{N}$ with $\operatorname{card} I < \infty$. Since $E \in (\mathrm{OC})$, there exists $A \subset \mathbb{N}$ with

$$\operatorname{card} A < \infty, \qquad \big\| \|g(\cdot)\|_X \, \chi_{\mathbb{N}\setminus A} \big\|_E < \varepsilon/16. \tag{32}$$

Moreover, by (31), there exists $N_1 \in \mathbb{N}$ such that $\big\| \|f_n(\cdot) - f_m(\cdot)\|_X \, \chi_A \big\|_E < \varepsilon/2$ for all $n, m \ge N_1$. So, as $\operatorname{sep}(f_n)_{E(X)} \ge \varepsilon$, we get $\operatorname{sep}\{(f_n \chi_{\mathbb{N}\setminus A})_{n=N_1}^{\infty}\}_{E(X)} \ge \varepsilon/2$. Consequently,

$$\big\| \|f_n(\cdot)\|_X \, \chi_{\mathbb{N}\setminus A} \big\|_E \ge \varepsilon/4 \text{ for every } n \ge N_1 \text{ excluding at most one element.} \tag{33}$$

Define $h_n(\cdot) = (\|f_{n+N_1}(\cdot)\|_X - \|g(\cdot)\|_X)\chi_{\mathbb{N}\setminus A}$. Then $h_n \to 0$ pointwise in E. It follows from Proposition 8 of [8] that if E is a reflexive Köthe space over a complete, σ-finite measure space (Ω, Σ, μ) and $x_n \to x$ μ-a.e., then $x_n \to x$ weakly in E. Consequently, $h_n \xrightarrow{w} 0$ in E. Furthermore, by (32) and (33), we get $\|h_n\|_E \ge \varepsilon/8$ for every $n \in \mathbb{N}$. Then, by the Hahn–Banach theorem, passing to a subsequence if necessary, we may assume that $\operatorname{sep}(h_n)_E \ge \varepsilon/16$. But $\operatorname{sep}(h_n)_E = \operatorname{sep}(\|f_{n+N_1}(\cdot)\|_X \, \chi_{\mathbb{N}\setminus A})_E \le \operatorname{sep}(\|f_{n+N_1}(\cdot)\|_X)_E$. Applying property (β) of E, we get

$$\big\| \|f(\cdot)\|_X + \|f_k(\cdot)\|_X \big\|_E \le 2(1-\delta)$$

for some $k > N_1$, where $\delta = \delta(\varepsilon/16)$ is from (3). Finally,

$$\|f + f_k\| \le \big\| \|f(\cdot)\|_X + \|f_k(\cdot)\|_X \big\|_E \le 2(1-\delta). \ \blacksquare$$

Remark 3. It is worth mentioning that property (β) does not lift from X to $E(X)$ in the case when E is a Köthe function space. Namely, consider the Lebesgue–Bochner space $L_p(\mu, X)$ with $1 < p < \infty$ and μ being the Lebesgue measure on $[0,1]$. Then $L_p(\mu, X)$ fails to have the uniform Kadec–Klee property whenever X is not uniformly convex (Theorem 3.4.9 in [29]). This also follows from Theorem 2 in [32]. Moreover, if $E = E(T, \Sigma, \mu)$ is a Köthe function space, i.e. μ is non-atomic, and X is a real Banach space, then $E(X)$ has property (β) iff X is uniformly convex and E has property (β) (see [19, Corollary 1]).

Recall that E is a *symmetric Köthe sequence space* if for any $x \in E$ and each permutation (n_k) of $\mathbb{N}$ we have $\widetilde{x} = \{x(n_k)\}_{k=1}^{\infty} \in E$ and $\|x\|_E = \|\widetilde{x}\|_E$.

COROLLARY 1. *Let E be a symmetric Köthe sequence space and X a Banach space. The following assertions are equivalent*:

(i) *$E(X)$ has property (β).*
(ii) *X and E have property (β).*
(iii) *X has property (β) and E is orthogonally uniformly convex.*

Proof. If E is a symmetric Köthe sequence space, then $E \in (\mathrm{UC}^{\perp})$ iff $E \in (\beta)$ ([20]). Thus, the assertion follows immediately from Theorems 3 and 4. ∎

In the last part of this paper we consider Musielak–Orlicz sequence spaces of Bochner type. We say a map $\Phi : \mathbb{R} \to \mathbb{R}_+$ is an *Orlicz function* if Φ is convex, even, vanishing at zero and not identically zero. Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ be a *Musielak–Orlicz function*, i.e. φ_i is an Orlicz function for every $i \in \mathbb{N}$. Any Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^{\infty}$ generates the *Musielak–Orlicz space*

$$l_{\varphi} = \Big\{x \in l_0 : I_{\varphi}(cx) = \sum_{i=1}^{\infty} \varphi_i(cu_i) < \infty \text{ for some } c > 0\Big\}.$$

We endow the space l_{φ} with the *Luxemburg norm* $\|x\|_{\varphi} = \inf\{\varepsilon > 0 : I_{\varphi}(x/\varepsilon) \le 1\}$. The symbol $\varphi > 0$ is used to indicate that the functions φ_i vanish only at zero for each $i \in \mathbb{N}$.

We say a Musielak–Orlicz function φ satisfies the *δ_2-condition* ($\varphi \in \delta_2$) if there are positive constants k, a and a sequence $(c_i)_{i=1}^{\infty}$ of positive reals with $\sum_{i=1}^{\infty} c_i < \infty$ such that $\varphi_i(2u) \le k\varphi_i(u) + c_i$ for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\varphi_i(u) \le a$.

Denote by φ_i^* the complementary function to φ_i and write $\varphi^* = (\varphi_i^*)_{i=1}^{\infty}$.

We say that an Orlicz function Φ is *strictly convex* on an interval $[a, b]$ if $\Phi((u + v)/2) < (\Phi(u) + \Phi(v))/2$ for all $u, v \in [a, b]$, $u \ne v$.

Given a Musielak–Orlicz function φ we define the function $h_i : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ for $i \in \mathbb{N}$ by

$$h_i(u, v) = \begin{cases} \dfrac{2\varphi_i((u + v)/2)}{\varphi_i(u) + \varphi_i(v)} & \text{if } \varphi_i(u) \vee \varphi_i(v) > 0, \\ 0 & \text{if } \varphi_i(u) \vee \varphi_i(v) = 0. \end{cases}$$

Let $c > 0$ be a positive number. A Musielak–Orlicz function $\varphi = (\varphi_i)_{i=1}^{\infty}$ is said to be *uniformly convex* in the c-neighbourhood of zero if for every $a \in [0, 1)$ there exist $\delta \in (0, 1)$ and a non-negative sequence $d = (d_i)$ with

$I_\varphi(d) < \infty$ and $\varphi_i(d_i) \le c$ for every $i \in \mathbb{N}$ such that

$$h_i(u, au) \le 1 - \delta$$

for all $u \in (d_i, \varphi_i^{-1}(c)]$, $i \in \mathbb{N}$. Let $N \subset \mathbb{N}$. We say that a family $(\varphi_i)_{i\in N}$ is uniformly convex in the c-neighbourhood of zero in the above definition "$i \in \mathbb{N}$" is replaced by "$i \in N$".

We say that φ satisfies *condition* $(*)$ if for every $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that $\varphi_i(u) < 1-\varepsilon$ implies $\varphi_i((1+\delta)u) \le 1$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$.

For more details and references see [12].

Taking $E = l_\varphi$ in Theorem 3 we get the following

COROLLARY 2. *Let φ be a Musielak–Orlicz function and $(X, \|\cdot\|_X)$ an infinite-dimensional Banach space. Then $l_\varphi(X)$ has property (β) if and only if*

(i) $\varphi \in \delta_2$, $\varphi^* \in \delta_2$, $\varphi > 0$ *and φ satisfies condition* $(*)$.

(ii) φ_i *is linear in a neighbourhood of zero for at most one* $i \in \mathbb{N}$.

(iii) *If φ_j is linear in a neighbourhood of zero for some $j \in \mathbb{N}$, then φ_i is strictly convex in $[0,1]$ for every $i \neq j$ and $(\varphi_i)_{i\neq j}$ is uniformly convex in the* 1-*neighbourhood of zero.*

(iv) $X \in (\beta)$.

Proof. It is known that $l_\varphi \in (\mathrm{UC}^\perp)$ iff conditions (i)–(iii) are satisfied (see [20, Theorem 6]). Therefore the corollary follows directly from Theorem 3. ∎

REMARK 4. It is worth mentioning that from Corollary 2 it follows that conditions $\varphi > 0$ and $\varphi \in (*)$, which were assumed in Theorem 2 in [14], are necessary for property (β) of Musielak–Orlicz sequence spaces of Bochner type. Moreover, the assumption that $\varphi_i(u)/u \to 0$ as $u \to 0$ for every $i \in \mathbb{N}$, which has also been assumed in [14], is essentially weakened here to the necessary one formulated in conditions (ii) and (iii).

COROLLARY 3. *Let φ be a Musielak–Orlicz function and $(X, \|\cdot\|_X)$ a finite-dimensional Banach space. Suppose that φ satisfies condition $(*)$. Then $l_\varphi(X)$ has property (β) if and only if $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$.*

Proof. It is known that if $\varphi \in (*)$, then $l_\varphi \in (\beta)$ iff $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$ (see [20]). So, the corollary follows from Theorem 4. ∎

REMARK 5. Notice that the results of Corollaries 2 and 3 have not been distinguished in Theorem 2 of [14], since the general assumptions that $\varphi > 0$ and $\varphi_i(u)/u \to 0$ as $u \to 0$ for any $i \in \mathbb{N}$ have been made there.

If $\varphi_i(u) = \Phi(u)$ for any $i \in \mathbb{N}$, then the Musielak–Orlicz sequence space becomes the Orlicz sequence space l_Φ. Notice that Orlicz sequence spaces are symmetric. Hence, by Corollary 1 and a result from [5] or [17], we get the criteria for property (β) in Orlicz–Bochner sequence spaces proved in [15].

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