STUDIA MATHEMATICA 162 (3) (2004)

L^1-convergence and hypercontractivity of diffusion semigroups on manifolds

by

Feng-Yu Wang (Beijing)

Abstract. Let P_t be the Markov semigroup generated by a weighted Laplace operator on a Riemannian manifold, with μ an invariant probability measure. If the curvature associated with the generator is bounded below, then the exponential convergence of P_t in $L^1(\mu)$ implies its hypercontractivity. Consequently, under this curvature condition L^1-convergence is a property stronger than hypercontractivity but weaker than ultracontractivity. Two examples are presented to show that in general, however, L^1-convergence and hypercontractivity are incomparable.

1. Introduction. Let M be a connected, complete, noncompact Riemannian manifold either without boundary or with a convex boundary ∂M. Consider the operator $L := \Delta + Z$, where Z is a C^1 vector field such that for some $K \geq 0$,

$$\mathrm{Ric}(X,X) - \langle \nabla_X Z, X\rangle \geq -K|X|^2, \quad X \in TM. \tag{1.1}$$

Then the (reflecting) L-diffusion process is non-explosive (see e.g. [10, Theorem 8.2]). Let P_t be the corresponding diffusion semigroup. We assume that P_t has a (unique) invariant probability measure μ (see [3] for a sufficient condition of its existence and uniqueness). In particular, if $Z = \nabla V$ for some $V \in C^2(M)$ such that $R := \int_M e^{V(x)}\,dx < \infty$, where dx stands for the Riemannian volume measure, then $\mu(dx) = R^{-1}e^{V(x)}dx$.

Our main purpose is to compare the L^1-convergence and hypercontractivity of P_t. Let us first explain that both properties are stronger than the L^2-exponential convergence of P_t.

It is well known that the log-Sobolev inequality implies the Poincaré inequality, and if P_t is symmetric then these two inequalities are equivalent, respectively, to the hypercontractivity and L^2-exponential convergence of P_t (see e.g. [6, 8]). Therefore, at least for the symmetric case, the hypercontractivity of P_t is stronger than its exponential convergence in $L^2(\mu)$. In fact, this implication is also true for the non-symmetric case as soon as (1.1)

2000 *Mathematics Subject Classification*: 47D07, 60J60.

Supported in part by NNSFC(10025105,10121101), TRAPOYT and the 973-Project.

holds, since according to [15, Theorem 2.1] (see also [19, Theorem 5.3]) if (1.1) holds then the hypercontractivity of P_t is equivalent to the log-Sobolev inequality as well, which in turn implies the Poincaré inequality and hence the L^2-exponential convergence of P_t.

On the other hand, suppose that P_t converges in $L^1(\mu)$, i.e. there is a positive function ξ on $[0,\infty)$ with $\xi(t)\downarrow 0$ as $t\uparrow\infty$ such that

$$\|P_t-\mu\|_{1\to 1}\le \xi(t),\quad t\ge 0,$$

where $\|\cdot\|_{p\to q}$ denotes the operator norm from $L^p(\mu)$ to $L^q(\mu)$, and $\mu(f):=\int_M f\,d\mu$ for $f\in L^1(\mu)$. Then, by the semigroup property, P_t converges in $L^1(\mu)$ exponentially fast, i.e. there exist $c,\lambda>0$ such that

$$\|P_t-\mu\|_{1\to 1}\le ce^{-\lambda t},\quad t\ge 0. \tag{1.2}$$

Since $\|P_t-\mu\|_{\infty\to\infty}\le 2$ for all $t\ge 0$, by Riesz–Thorin's interpolation theorem (see e.g. [5]) one has

$$\|P_t-\mu\|_{2\to 2}\le \sqrt{2c}\,e^{-\lambda t/2},\quad t\ge 0. \tag{1.3}$$

If, in particular, P_t is symmetric, then (1.2) implies

$$\|P_t-\mu\|_{\infty\to 2}=\|P_t-\mu\|_{2\to 1}\le\|P_t-\mu\|_{1\to 1}\le ce^{-\lambda t},\quad t\ge 0,$$

hence according to [14, Theorem 2.3],

$$\|P_t-\mu\|_{2\to 2}\le e^{-\lambda t},\quad t\ge 0.$$

Therefore, besides hypercontractivity, L^1-convergence also implies L^2-exponential convergence.

Our main result says that under condition (1.1), L^1-exponential convergence is a property between hypercontractivity and ultracontractivity. We refer to [15, 20] for explicit sufficient and necessary conditions for these two contractivity properties.

THEOREM 1.1. (1) *If* (1.1) *holds, then* (1.2) *implies the log-Sobolev inequality*: *there exists* $C>0$ *such that*

$$\mu(f^2\log f^2)\le C\mu(|\nabla f|^2),\qquad \mu(f^2)=1. \tag{1.4}$$

Consequently, *the* L^1*-convergence of* P_t *implies its hypercontractivity*, *i.e. for any* $t>0$ *there exists* $p_t>2$ *such that* $\|P_t\|_{2\to p_t}\le 1$.

(2) *If either* (1.1) *holds or* P_t *is symmetric*, *then the ultracontractivity of* P_t (*i.e.* $\|P_t\|_{1\to\infty}<\infty$ *for some* $t>0$) *implies* (1.2) *for some* $c,\lambda>0$.

REMARK 1.2. When P_t is symmetric, its L^1-convergence is equivalent to strong ergodicity:

$$\lim_{t\to 0}\sup_{\nu\in\mathcal{P}(M)}\|\nu P_t-\mu\|_{\rm var}=0,$$

where $\mathcal{P}(M)$ is the set of all probability measures on M, $\nu P_t\in\mathcal{P}(M)$ is defined by $(\nu P_t)(A):=\nu(P_t 1_A)$ for a measurable set A, and $\|\cdot\|_{\rm var}$ is the total variation norm defined by $\|\psi\|_{\mathrm{var}} := \sup_A \psi(A) - \inf_A \psi(A)$ for a set function ψ. In fact, if ν is absolutely continuous with respect to μ then (see e.g. [4, Theorem 5.7])

$$\frac{1}{2}\|\nu P_t - \mu\|_{\mathrm{var}} = \int_M \left(P_t \frac{d\nu}{d\mu} - 1\right)^+ d\mu = \frac{1}{2}\mu\left(\left|P_t \frac{d\nu}{d\mu} - 1\right|\right).$$

Since P_t ($t > 0$) has transition density (see e.g. [7, p. 79]) and since μ has strictly positive density with respect to the volume measure (see e.g. [3, Theorem 1.1(ii)]), νP_1 is absolutely continuous with respect to μ. Thus, for any $t > 1$ one has

$$\begin{aligned}\|P_t - \mu\|_{1\to 1} &= \sup_{f\ge 0,\, \mu(f)=1} \mu(|P_t f - 1|) \le \sup_{\nu\in\mathcal{P}(M)} \|\nu P_t - \mu\|_{\mathrm{var}} \\ &= \sup_{\nu\in\mathcal{P}(M)} \|(\nu P_1)P_{t-1} - \mu\|_{\mathrm{var}} \\ &= \sup_{\nu\in\mathcal{P}(M)} \mu\left(\left|P_{t-1}\frac{d(\nu P_1)}{d\mu} - 1\right|\right) \le \|P_{t-1} - \mu\|_{1\to 1}.\end{aligned}$$

Therefore, in other words, Theorem 1.1(1) means that under (1.1) the strong ergodicity of P_t implies the log-Sobolev inequality.

The proof of Theorem 1.1 is given in the next section, while two examples are presented in Section 3 to show that in general L^1-convergence and hypercontractivity are incomparable.

2. Proof of Theorem 1.1. To prove Theorem 1.1(1), we need the following interpolation theorem due to Peetre [13] (see also [9, Theorem A.1]). In the version below we give an explicit relationship between the relevant constants.

THEOREM 2.1 (Peetre's interpolation theorem). *Let ϕ_0, ϕ_1, ϕ_2 be three non-negative increasing functions defined on $[0,\infty)$ such that $\phi_1 = \phi_0\sigma(\phi_2/\phi_0)$ for a concave function σ and $\phi_i(2r) \le a\phi_i(r)$ for some $a > 0$ and all $r \ge 0$, $i = 0,1,2$. Let T be a linear operator defined on a space $\mathcal{D}(T) \supset O^{\phi_i} := \{f : \mu(\phi_i(|f|)) < \infty\}$, $i = 0,1,2$. There exists $c > 0$ such that if*

$$\int \phi_i(|Tf|)\,d\mu \le c_i \int \phi_i(|f|)\,d\mu, \quad f \in O^{\phi_i},\ i = 0,2, \tag{2.1}$$

for some $c_0, c_2 > 0$, then

$$\int \phi_1(|Tf|)\,d\mu \le c(c_0 \vee c_2)\int \phi_1(|f|)\,d\mu, \quad f \in O^{\phi_1}. \tag{2.2}$$

Proof. For $f \in \mathcal{D}(T)$, define

$$L(t,f) := \inf_{f=f_0+f_2}\{\mu(\phi_0(|f_0|)) + t\mu(\phi_2(|f_2|))\}, \quad t \ge 0.$$

By (A.4) in [9], there exist $C \in [1,\infty)$ and a positive measure ν on $[0,\infty)$ such that for any $f \in \mathcal{D}(T)$,

$$\frac{1}{C}\int_0^\infty L(t,f)\,\nu(dt) \le \mu(\phi_1(|f|)) \le C\int_0^\infty L(t,f)\,\nu(dt). \tag{2.3}$$

By (2.1),

$$\begin{aligned} L(t,Tf) &\le \inf_{f=f_0+f_2}\{\mu(\phi_0(|Tf_0|)) + t\mu(\phi_2(|Tf_2|))\} \\ &\le \inf_{f=f_0+f_2}\{c_0\mu(\phi_0(|f_0|)) + tc_2\mu(\phi_2(|f_2|))\} \le (c_0\vee c_2)L(t,f). \end{aligned}$$

Combining this with (2.3) we obtain (2.2). ■

Proof of Theorem 1.1. (1) By (1.1) we have (see [1, 17])

$$P_t(f^2\log f^2) \le \frac{2(\exp[2Kt]-1)}{K}\,P_t|\nabla f|^2 + (P_tf^2)\log(P_tf^2), \quad t>0.$$

This implies that

$$\begin{aligned} &\mu(f^2\log f^2) \\ &\quad\le \frac{2(\exp[2Kt]-1)}{K}\,\mu(|\nabla f|^2) + \mu((P_tf^2)\log P_tf^2), \quad t>0. \end{aligned} \tag{2.4}$$

To apply Theorem 2.1, let $\phi_0(r) = r$, $\phi_2(r) = r^2$ and $\phi_1(r) = r\log(1+r)$. We have $\sigma(r) = \log(1+r)$, which is concave. Applying Theorem 2.1 to $T := P_t - \mu$ and using (1.2) and (1.3), we obtain

$$\begin{aligned} &\mu(|P_tf^2-1|\log(1+|P_tf^2-1|)) \\ &\qquad\le c_2\mu(|f^2-1|\log(1+|f^2-1|))e^{-\lambda_2 t}, \quad \mu(f^2)=1, \end{aligned}$$

for some $c_2, \lambda_2 > 0$ and all $t \ge 0$. Therefore, there exists $c_3 > c_2$ such that

$$\mu(P_tf^2\log P_tf^2) \le c_3e^{-\lambda_2 t}\mu(f^2\log f^2) + c_3, \quad t\ge 0,\ \mu(f^2)=1.$$

Combining this with (2.4) for a proper choice of $t>0$, we obtain

$$\mu(f^2\log f^2) \le A\mu(|\nabla f|^2) + B, \quad \mu(f^2)=1, \tag{2.5}$$

for some $A, B > 0$. Therefore, to prove the hypercontractivity of P_t, it suffices to verify the following Poincaré inequality (see e.g. [6, Theorem 6.1.22(ii)]):

$$\mu(f^2) \le C\mu(|\nabla f|^2) + \mu(f)^2, \tag{2.6}$$

where $C>0$ is a constant. To this end, we make use of [14, Proposition 3.1], which involves the weak and super Poincaré inequalities. First, since $x\log x \ge Rx - e^{R-1}$ for all $x, R \ge 0$, we have (for $\mu(f^2)=1$)

$$\begin{aligned} \mu(f^2\log f^2) &= 2\mu(f^2\log|f|) \ge 2R - 2e^{R-1}\mu(|f|) \\ &\ge 2R - 1 - e^{2R}\mu(|f|)^2, \quad R>0. \end{aligned}$$

Combining this with (2.5) we arrive at

$$\mu(f^2) \le \frac{A\mu(|\nabla f|^2) + e^{2R}\mu(|f|)^2}{2R - B - 1}, \quad 2R > B + 1.$$

Thus, we have the following super Poincaré inequality for some $\beta : (0,\infty) \to (0,\infty)$:

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0.$$

On the other hand, by e.g. [3, Theorem 1.1(ii)] one has $\mu(dx) = e^{V(x)}dx$ for some $V \in C(M)$. Then [14, Theorem 3.1] implies the weak Poincaré inequality, i.e. there exists $\alpha : (0,\infty) \to (0,\infty)$ such that

$$\mu(f^2) \le \alpha(r)\mu(|\nabla f|^2) + r\|f\|_\infty^2, \quad r > 0,\ \mu(f) = 0.$$

Therefore, by [14, Proposition 1.3] we obtain (2.6) for some constant $C > 0$.

(2) If P_t is ultracontractive then (2.5) holds for some constants $A, B > 0$ (see e.g. [19, Theorem 5.3]). Thus, as explained above, (2.6) holds and hence $\|P_t - \mu\|_{2\to 2} \le e^{-t/C}$, $t \ge 0$. Therefore, if $\|P_{t_0} - \mu\|_{1\to 2} < \infty$ then for all $t > 0$ one has

$$\|P_{t+t_0} - \mu\|_{1\to 1} \le \|P_{t_0} - \mu\|_{1\to 2}\|P_t - \mu\|_{2\to 2} \le e^{-t/C}\|P_{t_0} - \mu\|_{1\to 2}. \ \blacksquare$$

3. Incomparability of L^1-convergence and hypercontractivity. To show that L^1-convergence and hypercontractivity are incomparable, let us first recall a result on strong ergodicity which is equivalent to L^1-convergence for the symmetric case according to Remark 1.2. By Tweedie [16, Theorem 2(iii)] it is well known that for irreducible Markov chains on $\mathbb{Z}_+$ strong ergodicity is equivalent to $\sup_{i\in\mathbb{Z}_+} E^i\tau_0 < \infty$, where τ_0 is the hitting time to 0 and E^i is the expectation with respect to the Markov chain starting from i. The same has been proved recently by Mao [12] for diffusion processes.

THEOREM 3.1 (Mao [12]). *Consider $L := a(x)d^2/dx^2 + b(x)d/dx$, where $a, b \in C^1([0,\infty))$ with $a(x) > 0$ for all $x \ge 0$. Let*

$$C(x) = \int_0^x \frac{b(r)}{a(r)}\, dr, \qquad x \in \mathbb{R}.$$

Assume that $\int_0^\infty (e^{C(r)}/a(r))\, dr < \infty$. Then the corresponding reflecting diffusion semigroup P_t is strongly ergodic if and only if

$$\delta := \int_0^\infty e^{-C(x)}\, dx \int_x^\infty \frac{e^{C(r)}}{a(r)}\, dr < \infty. \tag{3.1}$$

Proof. We include the proof for completeness. Let $\tau_0 := \inf\{t \ge 0 : x_t = 0\}$, where x_t is the reflecting L-diffusion process.

(a) (3.1) *is equivalent to* $\sup_{x>0} E^x\tau_0 < \infty$. Let

$$F(x) := \int_0^x e^{-C(r)}dr \int_r^\infty \frac{e^{C(s)}}{a(s)}ds, \quad x \geq 0.$$

We have $LF(x) = -1$ and hence for $x > 0$,

$$0 \leq E^x F(x_{\tau_0\wedge t}) = F(x) - E^x(\tau_0 \wedge t), \quad t > 0.$$

Letting $t \to \infty$ we obtain $E^x\tau_0 \leq F(x)$ and hence (3.1) implies $\sup_{x>0} E^x\tau_0 < \infty$.

Conversely, letting $\tau_n := \inf\{t \geq 0 : x_t \geq n\}$ we have

$$\begin{aligned} F(x) &= E^x\tau_0 \wedge \tau_n + E^x F(\tau_0 \wedge \tau_n) \\ &\leq F(n)P^x(\tau_n < \tau_0) + E^x\tau_0, \quad n > x. \end{aligned} \tag{3.2}$$

Since for $G(x) := \int_0^x e^{-C(r)}dr$ one has $LG = 0$, it follows that

$$F(x) = E^x G(x_{\tau_0\wedge\tau_n}) = G(n)P^x(\tau_n < \tau_0), \quad n > x.$$

Combining this with (3.2) we arrive at

$$F(x) \leq E^x\tau_0 + \frac{F(n)G(x)}{G(n)}, \quad n > x. \tag{3.3}$$

This implies that $F(\infty) < \infty$ provided $\sup_{x>0} E^x\tau_0 < \infty$. Indeed, if $F(\infty) = \infty$, then since $\int_0^\infty (e^{C(r)}/a(r))\,dr < \infty$, we have $G(\infty) = \infty$ and $F(n)/G(n) \to 0$ as $n \to \infty$. Thus, by letting $n \to \infty$, we see from (3.3) that $F(x) \leq E^x\tau_0$ for all $x > 0$ and hence $\sup_{x>0} E^x\tau_0 = \infty$.

(b) *Strong ergodicity implies* $\sup_{x>0} E^x\tau_0 < \infty$. If P_t is strongly ergodic, then there exists $t > 0$ such that $\inf_{x>1} P^x(x_t \leq 1) \geq \frac{1}{2}\mu([0,1]) =: c > 0$, where μ is the invariant probability measure. Thus,

$$\begin{aligned} P^x(\tau_1 > 2t) &\leq P^x(x_t > 1, x_{2t} > 1) = E^x 1_{\{x_t>1\}} P^{x_t}(x_t > 1) \\ &\leq (1-c)^2, \quad x > 1. \end{aligned}$$

Similarly, we have

$$P^x(\tau_1 > nt) \leq (1-c)^n, \quad x > 1.$$

Therefore,

$$E^x\tau_1 = \int_0^\infty P^x(\tau_1 > s)\,ds \leq \delta < \infty$$

for some $\delta > 0$ and all $x > 1$.

On the other hand, by the proof in (a) we see that

$$E^x\tau_1 = \int_1^x e^{-C(r)}\,dr \int_r^\infty \frac{e^{C(s)}}{a(s)}\,ds, \quad x > 1.$$

Then $\int_0^\infty (e^{C(s)}/a(s))\,ds < \infty$ and

$$\sup_{x>0} E^x\tau_0 = \sup_{x>1} E^x\tau_0 = \sup_{x>1} E^x\tau_1 + \int_0^1 e^{-C(r)}dr \int_r^\infty \frac{e^{C(s)}}{a(s)}\,ds < \infty.$$

(c) $\sup_{x>0} E^x\tau_0 < \infty$ *implies strong ergodicity.* For any $x > y \geq 0$, let (x_t, y_t) be a coupling of the reflecting L-diffusion process with $x_0 = x$, $y_0 = y$. We have

$$T := \inf\{t \geq 0 : x_t = y_t\} \leq \tau_0 := \inf\{t \geq 0 : x_t = 0\}.$$

As usual we let $x_t = y_t$ for $t > T$ so that for any measurable set A we have

$$|\delta_x P_t(A) - \delta_y P_t(A)| \leq E^{x,y}|1_A(x_t) - 1_A(y_t)| \leq P^{x,y}(x_t \neq y_t) \leq P^{x,y}(T > t).$$

Therefore,

$$\sup_{x>y\geq 0} \|\delta_x P_t - \delta_y P_t\|_{\mathrm{var}} \leq 2 \sup_{x>y\geq 0} P^{x,y}(T > t) \leq 2 \sup_{x>0} P^x(\tau_0 > t) \leq \frac{\sup_{x>0} E^x\tau_0}{t},$$

which goes to zero as $t \to \infty$. This means that P_t is strongly ergodic. ■

EXAMPLE 3.1. Consider the Ornstein–Uhlenbeck operator $L := d^2/dx^2 - xd/dx$ on $[0, \infty)$. It is well known that the semigroup P_t of the reflecting L-diffusion process is hypercontractive. But according to Theorem 3.1, P_t is not strongly ergodic since (3.1) does not hold. Therefore, P_t does not converge in the L^1-norm by Remark 1.2.

EXAMPLE 3.2. Let $M = [0, \infty)$ and consider $L := d^2/dx^2 + b(x)d/dx$, where

$$b(x) := -\frac{\gamma'(x)}{\gamma(x)} - \frac{1}{\gamma(x)}, \qquad x \geq 0,$$

with γ constructed as follows. For any $n \geq 1$, let $\phi_n \in C^\infty[0, \infty)$ be non-negative such that $\phi_n|_{[n,n+e^{-n}]^c} = 0$ and

$$\phi_n|_{[n+e^{-n}/4,\,n+3e^{-n}/4]} = \max \phi_n = e^n(1 + n)^{-2}.$$

Set $\gamma(r) = (1 + r)^{-2} + \sum_{n\geq 1} \phi_n(r),\ r \geq 0$. Then P_t is strongly ergodic and hence L^1-convergent but not hypercontractive.

Proof. We have

$$C(x) := \int_0^x b(r)\,dr = -\log \gamma(x) - \int_0^x \frac{dr}{\gamma(r)}, \qquad x \geq 0.$$

Then

$$\exp[-C(x)] = \gamma(x) \exp\left[\int_0^x \frac{dr}{\gamma(r)}\right], \quad \exp[C(x)] = -\frac{d}{dx} \exp\left[-\int_0^x \frac{dr}{\gamma(r)}\right].$$

Therefore, $\int_0^\infty e^{C(x)}\,dx = 1$ and

$$\int_0^\infty e^{-C(x)}\,dx \int_x^\infty e^{C(y)}\,dy = \int_0^\infty \gamma(x)\,dx \le \int_0^\infty \frac{dr}{1+r^2} + \sum_{n=1}^\infty \frac{1}{1+n^2} < \infty.$$

Thus, (3.1) holds and hence P_t is strongly ergodic.

On the other hand, we use [2, Theorem 1.1] to disprove the log-Sobolev inequality. Observe that

$$\frac{1}{3}(1+x)^3 \ge \int_0^x \frac{dr}{\gamma(r)} \ge \frac{1}{3}(1+x)^3 - \sum_{n\ge1} \int_n^{n+e^{-n}} (1+r)^2\,dr$$
$$\ge \frac{1}{3}(1+x)^3 - \sum_{n\ge1}(2+n)^2 e^{-n} =: \frac{1}{3}(1+x)^3 - c_1.$$

Then for $\mu([0,x]) := \int_0^x e^{C(r)}\,dr$,

$$\int_0^\infty e^{-C(x)}\mu([0,x])\,dx = \int_0^\infty \gamma(x)\Big(\exp\Big[\int_0^x \frac{dr}{\gamma(r)}\Big] - 1\Big)dx$$
$$\ge e^{-c_1}\int_0^\infty \frac{e^{(1+x)^3/3}}{(1+x)^2}\,dx - \int_0^\infty \gamma(x)\,dx = \infty$$

since $\int_0^\infty \gamma(x)dx < \infty$. Moreover,

$$I(n) := \Big(\int_0^{n+e^{-n}} e^{-C(y)}\,dy\Big)\Big(\int_{n+e^{-n}}^\infty e^{C(y)}dy\Big)\Big(\log\frac{1}{\int_{n+e^{-n}}^\infty e^{C(y)}dy}\Big)$$
$$= \Big(\int_0^{n+e^{-n}} \gamma(y)\exp\Big[\int_0^y \frac{dr}{\gamma(r)}\Big]dy\Big)\exp\Big[-\int_0^{n+e^{-n}} \frac{dr}{\gamma(r)}\Big]\Big(\int_0^{n+e^{-n}} \frac{dr}{\gamma(r)}\Big)$$
$$\ge c_2\Big(\int_{n+e^{-n}/4}^{n+3e^{-n}/4} \frac{e^n}{(1+n)^2}\exp[(1+y)^3/3]dy\Big)e^{-(1+n)^3/3}(1+n)^3$$
$$\ge c_3(1+n) - c_4$$

for some $c_2, c_3, c_4 > 0$. Thus $\lim_{n\to\infty} I(n) = \infty$. Therefore, according to [2, Theorem 1.1] the log-Sobolev inequality does not hold; see also [11] for a more general result. ∎

Acknowledgements. The author would like to thank Professors Mu-Fa Chen, Yong-Hua Mao and Liming Wu for useful conversations. He also thanks the referee for helpful comments.

References

[1] D. Bakry and M. Ledoux, *Lévy–Gromov's isoperimetric inequality for an infinite dimensional diffusion generator*, Invent. Math. 123 (1996), 259–281.

[2] S. G. Bobkov and F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal. 163 (1999), 1–28.
[3] V. I. Bogachev, M. Röckner and F. Y. Wang, *Elliptic equations for invariant measures on finite and infinite dimensional manifolds*, J. Math. Pures Appl. 80 (2001), 177–221.
[4] M. F. Chen, *From Markov Chains to Non-Equilibrium Particle Systems*, World Sci., Singapore, 1992.
[5] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1989.
[6] J. D. Deuschel and D. W. Stroock, *Large Deviations*, Pure Appl. Math. 137, Academic Press, San Diego, 1989.
[7] K. D. Elworthy, *Geometric aspects of diffusions on manifolds*, in: Lecture Notes in Math. 1362, 277–425, Springer, 1988, 277–425.
[8] L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups*, in: Lecture Notes in Math. 1563, Springer, 1993, 54–88.
[9] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. 60 (1977), 33–59.
[10] X. M. Li, *Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds*, Probab. Theory Related Fields 100 (1994), 485–511.
[11] Y. H. Mao, *Logarithmic Sobolev inequalities for birth-death processes and diffusion processes on the line*, Chinese J. Appl. Probab. Statist. 18 (2002), 94–100.
[12] —, *Strong ergodicity for Markov processes by coupling methods*, J. Appl. Probab. 39 (2002), 839–852.
[13] J. Peetre, *A new approach in interpolation spaces*, Studia Math. 34 (1970), 23–42.
[14] M. Röckner and F. Y. Wang, *Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups*, J. Funct. Anal. 185 (2001), 564–603.
[15] —, —, *Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds*, Forum Math. 15 (2003), 893–921.
[16] R. L. Tweedie, *Criteria for ergodicity, exponential ergodicity and strong ergodicity of Markov processes*, J. Appl. Probab. 18 (1981), 122–130.
[17] F. Y. Wang, *On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups*, Probab. Theory Related Fields 108 (1997), 87–101.
[18] —, *Functional inequalities for empty essential spectrum*, J. Funct. Anal. 170 (2000), 219–245.
[19] —, *Functional inequalities, semigroup properties and spectrum estimates*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000), 263–295.
[20] —, *Logarithmic Sobolev inequalities: conditions and counterexamples*, J. Operator Theory 46 (2001), 183–197.

Department of Mathematics
Beijing Normal University
Beijing 100875, P.R. China
E-mail: wangfy@bnu.edu.cn

Received August 12, 2002
Revised version December 1, 2003 (5016)