$L^1$-convergence and hypercontractivity of
diffusion semigroups on manifolds

by

FENG-YU WANG (Beijing)

Abstract. Let $P_t$ be the Markov semigroup generated by a weighted Laplace operator on a Riemannian manifold, with $\mu$ an invariant probability measure. If the curvature associated with the generator is bounded below, then the exponential convergence of $P_t$ in $L^1(\mu)$ implies its hypercontractivity. Consequently, under this curvature condition $L^1$-convergence is a property stronger than hypercontractivity but weaker than ultracontractivity. Two examples are presented to show that in general, however, $L^1$-convergence and hypercontractivity are incomparable.

1. Introduction. Let $M$ be a connected, complete, noncompact Riemannian manifold either without boundary or with a convex boundary $\partial M$. Consider the operator $L := \Delta + Z$, where $Z$ is a $C^1$ vector field such that for some $K \geq 0$,

$$\text{Ric}(X, X) - \langle \nabla_X Z, X \rangle \geq -K|X|^2, \quad X \in TM. \tag{1.1}$$

Then the (reflecting) $L$-diffusion process is non-explosive (see e.g. [10, Theorem 8.2]). Let $P_t$ be the corresponding diffusion semigroup. We assume that $P_t$ has a (unique) invariant probability measure $\mu$ (see [3] for a sufficient condition of its existence and uniqueness). In particular, if $Z = \nabla V$ for some $V \in C^2(M)$ such that $R := \int_M e^{V(x)} \, dx < \infty$, where $dx$ stands for the Riemannian volume measure, then $\mu(dx) = R^{-1} e^{V(x)} \, dx$.

Our main purpose is to compare the $L^1$-convergence and hypercontractivity of $P_t$. Let us first explain that both properties are stronger than the $L^2$-exponential convergence of $P_t$.

It is well known that the log-Sobolev inequality implies the Poincaré inequality, and if $P_t$ is symmetric then these two inequalities are equivalent, respectively, to the hypercontractivity and $L^2$-exponential convergence of $P_t$ (see e.g. [6, 8]). Therefore, at least for the symmetric case, the hypercontractivity of $P_t$ is stronger than its exponential convergence in $L^2(\mu)$. In fact, this implication is also true for the non-symmetric case as soon as (1.1)
holds, since according to [15, Theorem 2.1] (see also [19, Theorem 5.3]) if (1.1) holds then the hypercontractivity of $P_t$ is equivalent to the log-Sobolev inequality as well, which in turn implies the Poincaré inequality and hence the $L^2$-exponential convergence of $P_t$.

On the other hand, suppose that $P_t$ converges in $L^1(\mu)$, i.e. there is a positive function $\xi$ on $[0, \infty)$ with $\xi(t) \downarrow 0$ as $t \uparrow \infty$ such that
\[ \|P_t - \mu\|_{1-\rightarrow 1} \leq \xi(t), \quad t \geq 0, \]
where $\|\cdot\|_{p\rightarrow q}$ denotes the operator norm from $L^p(\mu)$ to $L^q(\mu)$, and $\mu(f) := \int_M f \, d\mu$ for $f \in L^1(\mu)$. Then, by the semigroup property, $P_t$ converges in $L^1(\mu)$ exponentially fast, i.e. there exist $c, \lambda > 0$ such that
\[ (1.2) \quad \|P_t - \mu\|_{1-\rightarrow 1} \leq ce^{-\lambda t}, \quad t \geq 0. \]
Since $\|P_t - \mu\|_{\infty \rightarrow \infty} \leq 2$ for all $t \geq 0$, by Riesz–Thorin’s interpolation theorem (see e.g. [5]) one has
\[ (1.3) \quad \|P_t - \mu\|_{2 \rightarrow 2} \leq \sqrt{2}ce^{-\lambda t/2}, \quad t \geq 0. \]
If, in particular, $P_t$ is symmetric, then (1.2) implies
\[ \|P_t - \mu\|_{\infty \rightarrow 2} = \|P_t - \mu\|_{2 \rightarrow 1} \leq \|P_t - \mu\|_{1 \rightarrow 1} \leq ce^{-\lambda t}, \quad t \geq 0, \]
hence according to [14, Theorem 2.3],
\[ \|P_t - \mu\|_{2 \rightarrow 2} \leq e^{-\lambda t}, \quad t \geq 0. \]
Therefore, besides hypercontractivity, $L^1$-convergence also implies $L^2$-exponential convergence.

Our main result says that under condition (1.1), $L^1$-exponential convergence is a property between hypercontractivity and ultracontractivity. We refer to [15, 20] for explicit sufficient and necessary conditions for these two contractivity properties.

**Theorem 1.1.** (1) If (1.1) holds, then (1.2) implies the log-Sobolev inequality: there exists $C > 0$ such that
\[ (1.4) \quad \mu(f^2 \log f^2) \leq C\mu(\|\nabla f\|^2), \quad \mu(f^2) = 1. \]
Consequently, the $L^1$-convergence of $P_t$ implies its hypercontractivity, i.e. for any $t > 0$ there exists $p_t > 2$ such that $\|P_t\|_{2 \rightarrow p_t} \leq 1$.

(2) If either (1.1) holds or $P_t$ is symmetric, then the ultracontractivity of $P_t$ (i.e. $\|P_t\|_{1 \rightarrow \infty} < \infty$ for some $t > 0$) implies (1.2) for some $c, \lambda > 0$.

**Remark 1.2.** When $P_t$ is symmetric, its $L^1$-convergence is equivalent to strong ergodicity:
\[ \lim_{t \rightarrow 0} \sup_{\nu \in \mathcal{P}(M)} \|\nu P_t - \mu\|_{\text{var}} = 0, \]
where $\mathcal{P}(M)$ is the set of all probability measures on $M$, $\nu P_t \in \mathcal{P}(M)$ is defined by $(\nu P_t)(A) := \nu(P_t1_A)$ for a measurable set $A$, and $\|\cdot\|_{\text{var}}$ is the
total variation norm defined by $\|\psi\|_{\text{var}} := \sup_A \psi(A) - \inf_A \psi(A)$ for a set function $\psi$. In fact, if $\nu$ is absolutely continuous with respect to $\mu$ then (see e.g. [4, Theorem 5.7])

$$
\frac{1}{2} \|\nu P_t - \mu\|_{\text{var}} = \int_M \left( P_t \frac{d\nu}{d\mu} - 1 \right)^+ d\mu = \frac{1}{2} \mu \left( \left| P_t \frac{d\nu}{d\mu} - 1 \right| \right).
$$

Since $P_t$ ($t > 0$) has transition density (see e.g. [7, p. 79]) and since $P_t$ has strictly positive density with respect to the volume measure (see e.g. [3, Theorem 1.1(ii)]), $\nu P_1$ is absolutely continuous with respect to $\mu$. Thus, for any $t > 1$ one has

$$
\|P_t - \mu\|_{1\rightarrow 1} = \sup_{f \geq 0, \mu(f) = 1} \mu(|P_t f - 1|) \leq \sup_{\nu \in \mathcal{P}(M)} \|\nu P_t - \mu\|_{\text{var}}
$$

$$
= \sup_{\nu \in \mathcal{P}(M)} \|(\nu P_1) P_{t-1} - \mu\|_{\text{var}}
$$

$$
= \sup_{\nu \in \mathcal{P}(M)} \mu \left( \left| P_{t-1} \frac{d(\nu P_1)}{d\mu} - 1 \right| \right) \leq \|P_{t-1} - \mu\|_{1\rightarrow 1}.
$$

Therefore, in other words, Theorem 1.1(1) means that under (1.1) the strong ergodicity of $P_t$ implies the log-Sobolev inequality.

The proof of Theorem 1.1 is given in the next section, while two examples are presented in Section 3 to show that in general $L^1$-convergence and hypercontractivity are incomparable.

2. Proof of Theorem 1.1. To prove Theorem 1.1(1), we need the following interpolation theorem due to Peetre [13] (see also [9, Theorem A.1]). In the version below we give an explicit relationship between the relevant constants.

**Theorem 2.1** (Peetre’s interpolation theorem). Let $\phi_0, \phi_1, \phi_2$ be three non-negative increasing functions defined on $[0, \infty)$ such that $\phi_1 = \phi_0 \sigma(\phi_2/\phi_0)$ for a concave function $\sigma$ and $\phi_i(2r) \leq a \phi_i(r)$ for some $a > 0$ and all $r \geq 0$, $i = 0, 1, 2$. Let $T$ be a linear operator defined on a space $\mathcal{D}(T) \supset O^{\phi_i} := \{ f : \mu(\phi_i(|f|)) < \infty \}, i = 0, 1, 2$. There exists $c > 0$ such that if

$$
\phi_i(|Tf|) d\mu \leq c_i \phi_i(|f|) d\mu, \quad f \in O^{\phi_i}, \quad i = 0, 2,
$$

for some $c_0, c_2 > 0$, then

$$
\phi_1(|Tf|) d\mu \leq c(c_0 \vee c_2) \phi_1(|f|) d\mu, \quad f \in O^{\phi_1}.
$$

**Proof.** For $f \in \mathcal{D}(T)$, define

$$
L(t, f) := \inf_{f=f_0+f_2} \{ \mu(\phi_0(|f_0|)) + t \mu(\phi_2(|f_2|)) \}, \quad t \geq 0.
$$
By (A.4) in [9], there exist $C \in [1, \infty)$ and a positive measure $\nu$ on $[0, \infty)$ such that for any $f \in \mathcal{D}(T)$,

$$
(2.3) \quad \frac{1}{C} \int_0^\infty L(t, f) \nu(dt) \leq \mu(\phi_1(|f|)) \leq C \int_0^\infty L(t, f) \nu(dt).
$$

By (2.1),

$$
L(t, Tf) \leq \inf_{f=f_0+f_2} \{\mu(\phi_0([Tf_0])) + t\mu(\phi_2([Tf_2]))\}
\leq \inf_{f=f_0+f_2} \{c_0\mu(\phi_0(|f_0|)) + tc_2\mu(\phi_2(|f_2|))\} \leq (c_0 \lor c_2)L(t, f).
$$

Combining this with (2.3) we obtain (2.2).

---

**Proof of Theorem 1.1.** (1) By (1.1) we have (see [1, 17])

$$
P_t(f^2 \log f^2) \leq \frac{2(\exp[2Kt] - 1)}{K} P_t|\nabla f|^2 + (P_tf^2) \log(P_tf^2), \quad t > 0.
$$

This implies that

$$
(2.4) \quad \mu(f^2 \log f^2) \leq \frac{2(\exp[2Kt] - 1)}{K} \mu(|\nabla f|^2) + \mu((P_tf^2) \log P_tf^2), \quad t > 0.
$$

To apply Theorem 2.1, let $\phi_0(r) = r$, $\phi_2(r) = r^2$ and $\phi_1(r) = r \log(1 + r)$. We have $\sigma(r) = \log(1 + r)$, which is concave. Applying Theorem 2.1 to $T := P_t - \mu$ and using (1.2) and (1.3), we obtain

$$
\mu(|P_tf^2 - 1| \log(1 + |P_tf^2 - 1|)) \\
\leq c_2\mu(|f^2 - 1| \log(1 + |f^2 - 1|))e^{-\lambda_2 t}, \quad \mu(f^2) = 1,
$$

for some $c_2, \lambda_2 > 0$ and all $t \geq 0$. Therefore, there exists $c_3 > c_2$ such that

$$
\mu(P_tf^2 \log P_tf^2) \leq c_3e^{-\lambda_2 t}\mu(f^2 \log f^2) + c_3, \quad t \geq 0, \quad \mu(f^2) = 1.
$$

Combining this with (2.4) for a proper choice of $t > 0$, we obtain

$$
(2.5) \quad \mu(f^2 \log f^2) \leq A\mu(|\nabla f|^2) + B, \quad \mu(f^2) = 1,
$$

for some $A, B > 0$. Therefore, to prove the hypercontractivity of $P_t$, it suffices to verify the following Poincaré inequality (see e.g. [6, Theorem 6.1.22(ii)]):

$$
(2.6) \quad \mu(f^2) \leq C\mu(|\nabla f|^2) + \mu(f)^2,
$$

where $C > 0$ is a constant. To this end, we make use of [14, Proposition 3.1], which involves the weak and super Poincaré inequalities. First, since $x \log x \geq Rx - e^{R-1}$ for all $x, R \geq 0$, we have (for $\mu(f^2) = 1$)

$$
\mu(f^2 \log f^2) = 2\mu(f^2 \log |f|) \geq 2R - 2e^{R-1}\mu(|f|) \geq 2R - 1 - e^{2R}\mu(|f|)^2, \quad R > 0.
$$
Combining this with (2.5) we arrive at
\[ \mu(f^2) \leq \frac{A\mu(\|\nabla f\|^2) + e^{2R}\mu(\|f\|^2)}{2R - B - 1}, \quad 2R > B + 1. \]

Thus, we have the following super Poincaré inequality for some \( \beta : (0, \infty) \to (0, \infty) \):
\[ \mu(f^2) \leq r\mu(\|\nabla f\|^2) + \beta(r)\mu(\|f\|^2), \quad r > 0. \]

On the other hand, by e.g. [3, Theorem 1.1(ii)] one has \( \mu(dx) = e^{V(x)}dx \) for some \( V \in C(M) \). Then [14, Theorem 3.1] implies the weak Poincaré inequality, i.e. there exists \( \alpha : (0, \infty) \to (0, \infty) \) such that
\[ \mu(f^2) \leq \alpha(r)\mu(\|\nabla f\|^2) + r\|f\|_\infty^2, \quad r > 0, \quad \mu(f) = 0. \]

Therefore, by [14, Proposition 1.3] we obtain (2.6) for some constant \( C > 0 \).

(2) If \( P_t \) is ultracontractive then (2.5) holds for some constants \( A, B > 0 \) (see e.g. [19, Theorem 5.3]). Thus, as explained above, (2.6) holds and hence \( \|P_t - \mu\|_2 \leq e^{-t/C}, \ t \geq 0 \). Therefore, if \( \|P_0 - \mu\|_1 < \infty \) then for all \( t > 0 \) one has
\[ \|P_{t+t_0} - \mu\|_1 \leq \|P_{t_0} - \mu\|_1 \|P_t - \mu\|_2 \|P_t - \mu\|_2 \leq e^{-t/C}\|P_0 - \mu\|_1. \]

3. Incomparability of \( L^1 \)-convergence and hypercontractivity.
To show that \( L^1 \)-convergence and hypercontractivity are incomparable, let us first recall a result on strong ergodicity which is equivalent to \( L^1 \)-convergence for the symmetric case according to Remark 1.2. By Tweedie [16, Theorem 2(iii)] it is well known that for irreducible Markov chains on \( \mathbb{Z}_+ \) strong ergodicity is equivalent to \( \sup_{i \in \mathbb{Z}_+} E^i \tau_0 < \infty \), where \( \tau_0 \) is the hitting time to 0 and \( E^i \) is the expectation with respect to the Markov chain starting from \( i \). The same has been proved recently by Mao [12] for diffusion processes.

**Theorem 3.1 (Mao [12]).** Consider \( L := a(x)d^2/dx^2 + b(x)d/dx \), where \( a, b \in C^1([0, \infty)) \) with \( a(x) > 0 \) for all \( x \geq 0 \). Let
\[ C(x) = \int_0^x \frac{b(r)}{a(r)} \, dr, \quad x \in \mathbb{R}. \]
Assume that \( \sum_0^\infty (e^{C(r)}/a(r)) \, dr < \infty \). Then the corresponding reflecting diffusion semigroup \( P_t \) is strongly ergodic if and only if
\[ \delta := \int_0^\infty e^{-C(x)} \, dx \int_x^\infty \frac{e^{C(r)}}{a(r)} \, dr < \infty. \]

**Proof.** We include the proof for completeness. Let \( \tau_0 := \inf\{t \geq 0 : x_t = 0\} \), where \( x_t \) is the reflecting \( L \)-diffusion process.
(a) (3.1) is equivalent to $\sup_{x>0} E^x \tau_0 < \infty$. Let

$$F(x) := \int_0^x e^{-C(r)} dr \int_r^\infty \frac{e^C(s)}{a(s)} ds, \quad x \geq 0.$$  

We have $LF(x) = -1$ and hence for $x > 0$,

$$0 \leq E^x F(x_{\tau_0 \wedge t}) = F(x) - E^x (\tau_0 \wedge t), \quad t > 0.$$

Letting $t \to \infty$ we obtain $E^x \tau_0 \leq F(x)$ and hence (3.1) implies $\sup_{x>0} E^x \tau_0 < \infty$.

Conversely, letting $\tau_n := \inf\{t \geq 0 : x_t \geq n\}$ we have

$$F(x) = E^x \tau_0 \wedge \tau_n + E^x F(\tau_0 \wedge \tau_n) \leq F(n) P^x (\tau_n < \tau_0) + E^x \tau_0, \quad n > x.$$  

Since for $G(x) := \int_0^x e^{-C(r)} dr$ one has $L^x G = 0$, it follows that

$$F(x) = E^x G(x_{\tau_0 \wedge \tau_n}) = G(n) P^x (\tau_n < \tau_0), \quad n > x.$$  

Combining this with (3.2) we arrive at

$$F(x) \leq E^x \tau_0 + \frac{F(n) G(x)}{G(n)}, \quad n > x.$$  

This implies that $F(\infty) < \infty$ provided $\sup_{x>0} E^x \tau_0 < \infty$. Indeed, if $F(\infty) = \infty$, then since $\int_0^\infty (e^{C(r)} / a(r)) dr < \infty$, we have $G(\infty) = \infty$ and $F(n)/G(n) \to 0$ as $n \to \infty$. Thus, by letting $n \to \infty$, we see from (3.3) that $F(x) \leq E^x \tau_0$ for all $x > 0$ and hence $\sup_{x>0} E^x \tau_0 = \infty$.

(b) Strong ergodicity implies $\sup_{x>0} E^x \tau_0 < \infty$. If $P_t$ is strongly ergodic, then there exists $t > 0$ such that $\inf_{x>1} P^x (x_t \leq 1) \geq \frac{1}{2} \mu([0,1]) =: c > 0$, where $\mu$ is the invariant probability measure. Thus,

$$P^x (\tau_1 > 2t) \leq P^x (x_t > 1, x_{2t} > 1) = E^x 1_{\{x_t > 1\}} P^x_t (x_t > 1) \leq (1 - c)^2, \quad x > 1.$$  

Similarly, we have

$$P^x (\tau_1 > nt) \leq (1 - c)^n, \quad x > 1.$$  

Therefore,

$$E^x \tau_1 = \int_0^\infty P^x (\tau_1 > s) ds \leq \delta < \infty$$  

for some $\delta > 0$ and all $x > 1$.

On the other hand, by the proof in (a) we see that

$$E^x \tau_1 = \int_1^x e^{-C(r)} dr \int_r^\infty \frac{e^C(s)}{a(s)} ds, \quad x > 1.$$  

Then $\int_0^\infty (e^{C(s)}/a(s)) \, ds < \infty$ and

$$\sup_{x>0} E^x \tau_0 = \sup_{x>1} E^x \tau_0 = \sup_{x>1} E^x \tau_1 + \int_0^1 e^{C(r)} \, dr \int_r^\infty e^{C(s)} \, ds < \infty.$$ 

(c) $\sup_{x>0} E^x \tau_0 < \infty$ implies strong ergodicity. For any $x > y \geq 0$, let $(x_t, y_t)$ be a coupling of the reflecting $L$-diffusion process with $x_0 = x$, $y_0 = y$. We have

$$T := \inf \{ t \geq 0 : x_t = y_t \} \leq \tau_0 := \inf \{ t \geq 0 : x_t = 0 \}.$$ 

As usual we let $x_t = y_t$ for $t > T$ so that for any measurable set $A$ we have

$$|\delta_x P_t(A) - \delta_y P_t(A)| \leq E^{x,y}|1_A(x_t) - 1_A(y_t)| \leq P^{x,y}(x_t \neq y_t) \leq P^{x,y}(T > t).$$

Therefore,

$$\sup_{x>y \geq 0} \| \delta_x P_t - \delta_y P_t \|_{\text{var}} \leq 2 \sup_{x>y \geq 0} P^{x,y}(T > t) \leq 2 \sup_{x>0} P^{x}(\tau_0 > t) \leq \frac{\sup_{x>0} E^x \tau_0}{t},$$

which goes to zero as $t \to \infty$. This means that $P_t$ is strongly ergodic. □

**Example 3.1.** Consider the Ornstein–Uhlenbeck operator $L := d^2/\, dx^2 - xd/\, dx$ on $[0, \infty)$. It is well known that the semigroup $P_t$ of the reflecting $L$-diffusion process is hypercontractive. But according to Theorem 3.1, $P_t$ is not strongly ergodic since (3.1) does not hold. Therefore, $P_t$ does not converge in the $L^1$-norm by Remark 1.2.

**Example 3.2.** Let $M = [0, \infty)$ and consider $L := d^2/\, dx^2 + b(x)d/\, dx$, where

$$b(x) := -\frac{\gamma'(x)}{\gamma(x)} - \frac{1}{\gamma(x)}, \quad x \geq 0,$$

with $\gamma$ constructed as follows. For any $n \geq 1$, let $\phi_n \in C^\infty[0, \infty)$ be non-negative such that $\phi_n|_{[n, n+e^{-n}]} = 0$ and

$$\phi_n|_{[n+e^{-n}/4, n+3e^{-n}/4]} = \max \phi_n = e^n (1 + n)^{-2}.$$ 

Set $\gamma(r) = (1 + r)^{-2} + \sum_{n \geq 1} \phi_n(r)$, $r \geq 0$. Then $P_t$ is strongly ergodic and hence $L^1$-convergent but not hypercontractive.

**Proof.** We have

$$C(x) := \int_0^x b(r) \, dr = -\log \gamma(x) - \int_0^x \frac{dr}{\gamma(r)}, \quad x \geq 0.$$ 

Then

$$\exp[-C(x)] = \gamma(x) \exp \left[ \int_0^x \frac{dr}{\gamma(r)} \right], \quad \exp[C(x)] = -\frac{d}{dx} \exp \left[ -\int_0^x \frac{dr}{\gamma(r)} \right].$$
Therefore, \( \int_0^\infty e^{C(x)} \, dx = 1 \) and
\[
\int_0^\infty e^{-C(x)} \, dx \int_x^\infty e^{C(y)} \, dy = \int_0^\infty \gamma(x) \, dx \leq \int_0^\infty \frac{dr}{1 + r^2} + \sum_{n=1}^{\infty} \frac{1}{1 + n^2} < \infty.
\]
Thus, (3.1) holds and hence \( P_t \) is strongly ergodic.

On the other hand, we use [2, Theorem 1.1] to disprove the log-Sobolev inequality. Observe that
\[
\int_0^\infty \frac{dr}{\gamma(r)} \geq \frac{1}{3} (1 + x)^3 - \sum_{n \geq 1} (2 + n)^2 e^{-n} =: \frac{1}{3} (1 + x)^3 - c_1.
\]
Then for \( \mu([0, x]) := \int_0^x e^{C(r)} \, dr \),
\[
\int_0^\infty e^{-C(x)} \mu([0, x]) \, dx = \int_0^\infty \gamma(x) \left( \exp \left[ \int_0^x \frac{dr}{\gamma(r)} \right] - 1 \right) \, dx 
\geq e^{-c_1} \int_0^\infty \frac{e^{(1+x)^3/3}}{(1 + x)^2} \, dx - \int_0^\infty \gamma(x) \, dx = \infty
\]
since \( \int_0^\infty \gamma(x) \, dx < \infty \). Moreover,
\[
I(n) := \left( \int_0^{n+e^{-n}} e^{-C(y)} \, dy \right) \left( \int_0^{n+e^{-n}} e^{C(y)} \, dy \right) \left( \log \frac{1}{\int_0^{n+e^{-n}} e^{C(y)} \, dy} \right)
\geq c_2 \left( \int_0^{n+3e^{-n}/4} e^{n+3e^{-n}/4} \exp[(1 + y)^3/3] \, dy \right) \left( \frac{n+e^{-n}}{\gamma(r)} \right) \frac{dr}{\gamma(r)}
\geq c_3 (1 + n) - c_4
\]
for some \( c_2, c_3, c_4 > 0 \). Thus \( \lim_{n \to \infty} I(n) = \infty \). Therefore, according to [2, Theorem 1.1] the log-Sobolev inequality does not hold; see also [11] for a more general result.

**Acknowledgements.** The author would like to thank Professors Mu-Fa Chen, Yong-Hua Mao and Liming Wu for useful conversations. He also thanks the referee for helpful comments.

**References**

diffusion semigroups


Department of Mathematics
Beijing Normal University
Beijing 100875, P.R. China
E-mail: wangfy@bnu.edu.cn

Received August 12, 2002
Revised version December 1, 2003 (5016)